A circular–circular regression model

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Outline

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Bivariate Circular Data

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Bivariate circular data is data which can be expressed as two angles $[0, 2\pi)^2$.

Example

- Wind directions in Milwaukee at 6 a.m. and noon (Downs & Mardia, 2002).
- Time of low tide and spawning time of certain fish (Lund, 1999).

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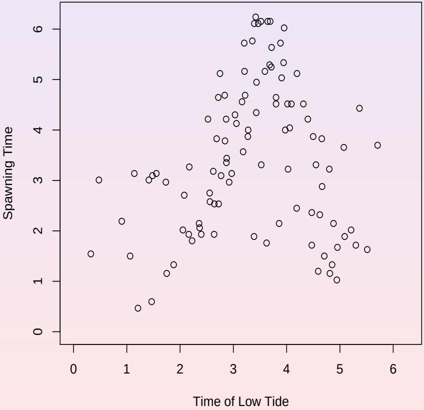

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Time of low tide and spawning time of certain fish (Lund, 1999).

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How can one regress the spawning time on the time of low tide?

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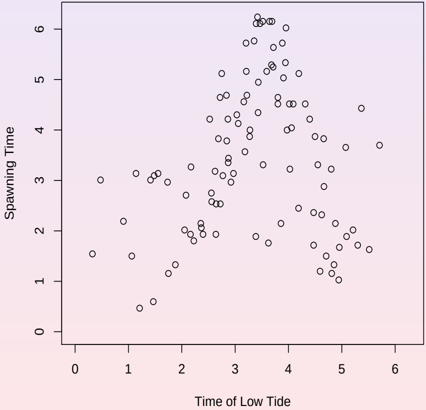

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Question

How can one regress the spawning time on the time of low tide?

What is a Circular–Circular Regression Model?

Definition

A circular–circular regression model is a regression model in which both independent and dependent variables take values on the circle.

Existing models

- Rivest (1997)
- Downs & Mardia (2002)

Purpose of the Research

Our goal

It would be ideal to have a regression model with following properties:

1. flexible regression curve,
2. mathematical tractability,
3. easy interpretation of parameters.

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Regression Curve

Definition

Let

U : unit circle in the complex plane, $U = \{z \in \mathbb{C}; |z| = 1\}$,
u : U-valued independent variable.

The regression curve $v : U \to U$ of the proposed model is defined by

$$v(u) = \beta_0 \frac{u + \beta_1}{1 + \overline{\beta_1} u}, \quad u \in U, \tag{1}$$

where $\beta_0 \in U, \ \beta_1 \in \mathbb{C}$.

The mapping (1) is called a Möbius transformation.

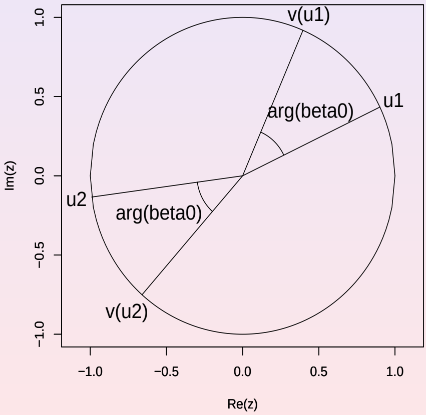
Properties of Regression Curve

$$v(u) = \beta_0 \frac{u + \beta_1}{1 + \overline{\beta_1} u}, \quad u \in U; \quad \beta_0 \in U, \ \beta_1 \in \mathbb{C}. \tag{1}$$

Properties

- For $|\beta_1| \neq 1$, the regression curve (1) is a one–to–one mapping from U onto itself.
- For $|\beta_1| = 1$, (1) maps U onto a point $\beta_0 \beta_1$, i.e. $v(u) = \beta_0 \beta_1$ for any u.

