HMM for precipitation

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Outline

- Basic HMM
  - Model description
  - Limitations
- Extensions
  - HMM with truncated Gaussian distributions
- Conclusion and perspectives

Basic HMM

Model description

- First proposed by Zucchini & Guttorp (1991)
  - Generalized to a nonhomogeneous HMM
    - Hughes et al. (1994)
- <u>Observed process:</u> $Y_t = (Y_t(1),\ldots,Y_t(K))$
  - $Y_t(k) \in R^+$: rainfall during day t at location k
- Existence of "weather type"
  - High pressure systems, frontal systems, …
- …Introduced as a <u>hidden process</u> $S_t \in \{1\ldots Q\}$
  - Common to all locations

Basic HMM

Model description

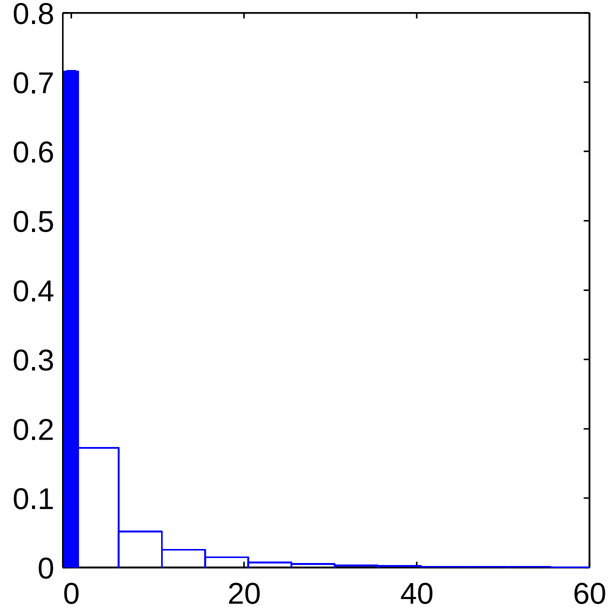
- Conditional independence assumptions
  - Temporal structure (HMM)
    - $P(S_t|Y_1,\ldots,Y_{t-1}, S_1,\ldots,S_{t-1}) = P(S_t|S_{t-1})$
    - $P(Y_t|Y_1,\ldots,Y_{t-1}, S_1,\ldots,S_t) = P(Y_t|S_t)$
  - Dynamics induced only by $\{S_t\}$
  - Spatial structure (conditional independence)
    - $p(Y_t(1),Y_t(2)\ldots,Y_t(K)|S_t)$

      $= p(Y_t(1)|S_t)\, p(Y_t(2)|S_t)\ldots\, p(Y_t(K)|S_t)$
  - Spatial dependence induced only by $\{S_t\}$

Basic HMM

Model description

- Conditional distributions $p(Y_t(k)|S_t=s)$
  - Two components
    - $Y_t\ (k)=0$ if no rainfall occurs
    - $Y_t\ (k)>0$ if a rainfall occurs
  - Mixed discrete-continuous distribution



Marginal distribution of $\{Y_t\ (k)\}$
(Winchmore, New Zealand)

Basic HMM

Model description

- Conditional distributions $p(Y_t(k)|S_t=s)$

$$P(Y_t(k)\in dy|S_t=s)=\begin{cases}1-p_k^{(s)} & \text{if } 0\in dy\\ p_k^{(s)} f(y;\alpha_k^{(s)},\beta_k^{(s)})dy & \text{if } 0\notin dy\end{cases}$$

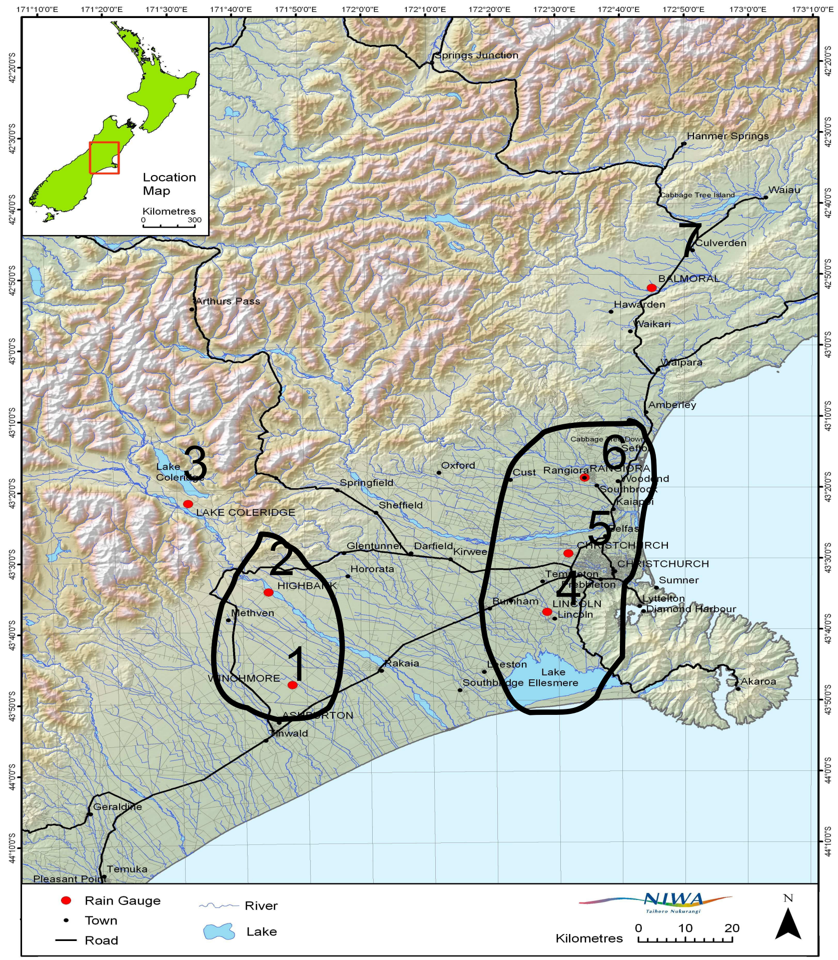
  - $p_k^{(s)}\in[0,1]$, $\alpha_k^{(s)}>0$, $\beta_k^{(s)}>0$
  - $f(y;\alpha,\beta)=\dfrac{1}{\Gamma(\alpha)\beta^{\alpha}}y^{\alpha-1}\exp(-\dfrac{y}{\beta})$

