AGGREGATE CLAIMS, SOLVENCY AND REINSURANCE

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Basic General Insurance Risk Model

$$S = \sum_{i=1}^{N} X_i$$

where

- S represents the aggregate amount of claims in a fixed period, e.g. one year
- N is a counting variable representing the number of claims
- X_i = amount of the i-th claim
- $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables

Standard Problem - find the distribution function of S

Reasons

- Premium setting

- Setting appropriate levels of reinsurance

- Solvency, e.g. for a given premium P, find the capital u such that

$$\Pr(u + P > S) = 0.99$$

How to fit our model for aggregate claims?

- Model for the number of claims, e.g. Poisson, negative binomial, or zero-modified versions

- Model for claim amounts, e.g. lognormal, Pareto

- Parameter estimation by maximum likelihood, standard goodness of fit tests

Excellent reference: "Loss Models - from Data to Decisions" by Klugman, Panjer & Willmot

Computational Issues

- Distribution function of S is

$$\text{Pr}(S \leq x) = \sum_{n=0}^{\infty} \text{Pr}(N = n) F^{n*}(x)$$

where $F(x) = \text{Pr}(X_i \leq x)$, and F^{n*} is the n-fold convolution

- Exact computation is difficult

- Approximations are often used - especially moment based approximations - but estimation of counting and claim amount distributions is important

Insurer's surplus process before reinsurance

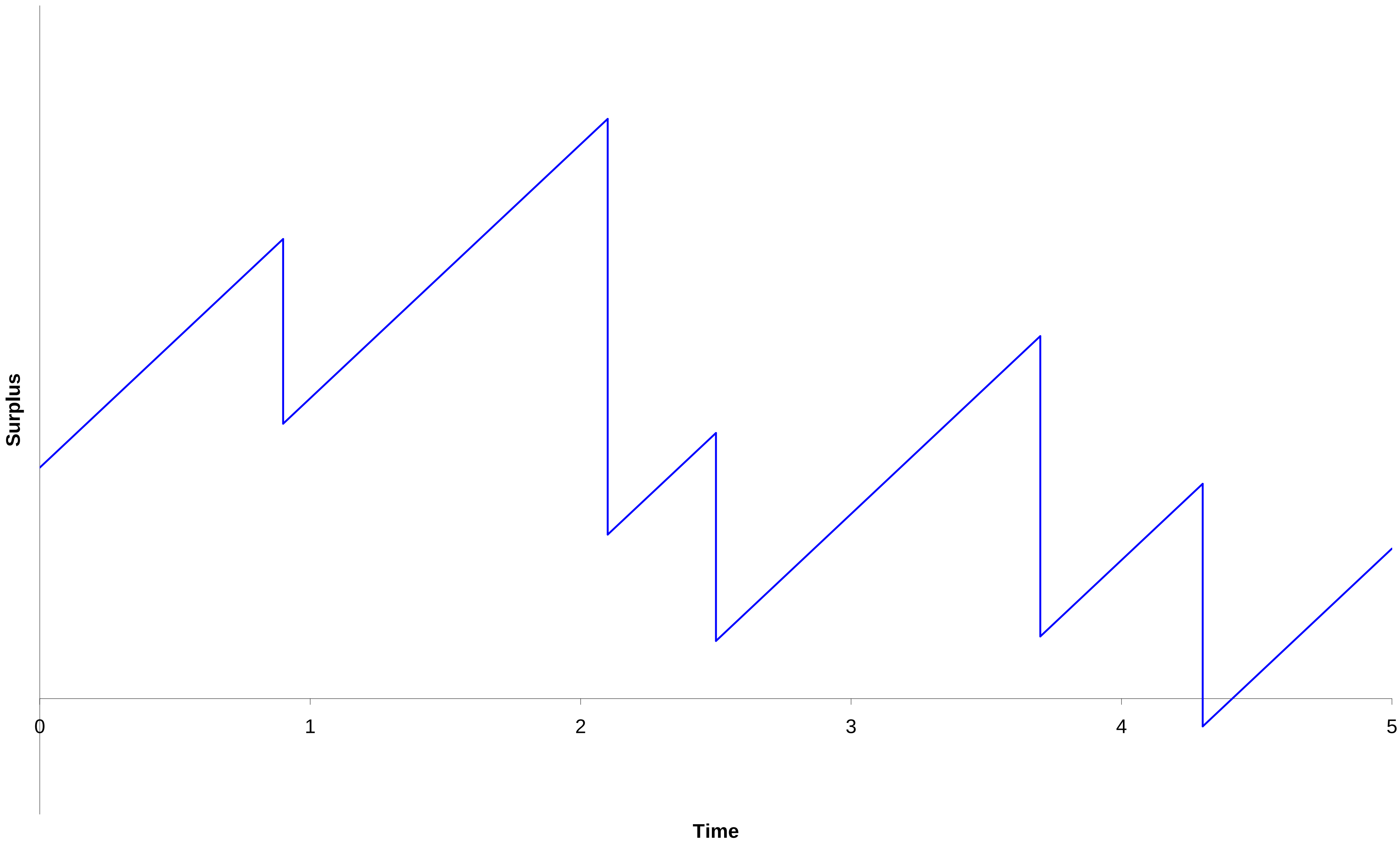
$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$$