Interpretation of Parameters



$$v(u) = \beta_0 u, \quad u \in U; \quad \beta_0 \in U.$$

Interpretation of β_0

The parameter β_0 is a rotation parameter

- If $\beta_0 = 1$, v is an identity mapping, i.e. $v(u) = u$.
- If $\beta_0 = -1$, u is rotated by π (rad).

Interpretation of Parameters

$$v = \frac{u + \beta_1}{1 + \overline{\beta_1} u}, \quad u \in U;\ \beta_1 \in \mathbb{C}.$$

(i) $|\beta_1| < 1$

Following properties hold for the parameter β_1:

1. $|\beta_1| \to 0 \quad \Longrightarrow \quad v \to u,$
2. $|\beta_1| \to 1,\ u \neq -\beta_1/|\beta_1| \quad \Longrightarrow \quad v \to \beta_1,$
3. $v_j = (u + \beta_{1j})/(1 + \overline{\beta_{1j}} u),\ \beta_{1j} = r_j e^{i\theta},\ j = 1, 2,$

   $r_1 > r_2 \geq 0,\ 0 \leq \theta < 2\pi$

   $\Longrightarrow\ |\arg(v_1) - \theta| \leq |\arg(v_2) - \theta|,$

Interpretation of Parameters

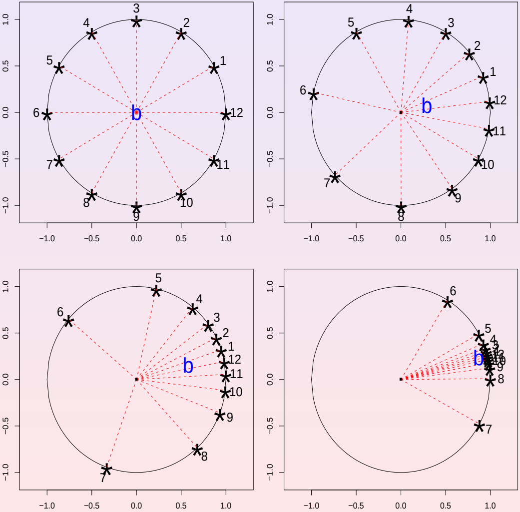
$$v = \frac{u + \beta_1}{1 + \overline{\beta_1} u}, \quad u \in U; \; \beta_1 \in \mathbb{C}.$$

(i) $|\beta_1| < 1$

Following properties hold for the parameter β_1:

4. $|\arg(u) - \arg(\beta_1)| \geq |\arg(v) - \arg(\beta_1)|$,
5. $u_1 = \beta_1/|\beta_1| \implies v(u_1) = \beta_1/|\beta_1|$,
6. $u_2 = -\beta_1/|\beta_1|, \; |\beta_1| \neq 1 \implies v(u_2) = -\beta_1/|\beta_1|$,
7. $u_1 = \theta\beta_1/|\beta_1|, \; u_2 = \overline{\theta}\beta_1/|\beta_1|, \; |\theta| = 1$
$$\implies v(u_1)\overline{\theta} = \overline{v(u_2)}\theta.$$

Interpretation of Parameters



$$v(u) = \frac{u + \beta_1}{1 + \overline{\beta_1} u},$$

$$u = exp(2\pi k i/12),$$

$$k = 1, \ldots, 12,$$

$$\arg(\beta_1) = \pi/12,$$

1. $|\beta_1| = 0$ (above left),
2. $|\beta_1| = 0.3$ (above right),
3. $|\beta_1| = 0.6$ (below left),
4. $|\beta_1| = 0.9$ (below right).