- $3KQ+Q(Q-1)$ parameters

Basic HMM

Data

- Rainfall data in New Zealand
  - Daily rainfall
  - 7 locations
  - 26 years
    - Focus on April



Basic HMM

Parameter estimation

- EM algorithm
- Model selection
  - First selection with AIC and BIC

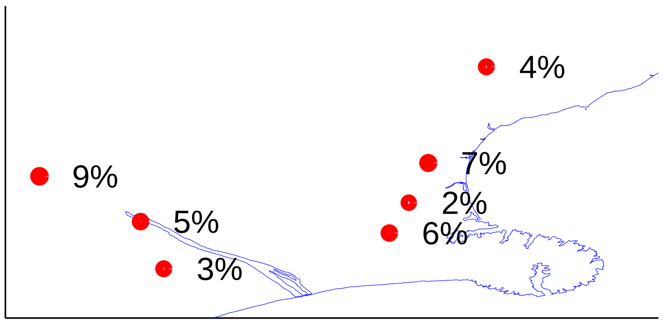
Q	1	2	3	4	5	6
AIC	17404	14317	13436	13213	13144	**12990**
BIC	17502	14523	13760	**13663**	13731	13722

  - Final selection according to
    - Meteorological interpretability
    - Ability to simulate realistic rainfalls
  - Focus on the model with $Q=4$