$u =$ initial surplus,

$c =$ rate of premium income per unit time (year),

$N(t) =$ number of claims in $[0, t]$, $N(t) \sim$ Poisson(λt),

$X_i =$ amount of the i-th claim.


Surplus
0
1
2
3
4
5
Time

Let θ denote a general reinsurance arrangement.

Then the insurer's surplus process **after reinsurance** is

$$U_\theta(t) = u + c_\theta t - S_\theta(t)$$

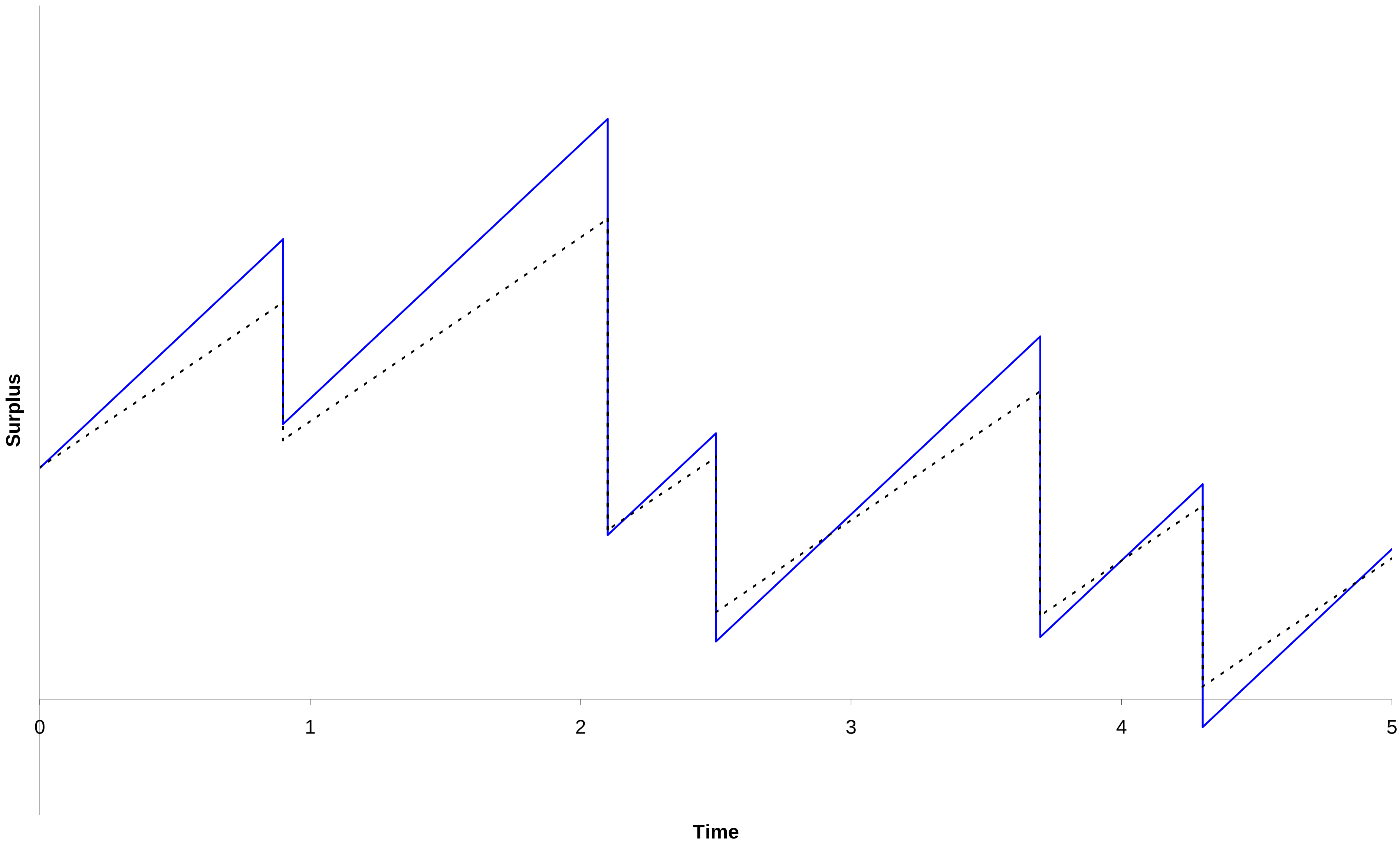
where

$U_\theta(t) =$ insurer's surplus at time t,

$c_\theta =$ rate of premium income to the insurer net of reinsurance,

$S_\theta(t) =$ net aggregate claims for the insurer in $[0, t]$,

$S_\theta(t)$ has cdf $G_\theta(x; t)$ and pdf $g_\theta(x; t)$ for $x > 0$.