Interpretation of Parameters

$$v(u) = \frac{u + \beta_1}{1 + \overline{\beta_1} u}, \quad u \in U; \ \beta_1 \in \mathbb{C}. \qquad (1)$$

(ii) $|\beta_1| = 1$

For $|\beta_1| = 1$, (1) maps U onto a point β_1:

$$v(u) = \beta_1 \quad \text{for any } u \in U.$$

Interpretation of Parameters

(iii) $|\beta_1| > 1$

For $|\beta_1| > 1$, regression curve can be expressed as

$$v = \frac{u + \beta_1}{1 + \overline{\beta_1} u} = \frac{u' + {\beta_1}'}{1 + \overline{{\beta_1}'} u'}, \tag{2}$$

where $u' = (\beta_1/|\beta_1|)(\beta_1 \overline{u}/|\beta_1|)$ and ${\beta_1}' = 1/\overline{\beta_1}$.

(2) shows that (1) consists of two types of transformations:

1. reflection with respect to $\overline{\beta_1} z - \beta_1 \overline{z} = 0$,
2. Möbius transformation with β_1' $(|\beta_1'| < 1)$.

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Distribution for Angular Error

Definition

The wrapped Cauchy (WC) distribution is defined by pdf

$$f(z) = \frac{1}{2\pi} \frac{\left|1 - |\phi|^2\right|}{|z - \phi|^2}, \quad z \in U, \tag{3}$$

where $\phi \in \mathbb{C} \setminus U$.

We write $Z \sim C^* (\phi)$ if random variable Z has the density (3).

Properties of Wrapped Cauchy Distribution

$$f(z) = \frac{1}{2\pi} \frac{\left|1 - |\phi|^2\right|}{|z - \phi|^2}, \quad u \in U; \quad \phi \in \mathbb{C} \setminus U.$$

Basic properties

The WC distribution has following properties:

- unimodality,
- symmetry about $z = \phi/|\phi|$,
- mode at $z = \phi/|\phi|$, antimode at $z = -\phi/|\phi|$,
- mean direction: $\arg\{E(Z)\} = \arg(\phi)$,
- concentration: $|E(Z)| = |\phi|$.

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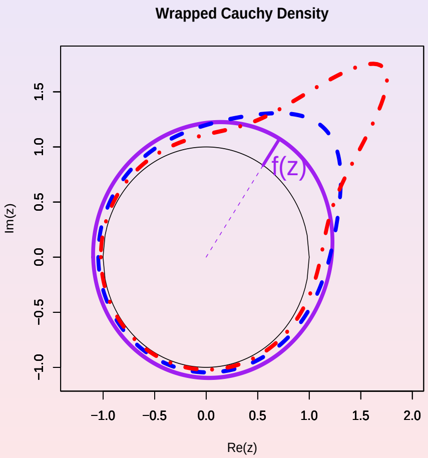
Wrapped Cauchy Distribution



Wrapped Cauchy density

$$f(z) = \frac{1}{2\pi} \frac{\left|1 - |\phi|^2\right|}{|z - \phi|^2}, \quad u \in U,$$

$$\arg(\phi) = \pi/4,$$

1. ——————— $|\phi| = 0.3$,
2. – – – – – – – $|\phi| = 0.6$,
3. — · — · — $|\phi| = 0.8$.

Properties of Wrapped Cauchy Distribution

Properties

Following properties hold for WC distribution:

- closed under rotation,

$$Z \sim C^*(\phi) \implies \beta_0 Z \sim C^*(\beta_0 \phi), \quad \beta_0 \in U,$$

- additive property,

$$Z_1 \sim C^*(\phi_1), \ Z_2 \sim C^*(\phi_2), \ Z_1 \perp Z_2 \implies Z_1 Z_2 \sim C^*(\phi_1 \phi_2),$$

- closed under Möbius transformation,

$$Z \sim C^*(\phi) \implies \frac{Z + \beta_1}{1 + \overline{\beta_1} Z} \sim C^*\left(\frac{\phi + \beta_1}{1 + \overline{\beta_1}\phi}\right), \quad \beta_1 \notin U.$$

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Derivation of Wrapped Cauchy Distribution

Derivation

Let

$X \sim$ Cauchy distribution on the real line $C(\mu, \sigma^2)$.