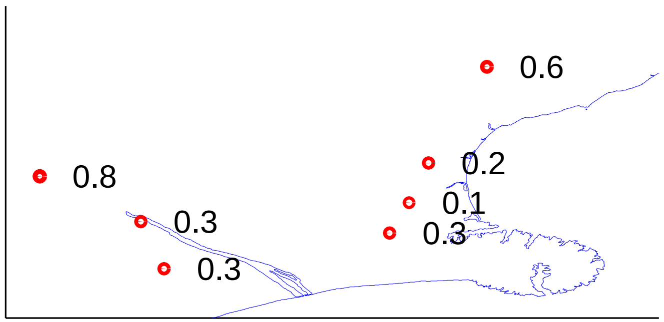
Basic HMM

Meteorological interpretability

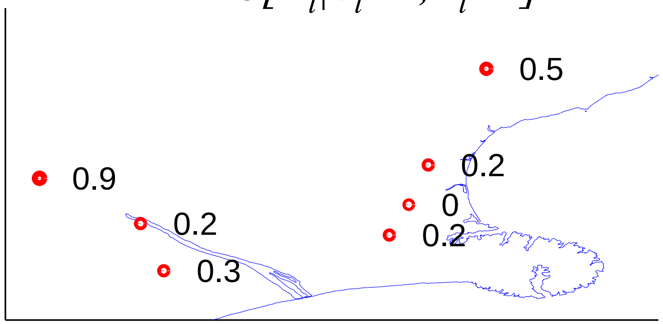
$P[Y_t>0|S_t=1]$



$E[Y_t|S_t=1,Y_t>0]$



$\sigma[Y_t|S_t=1,Y_t>0]$



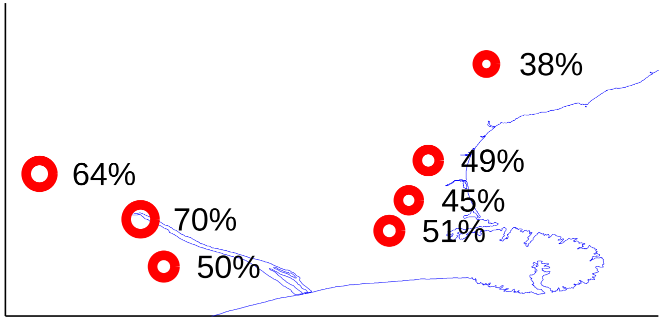
<u>Regime 1</u>

- Low probability of rainfall occurrence
- Low amount, higher at location 3

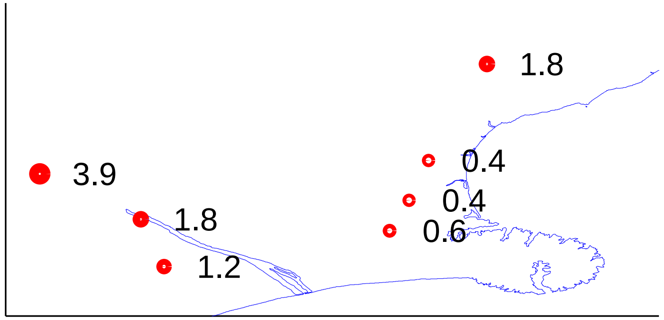
Basic HMM

Meteorological interpretability

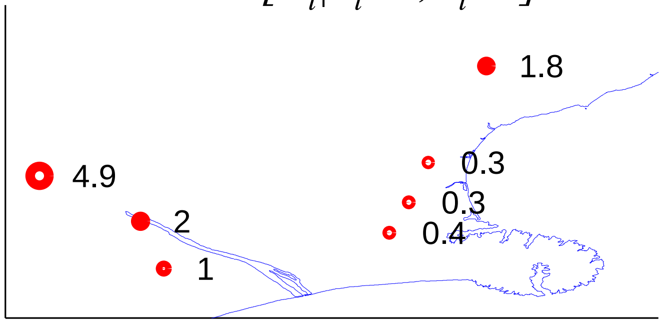
$P[Y_t>0|S_t=2]$



$E[Y_t|S_t=2,Y_t>0]$



$\sigma[Y_t|S_t=2,Y_t>0]$

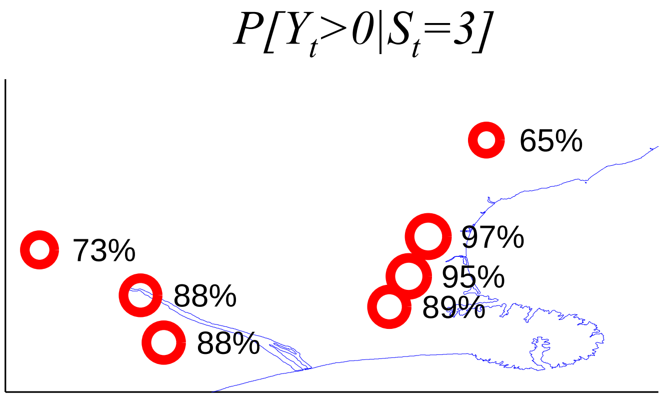


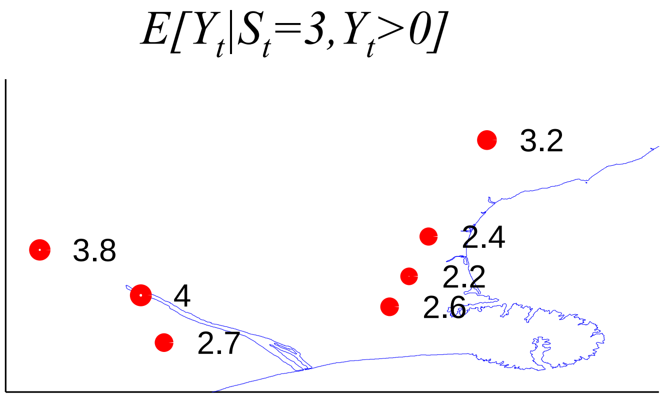
<u>Regime 2</u>

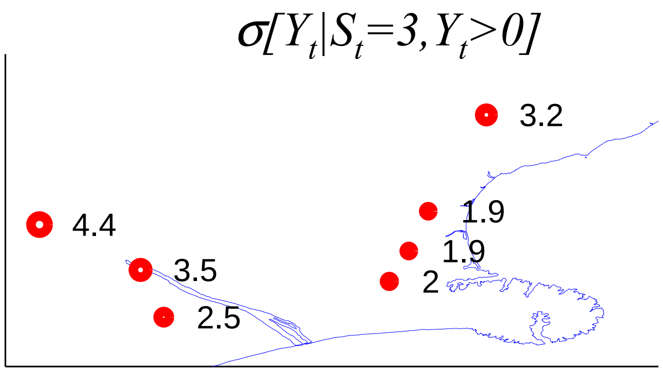
- Moderate probability of rainfall occurrence, higher in the west part
- Moderate amounts in the west part, low in the south-east

Basic HMM

Meteorological interpretability







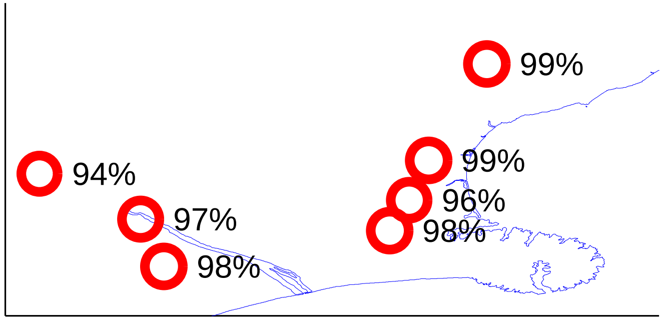
<u>Regime 3</u>

- High probability of rainfall occurrence, higher in the south-east
- Moderate amounts, lower in the south-east