Surplus
0
1
2
3
4
5
Time

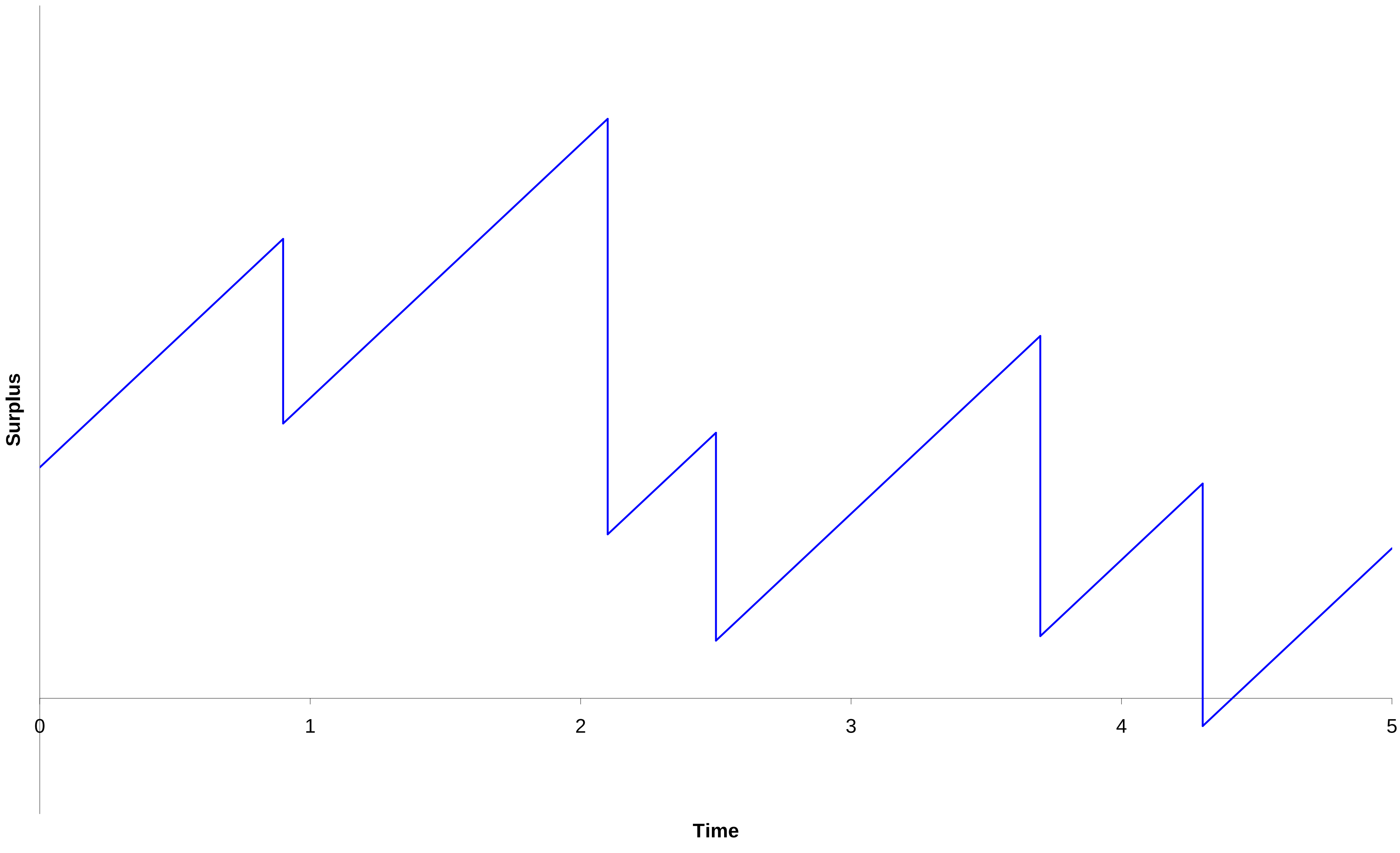
Probability of ruin - definitions

Discrete time:

$$\psi_\theta(u,t) = \text{Pr}[U_\theta(s) < 0 \text{ for some } s,\ s = 1, 2, \ldots, t]$$

Continuous time:

$$\psi_\theta(u,t) = \text{Pr}[U_\theta(s) < 0 \text{ for some } s,\ s \in (0, t]]$$


Surplus
0
1
2
3
4
5
Time

The problem: Determine:

$$\hat{\psi}(u,t) = \inf_{\theta \in \Theta} \psi_\theta(u,t)$$

where:

- Θ is a set of admissible reinsurance arrangements.
  In our study, admissible means:
  - either proportional or excess of loss reinsurance throughout the period,
  - the insurer's annual net income exceeds net expected claims.
- θ can be changed at the start of each year.
- Ruin can be in discrete or continuous time.

Proportional reinsurance

Let X_i denote the amount of the i-th individual claim. Under a proportional reinsurance arrangement with proportion retained a,

the insurer pays aX_i

the reinsurer pays $(1 - a)X_i$.

Excess of Loss Reinsurance