Then

1. $Z_1 = \exp(iX)$ has WC distribution $C^* \left\{ \exp(-\sigma^2 + i\mu) \right\}$,
2. $Z_2 = \dfrac{1 + iX}{1 - iX}$ has WC distribution $C^* \left(\dfrac{1 + i\theta}{1 - i\theta} \right)$,

   where $\theta = \mu + i\sigma$.

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   where $\theta = \mu + i\sigma$.

Circular–Circular Regression Model

Definition

Let

Y: U-valued dependent variable,
x: U-valued independent variable.

We propose a circular–circular regression model defined by

$$Y = \beta_0 \frac{x + \beta_1}{1 + \overline{\beta_1} x} \, \varepsilon, \quad x \in U, \tag{4}$$

where $\varepsilon \sim C^*(\varphi),\ 0 \le \varphi < 1,\ \beta_0 \in U,\ \beta_1 \in \mathbb{C}.$

Properties of Proposed Model

$$Y \,|\, x \sim C^{*}\left(\beta_0 \frac{x+\beta_1}{1+\overline{\beta_1} x}\, \varphi\right), \; x \in U, \; \beta_0 \in U, \; \beta_1 \in \mathbb{C}, \; 0 \leq \varphi < 1.$$

Properties

- The k th trigonometric moment of $Y \,|\, x$

$$E\left(Y^k \,|\, x\right) = \left(\beta_0 \frac{x+\beta_1}{1+\overline{\beta_1} x}\, \varphi\right)^k .$$

Properties of Proposed Model

$$Y \mid x \sim C^* \left(\beta_0 \frac{x+\beta_1}{1+\overline{\beta_1} x} \varphi \right), \quad x \in U, \ \beta_0 \in U, \ \beta_1 \in \mathbb{C}, \ 0 \le \varphi < 1.$$

Properties

- Mean direction

$$\arg\{E(Y \mid x)\} = \arg\left(\beta_0 \frac{x+\beta_1}{1+\overline{\beta_1} x} \right).$$

- Concentration

$$|E(Y \mid x)| = \varphi.$$

Properties of Proposed Model

$$Y \,|\, x \sim C^* \left(\beta_0 \frac{x + \beta_1}{1 + \overline{\beta_1} x} \varphi \right), \; x \in U, \; \beta_0 \in U, \; \beta_1 \in \mathbb{C}, \; 0 \le \varphi < 1.$$

Properties

- Closed under Möbius transformation:

$$\gamma_0 \frac{Y + \gamma_1}{1 + \overline{\gamma_1} Y} \,\Bigg|\, x \sim C^* \left(\frac{ax + b}{cx + d} \right),$$

where $\gamma_0 \in U, \quad \gamma_1 \in \mathbb{C},$

$$a = \gamma_0(\beta_0 \varphi + \gamma_1 \overline{\beta_1}), \quad b = \gamma_0(\gamma_1 + \beta_0 \beta_1 \varphi),$$
$$c = \overline{\beta_1} + \overline{\gamma_1} \beta_0 \varphi, \quad d = 1 + \overline{\gamma_1} \beta_0 \beta_1 \varphi.$$

$$Y \,|\, x \sim C^* \left(\beta_0 \frac{x + \beta_1}{1 + \overline{\beta_1} x} \varphi \right), \quad x \in U, \ \beta_0 \in U, \ \beta_1 \in \mathbb{C}, \ 0 \le \varphi < 1.$$

Properties

- Closed under Möbius transformation:

$$Y \left| \frac{\gamma_{00} x + \gamma_{01}}{\gamma_{10} x + \gamma_{11}} \right. \sim C^* \left(\frac{ax + b}{cx + d} \right),$$

where $\gamma_{jk} \in \mathbb{C}, \quad j, k = 0, 1,$

$$a = \beta_0 (\gamma_{00} + \beta_1 \gamma_{10} \varphi), \quad b = \beta_0 (\gamma_{01} + \beta_1 \gamma_{11}) \varphi,$$
$$c = \gamma_{10} + \overline{\beta_1} \gamma_{00}, \quad d = \gamma_{11} + \overline{\beta_1} \gamma_{01}.$$