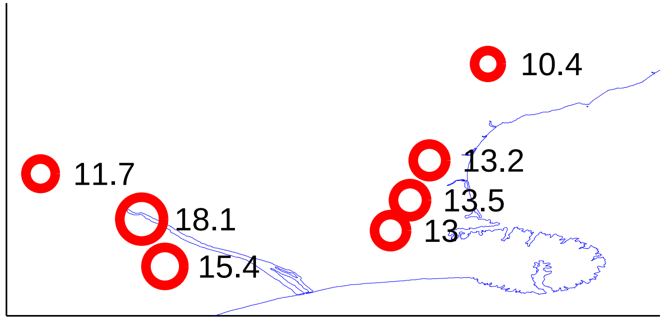
Basic HMM

Meteorological interpretability

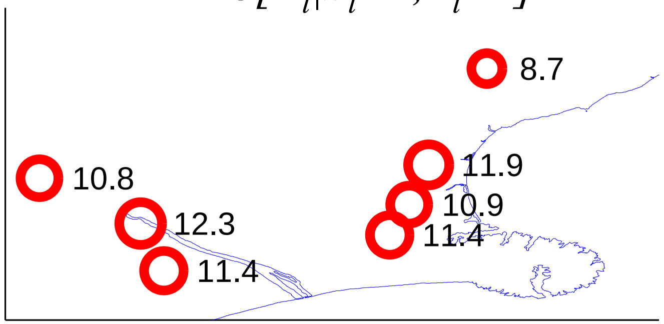
$P[Y_t>0|S_t=4]$



$E[Y_t|S_t=4,Y_t>0]$



$\sigma[Y_t|S_t=4,Y_t>0]$



<u>Regime 4</u>

- High probability of rainfall occurrence
- High amounts

Basic HMM

Meteorological interpretability

- Transition matrix, stationary distribution, mean durations

0.70	0.15	0.09	0.05
0.49	0.18	0.20	0.12
0.35	0.31	0.17	0.16
0.21	0.29	0.25	0.25

0.56
0.20
0.14
0.10

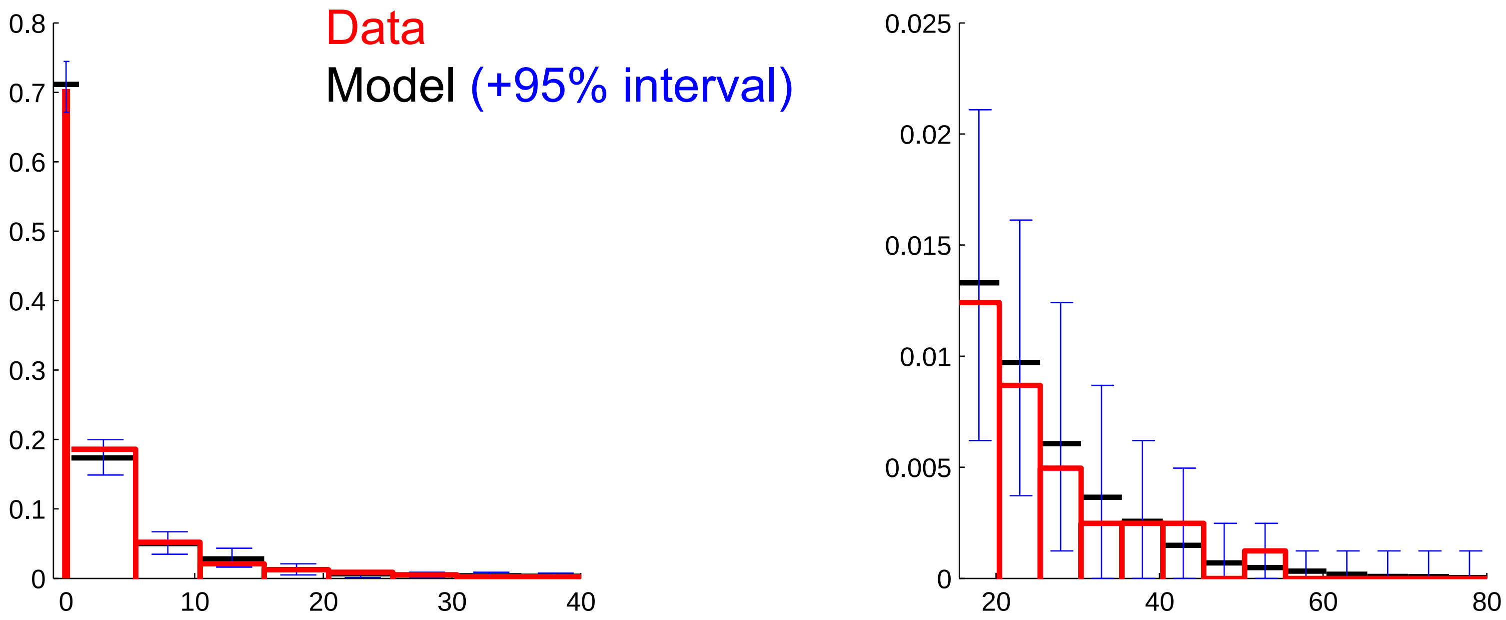
3.33
1.22
1.20
1.33

- Summary:
  - **Regime 1:** dry conditions, long persistence
  - **Regime 2 and 3:** intermediate patterns, regional differences, higher rainfall in regime 3, short persistence
  - **Regime 4:** heavy rainfall
- Similar meteorological interpretation for other datasets