Let X_i denote the amount of the i-th individual claim. Under an excess of loss reinsurance arrangement with retention level M,

the insurer pays $\min(X_i, M)$

the reinsurer pays $\max(0, X_i - M)$.

Discrete time – formulae

$$
\begin{aligned}
\hat{\psi}(u,1) &= \inf_{\theta\in\Theta}(1 - G_\theta(u+c_\theta;1)) \\
\psi_\theta(u,n) &= \psi_\theta(u,1) \\
&\quad + \int_0^{u+c_\theta} g_\theta(u+c_\theta-x;1)\,\hat{\psi}(x,n-1)\,dx \\
&\quad + e^{-\lambda}\hat{\psi}(u+c_\theta,n-1) \\
\hat{\psi}(u,n) &= \inf_{\theta\in\Theta}\psi_\theta(u,n)
\end{aligned}
$$

(Bellman Principle)

Calculating ruin probabilities

Translated gamma distribution: For each θ calculate $\alpha_\theta, \beta_\theta, \kappa_\theta$ such that:

$$\mathbf{E}[(Y_\theta + \kappa_\theta)^i] = \mathbf{E}[(S_\theta(1))^i] \quad i = 1, 2, 3$$

where $Y_\theta \sim \Gamma(\alpha_\theta, \beta_\theta)$ with cdf $\Gamma_\theta(x; \alpha_\theta, \beta_\theta)$ and pdf $\gamma_\theta(x; \alpha_\theta, \beta_\theta)$.