Properties of Proposed Model

Properties

Let

$f(s,t)$: a continuous real function on the closed unit disc satisfying

$$\Delta f = \frac{\partial^2 f}{\partial s^2} + \frac{\partial^2 f}{\partial t^2} = 0,$$

for all $y = s + it$ in the open disc. Then

$$Y \,|\, x \sim C^* \left(\beta_0 \frac{x + \beta_1}{1 + \overline{\beta_1} x} \varphi \right) \Longrightarrow E\left\{ f(Y) | x \right\} = f\left(\beta_0 \frac{x + \beta_1}{1 + \overline{\beta_1} x} \varphi \right).$$

Existing Models

Planar–linear regression model (McCullagh, 1996)

Y: $\mathbb{R}$-valued dependent variable,
z: $\mathbb{C}$-valued independent variable.

The regression model of McCullagh (1996) is defined by

$$Y \mid z \sim C\left(\frac{\beta_{00} z + \beta_{01}}{\beta_{10} z + \beta_{11}}\right),$$

where $C(\theta)$ is real Cauchy distribution with median $\mathrm{Re}(\theta)$ and scale parameter $\mathrm{Im}(\theta)$, and

$$\begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

Existing Models

Circular–circular regression model (Downs & Mardia, 2002)

Y : $[0, 2\pi)$-valued dependent variable,
x : $[0, 2\pi)$-valued independent variable.

Downs & Mardia (2002) proposed a regression model as follows:

$$Y \mid x \sim VM\left\{\mu(x), \kappa\right\},$$

where $\kappa \geq 0$,

$$\tan\left[\left\{\mu(x) - \nu\right\}/2\right] = \omega \tan\left\{(x - \eta)/2\right\},$$

$$0 \leq \nu, \eta < 2\pi, \quad -1 \leq \omega \leq 1.$$

Existing Models

$$Y \mid x \sim VM\left\{\mu(x), \kappa\right\},$$

$$\tan\left\{\frac{\mu(x)-\nu}{2}\right\} = \omega \tan\left(\frac{x-\eta}{2}\right).$$

Von Mises distribution

The von Mises distribution $VM(\mu, \kappa)$ has the density

$$f(\theta) = \frac{1}{2\pi I_0(\kappa)} \exp\left\{\kappa \cos(\theta-\mu)\right\}, \quad 0 \le \theta < 2\pi,$$

where $\quad 0 \le \mu < 2\pi, \quad \kappa \ge 0,$

I_0 : Bessel function of the first kind and order 0.

Parameter Estimation

Maximum likelihood estimation

$$Y_j \mid x_j \sim C^* \left(\beta_0 \frac{x_j + \beta_1}{1 + \overline{\beta_1} x_j} \varphi \right), \quad j = 1, \ldots, n.$$

$$Y_i \mid x_i \perp Y_j \mid x_j, \quad i \neq j.$$

The log-likelihood function is given by

$$\log L = C + \sum_{j=1}^{n} \left\{ \log \left| 1 - \varphi^2 \right| - 2 \log \left| y_j - \beta_0 \frac{x_j + \beta_1}{1 + \overline{\beta_1} x_j} \varphi \right| \right\}.$$

If β_1 is known, the algorithm by Kent & Tyler (1988) is available to estimate the parameters.