Basic HMM

Realism of simulated sequences

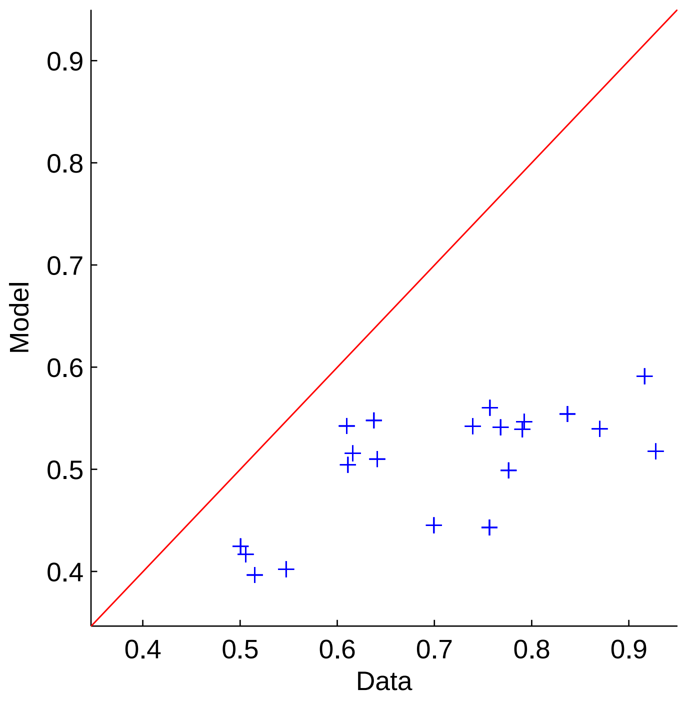
- Univariate marginal distributions (location 1, Winchmore)