Then for $x \geq 0$

$$\begin{aligned} G_\theta(x; 1) &\approx \Pr[Y_\theta + \kappa_\theta \leq x] \\ &= \Gamma_\theta(x - \kappa_\theta; \alpha_\theta, \beta_\theta). \end{aligned}$$

In particular,

$$G_\theta(0; 1) = e^{-\lambda} \approx \Pr[Y_\theta + \kappa_\theta \leq 0] = \Gamma_\theta(-\kappa_\theta; \alpha_\theta, \beta_\theta).$$

The pdf approximation is $g_\theta(x; 1) \approx \gamma_\theta(x - \kappa_\theta; \alpha_\theta, \beta_\theta)$.

Translated gamma distribution approximation to $\psi_\theta(u,n)$

1. Original (compound Poisson) process:

$$\begin{aligned}&\psi_\theta(u,1)\\&+\int_0^{u+c_\theta} g_\theta(u+c_\theta-x;1)\,\hat{\psi}(x,n-1)\,dx\\&+e^{-\lambda}\hat{\psi}(u+c_\theta,n-1)\end{aligned}$$

2. Translated gamma distribution approximation:

$$\begin{aligned}&1-\Gamma_\theta(u+c_\theta-\kappa_\theta;\alpha_\theta,\beta_\theta)\\&+\int_0^{u+c_\theta} \gamma_\theta(u+c_\theta-\kappa_\theta-x;\alpha_\theta,\beta_\theta)\,\hat{\psi}(x,n-1)\,dx\\&+\Gamma_\theta(-\kappa_\theta;\alpha_\theta,\beta_\theta)\hat{\psi}(u+c_\theta,n-1)\end{aligned}$$

Computational Issues

Need to use numerical integration.

Can reduce the amount of calculation required by introducing a truncation procedure - essentially we set very small ruin probabilities to zero.

A grid search is required for the optimal retention level - we limit our set of possible retention levels, e.g. 0.01,0.02,...,0.99,1 for proportional reinsurance.

Computations for discrete time ruin are considerably faster than for continuous time - factor may depend on programming language.

It does not appear possible to find explicit solutions!

Questions to be answered

To what extent is a dynamic reinsurance policy better than a static policy?

To what extent does the time horizon affect the optimal reinsurance strategy?