Parameter Estimation

Method of moments estimation

$$Y_j \,|\, x_j \sim i.i.d.\ C^*\left(\beta_0 \frac{x_j + \beta_1}{1 + \overline{\beta_1} x_j} \varphi\right), \quad j = 1, \ldots, n.$$

$$Y_i \,|\, x_i \perp Y_j \,|\, x_j, \quad i \neq j.$$

If β_1 is known, the moment estimates are

$$\hat{\beta}_0 = \sum_{j=1}^{n} y_j \frac{\overline{x_j} + \overline{\beta_1}}{1 + \beta_1 \overline{x_j}} \Bigg/ \left| \sum_{j=1}^{n} y_j \frac{\overline{x_j} + \overline{\beta_1}}{1 + \beta_1 \overline{x_j}} \right|,$$

$$\hat{\varphi} = \frac{1}{n} \left| \sum_{j=1}^{n} y_j \frac{\overline{x_j} + \overline{\beta_1}}{1 + \beta_1 \overline{x_j}} \right|.$$

A test of independence

A test of independence

$$H_0 : |\beta_1| = 1 \quad \text{v.s.} \quad H_1 : |\beta_1| \neq 1.$$

The likelihood ratio test gives the test statistic as

$$T = -2 \log \frac{\max L_0}{\max L_1},$$

where L_i is the likelihood function under H_i, $i = 0, 1$.

$\max L_0$ is obtained using the algorithm by Kent & Tyler (1988).

Related Bivariate Circular Distribution

Definition

We propose a bivariate circular distribution with pdf

$$f(x,y) = \frac{1}{(2\pi)^2} \frac{\left|1-\varphi^2\right|}{\left|y-\beta_0(x+\beta_1)\varphi/(1+\overline{\beta_1}x)\right|^2} \frac{\left|1-|\phi|^2\right|}{|x-\phi|^2},$$

$$|x|=|y|=1;\ \beta_0 \in U,\ \beta_1 \in \mathbb{C},\ \phi \in \mathbb{C}\setminus U,\ 0 \le \varphi < 1.$$

Property

- Conditional distribution

$$Y\,|\,x \sim C^*\left(\beta_0 \frac{x+\beta_1}{1+\overline{\beta_1}x}\varphi\right).$$

Properties of Bivariate Circular Distribution

$$f(x,y) = \frac{1}{(2\pi)^2} \frac{\left|1-\varphi^2\right|}{\left|y-\beta_0(x+\beta_1)\varphi/(1+\overline{\beta_1}x)\right|^2} \frac{\left|1-|\phi|^2\right|}{|x-\phi|^2},$$

Properties

The model has following properties:

- $X \sim C^*(\phi)$,
- $Y \sim C^*\left(\beta_0 \dfrac{\phi+\beta_1}{1+\overline{\beta_1}\phi}\varphi\right)$,
- $|\beta_1| = 1 \implies X \sim C^*(\phi),\ Y \sim C^*(\beta_0\beta_1\varphi),\ X \perp Y$,
- $\varphi = 0 \implies X \sim C^*(\phi),\ Y \sim C^*(0),\ X \perp Y$.

Multiple Circular Regression

Definition

Let

Y: U-valued dependent variable,

$x_1, \dots, x_p$: U-valued independent variables.

We propose a multiple circular regression model defined by

$$Y = \beta_0 \prod_{j=1}^{p} \frac{x_j + \beta_j}{1 + \overline{\beta_j} x_j} \, \varepsilon, \quad x_j \in U,$$

where $\varepsilon \sim C^*(\varphi)$, $0 \le \varphi < 1$, $\beta_0 \in U$, $\beta_j \in \mathbb{C}$, $j = 1, \dots, p$.

Conclusion

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- We proposed a circular–circular regression model with flexible regression curve and easy interpretation of the parameters.
- Some properties of the model are obtained using the theory of Möbius transformation and wrapped Cauchy distribution.
- As related topics, a bivariate circular distribution and a multiple circular regression model are introduced.

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Conclusion

- We proposed a circular–circular regression model with flexible regression curve and easy interpretation of the parameters.
- Some properties of the model are obtained using the theory of Möbius transformation and wrapped Cauchy distribution.
- As related topics, a bivariate circular distribution and a multiple circular regression model are introduced.

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