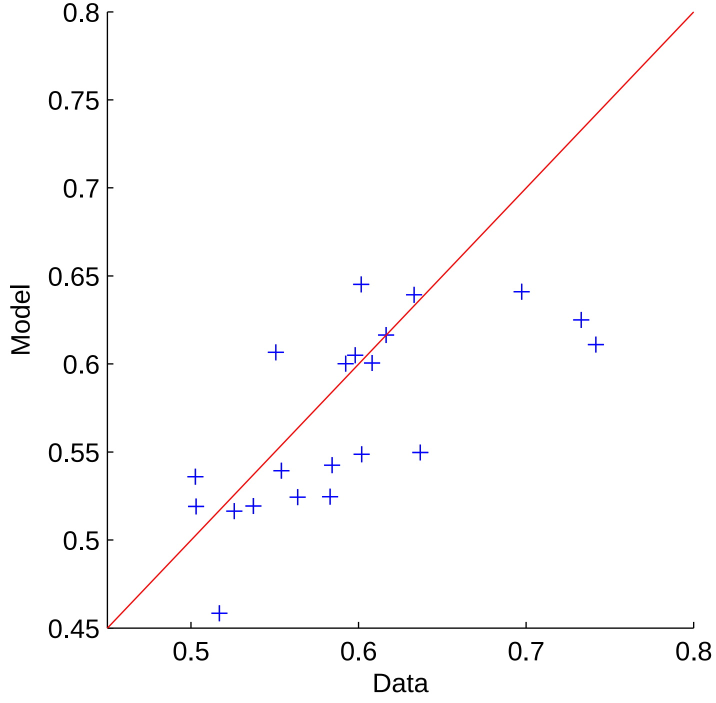
Basic HMM

Realism of simulated sequences

- Spatial pair-wise correlations



Amounts

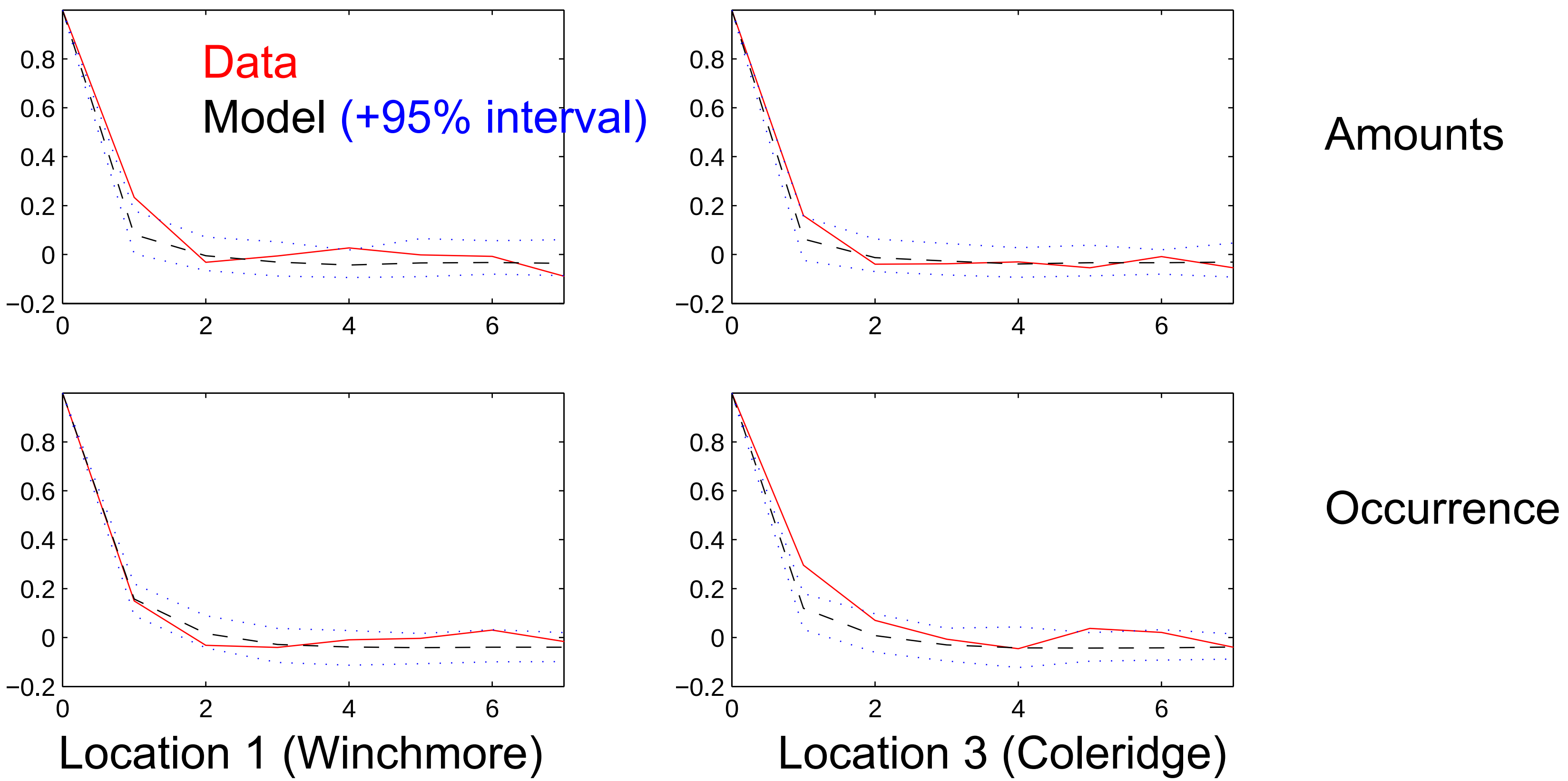


Occurrence

Basic HMM

Realism of simulated sequences

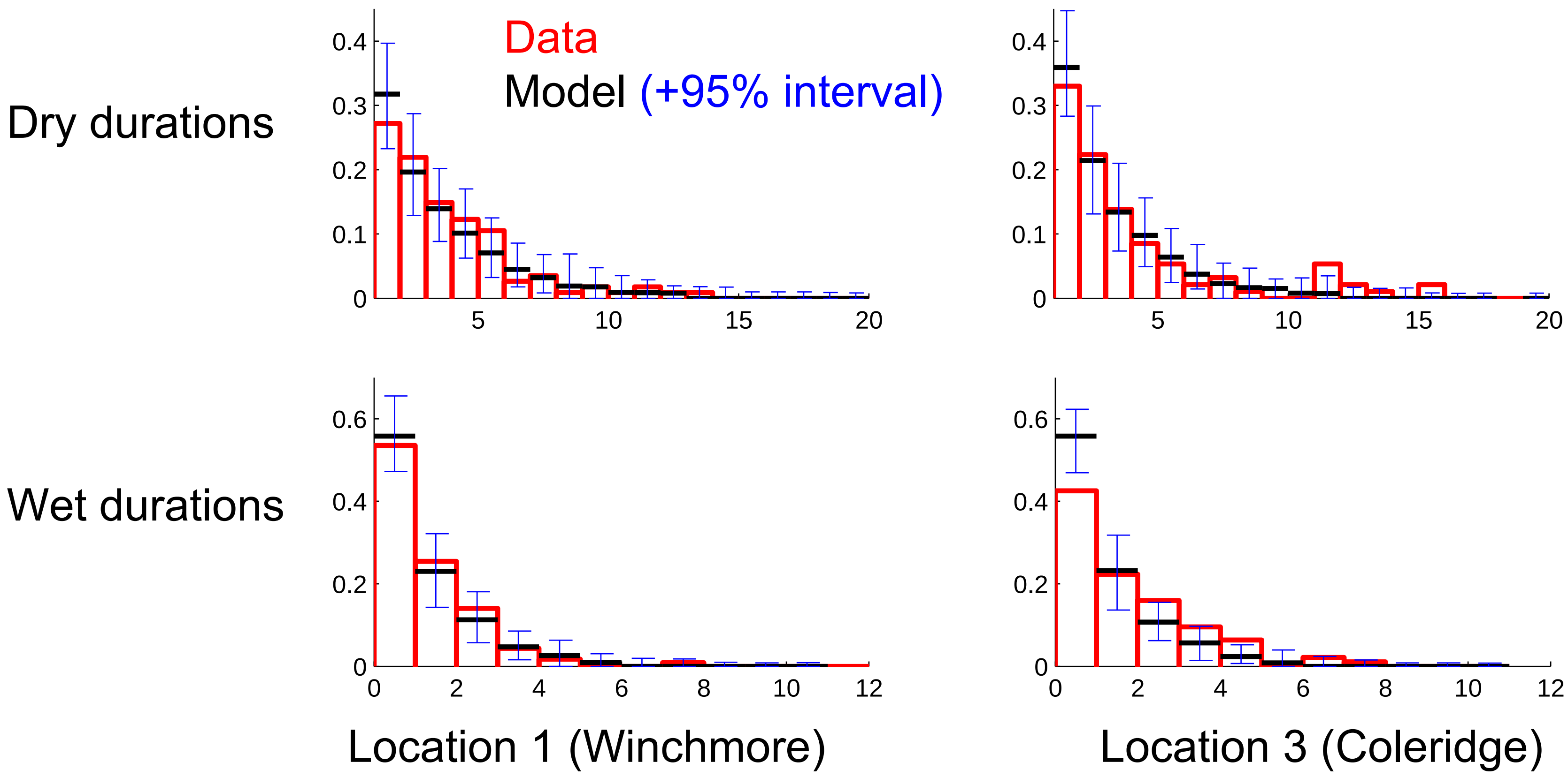
- Autocorrelation functions



Basic HMM

Realism of simulated sequences

- Dry/wet durations



Basic HMM

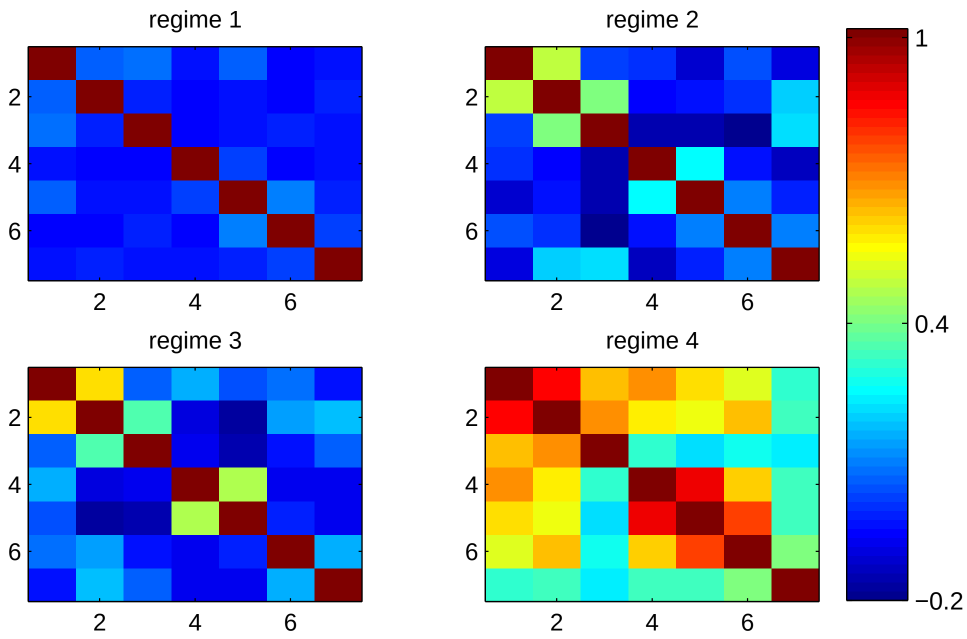
Conclusion

- ☺ Meteorological interpretation
- ☺ Reproduces marginal distribution
- ☹ Fails to reproduce…
  - Spatial structure
  - Dynamics at some locations
- …Need for a more sophisticated model
  - Focus on spatial aspects in the next part

Extensions

- Conditional independence assumption unrealistic

Empirical correlation matrices in the different weather types (identified via the Viterbi algo.)



➤ Residual spatial structure within the weather types

Extensions

- Add spatial structure in the emission probabilities
  - Need model for multivariate mixed discrete-continuous data
- **Markov random fields**
  - For the binary occurrence/non-occurrence process
    - Autologistic model (Hughes et al. (1999))
    - Chow-Liu trees (Kirshner (2005))
  - Generalization to include positive amounts?
- Introducing a **“local” weather type** to relate regional patterns to local rainfall
  - Thompson et al. (2005)
- **Truncated Gaussian random fields**
  - Allcroft et al. (2003) (without Markov switching)