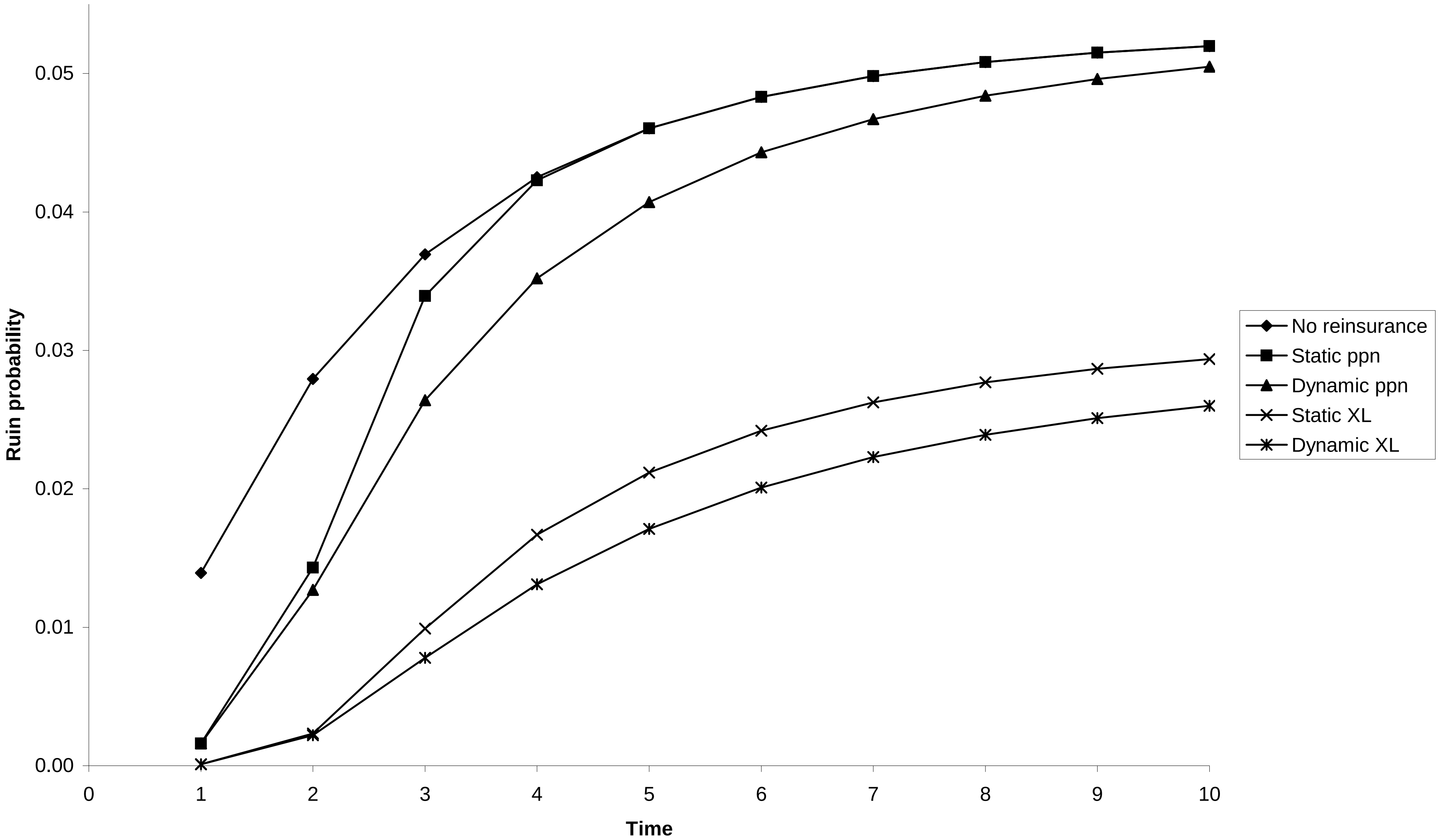
Is minimising the probability of ruin (in finite time) a sensible optimisation criterion?

Numerical example

- Individual claim amount distribution is Exponential(1)
- Premium loading factors: 0.1 (insurer) & 0.2 (reinsurer)
- Poisson parameter: $\lambda = 100$
- Initial surplus is $u = 23$ in graphical illustration

Surplus, u		Remaining term, t								
		9	8	7	6	5	4	3	2	1
10	M	2.1	2.1	2.1	2.0	2.0	1.8	1.7	1.3	0.7
	$\hat{\psi}(u,t)$	0.128	0.126	0.124	0.121	0.116	0.110	0.099	0.081	0.039
20	M	1.4	1.4	1.4	1.3	1.3	1.1	1.0	0.7	0.7
	$\hat{\psi}(u,t)$	0.037	0.036	0.034	0.031	0.028	0.023	0.016	0.006	0.000
30	M	1.4	1.4	1.3	1.2	1.1	0.9	0.7	0.7	0.7
	$\hat{\psi}(u,t)$	0.010	0.009	0.008	0.007	0.005	0.003	0.001	0.000	0.000
40	M	1.3	1.2	1.2	1.0	0.8	0.7	0.7	0.7	0.7
	$\hat{\psi}(u,t)$	0.002	0.002	0.002	0.001	0.001	0.000	0.000	0.000	0.000

Optimal strategy: Exponential claims, loadings 10%/20%, excess loss, discrete time ruin, distributional approximation.


Ruin probability
0.00
0.01
0.02
0.03
0.04
0.05
Time
0
1
2
3
4
5
6
7
8
9
10
No reinsurance
Static ppn
Dynamic ppn
Static XL
Dynamic XL

Questions to be answered

To what extent is a dynamic reinsurance policy better than a static policy?

– Significantly.

To what extent does the time horizon affect the optimal reinsurance strategy?

– Significantly.

Is minimising the probability of ruin (in finite time) a sensible optimisation criterion?

– Answered later!

Continuous time – formulae

Let $\delta_\theta(u,t) = 1 - \psi_\theta(u,t)$ be the survival probability to time t (years).

Prahbu's formula is

$$\delta_\theta(u,1) = G_\theta(u + c_\theta, 1) - c_\theta \int_0^1 \delta_\theta(0, 1-s) g_\theta(u + c_\theta s, s)\, ds$$

with

$$\delta_\theta(0,t) = \frac{1}{c_\theta t} \int_0^{c_\theta t} G_\theta(y,t)\, dy.$$

Then for $n > 1$ we have

$$\hat{\psi}(u,n) = \inf_{\theta} \psi_\theta(u,n),$$

where, for a given value of θ,

$$\begin{aligned} \psi_\theta(u,n) &= \psi_\theta(u,1) + e^{-\lambda}\,\hat{\psi}(u+c_\theta, n-1) + \int_0^{c_\theta} \delta_\theta(u,1,y)\,\hat{\psi}(y,n-1)\,dy \\ &\quad + \int_{c_\theta}^{u+c_\theta} g_\theta(u+c_\theta-y,1)\,\hat{\psi}(y,n-1)\,dy \end{aligned}$$

and

$$\delta_\theta(u,1,y) = g_\theta(u+c_\theta-y,1) - c_\theta \int_0^{1-y/c_\theta} g_\theta(u+c_\theta s, s)\delta_\theta(0,1-s,y)ds$$

with

$$\delta_\theta(0,t,y) = \frac{y}{c_\theta t}\, g_\theta(c_\theta t - y, t).$$

Translated gamma distribution approximation

This can be applied with ease

Formulae are relatively straightforward.

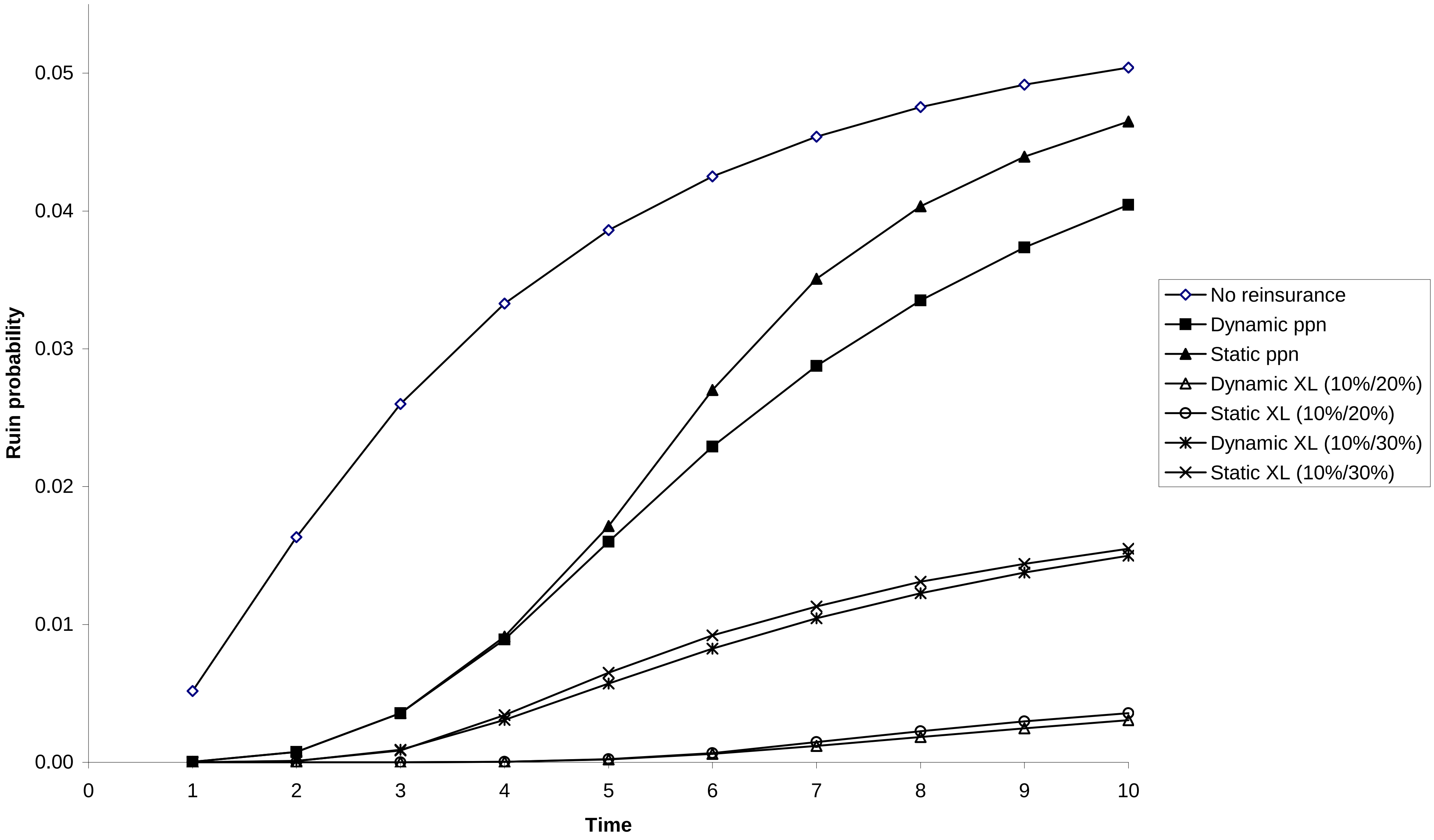
Numerical example

- Initial surplus is $u = 49$

- Individual claim amount distribution is Pareto(4,3) (mean is 1)

- Premium loading factors: 0.1 (insurer) & 0.2 or 0.3 (reinsurer)

- Poisson parameter: $\lambda = 100$



Ruin probability
0.00
0.01
0.02
0.03
0.04
0.05
0
1
2
3
4
5
6
7
8
9
10
Time
No reinsurance
Dynamic ppn
Static ppn
Dynamic XL (10%/20%)
Static XL (10%/20%)
Dynamic XL (10%/30%)
Static XL (10%/30%)

Questions to be answered

To what extent is a dynamic reinsurance policy better than a static policy?

– Clearly better for proportional reinsurance, but only at the longer time horizons.

– Marginally better for excess loss reinsurance for both reinsurance loadings.

To what extent does the time horizon affect the optimal reinsurance strategy?

– Significantly: for example, with excess of loss reinsurance and a 30% reinsurance loading factor we get the following values of M for the first year:

Term	10	8	6	4	2
Dynamic	3.0	2.8	2.4	1.7	1.4
Static	2.9	2.6	2.3	1.5	1.3

Questions to be answered

Is minimising the probability of ruin (in finite time) a sensible optimisation criterion?

– Yes, but it can lead to strategies that might not appear sensible.

For example, in the previous table, with $u = 49$ and a remaining term of 1 year, why should the insurer cede so much risk?

Alternative criteria, e.g. minimise the ruin probability over a fixed but rolling time horizon?

– Cannot reduce the ruin probability over the original time horizon.

Concluding Remarks

A dynamic strategy appears to have a greater effect if we consider discrete time ruin.

If we consider discrete time only, the translated gamma approximation still applies for other models, e.g. in the case of a compound negative binomial distribution for claim numbers.

It is computationally much faster to compute the optimal static strategy, especially in continuous time. Ruin probabilities under a static strategy bound those under a dynamic strategy.

Our study contained a second approach to calculation using translated gamma processes. Results were virtually identical - confidence in robustness of approaches.

Acknowledgement: co-author is Prof Howard Waters, Heriot-Watt University, Edinburgh

Paper is at www.economics.unimelb.edu.au/actwww/