HMM with truncated Gaussian fields

Model description

- If $S_t = s$ then

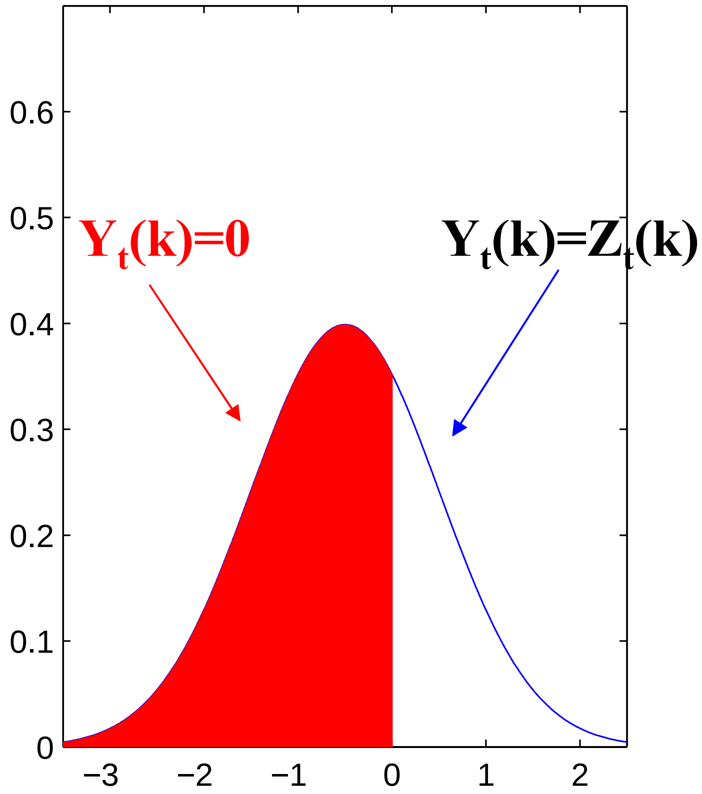
$$Y_t(k) = \max(Z_t(k), 0)$$

with

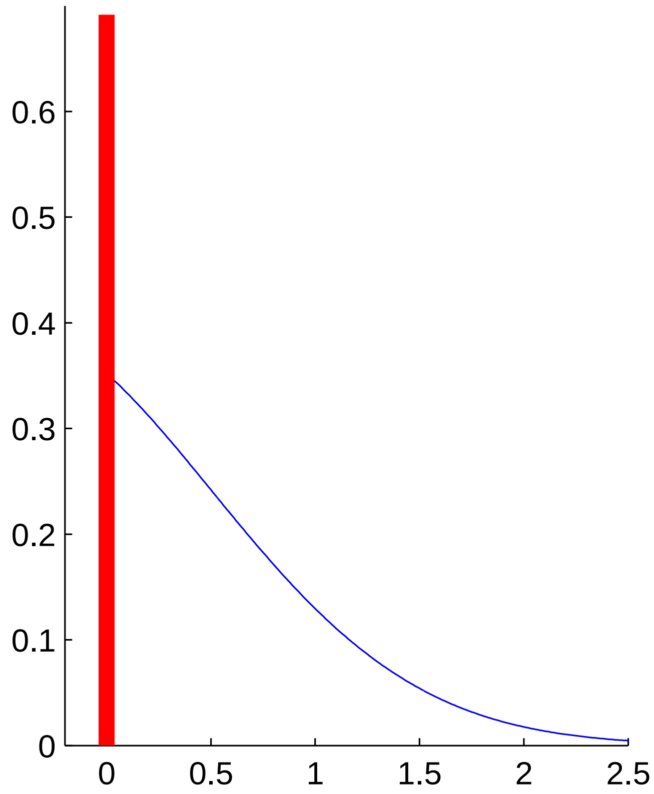
- $Z_t = (Z_t(1), \ldots, Z_t(K))' = \mu^{(s)} + S^{(s)} E_t$
- $\mu^{(s)} \in R^K$, $\Sigma^{(s)} = S^{(s)} S^{(s)}{}' \in R^{K*K}$
- $E_t \sim N(0, I_K)\ i.i.d$

HMM with truncated Gaussian fields

Model description



Conditional distribution of $Z_t(k)$



Conditional distribution of $Y_t(k)$

HMM with truncated Gaussian fields

Model description

- Assumptions on the covariance matrices

$$\Sigma^{(s)} = diag((\sigma_1^{(s)})^2, \ldots, (\sigma_K^{(s)})^2) \quad \text{(HMMCI)}$$

$$\Sigma^{(s)}(i,j) = \sigma_i^{(s)}\sigma_j^{(s)} \exp(-\lambda^{(s)}\, dist\,(x_i, x_j)) \quad \text{(HMMdist)}$$

$$\Sigma^{(s)}(i,j) = \sigma_i^{(s)}\sigma_j^{(s)} \exp(-\lambda_i^{(s)}\lambda_j^{(s)}\, dist\,(x_i, x_j)) \quad \text{(HMMloc)}$$

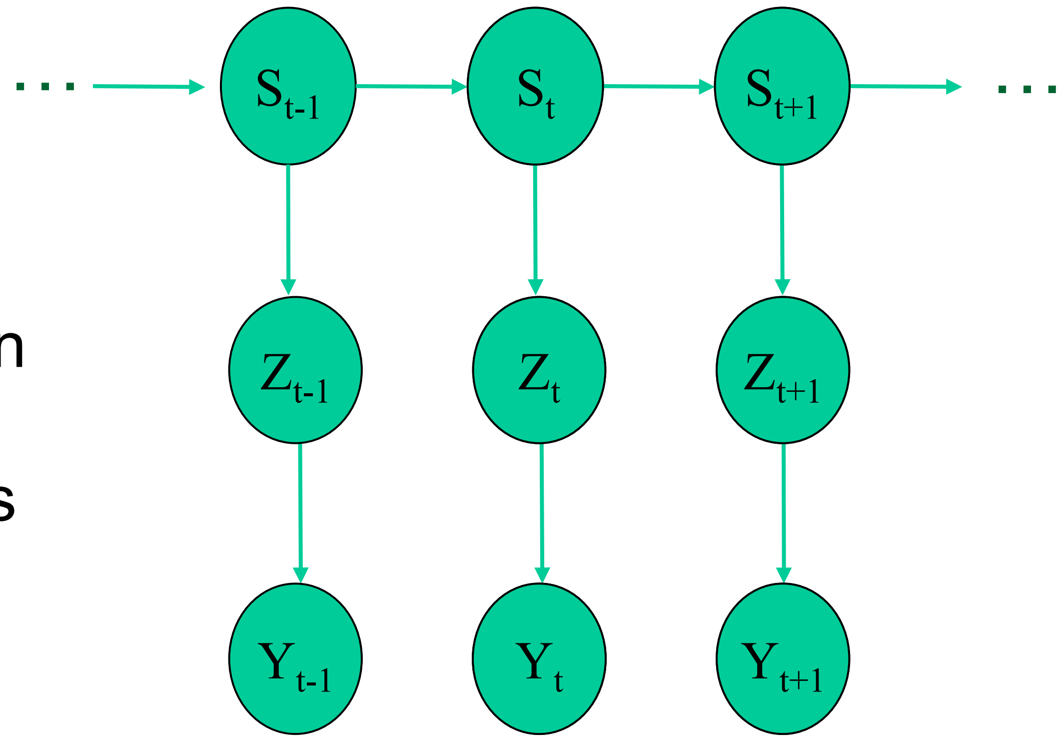
HMM with truncated Gaussian fields

Parameter estimation

Weather type
- Not observed
- Finite values

Conditionally Gaussian
- Partially observed
- Continuous values

Precipitation
- Observed
- Mixed continuous-discrete



HMM with truncated Gaussian fields

Parameter estimation

- E-step (Forward-Backward algorithm)
  - Computation of $p(Y_t=y_t|S_t=s;\Theta)$
  - If $y_t=(0,\ldots,0,y_t(k+1),\ldots y_t(K))$ with $y_t(k+1)>0,\ldots, y_t(K)>0$

$$p(Y_t = y \mid S_t = s;\Theta)$$
$$= \int_{]-\infty,0]^k} f(z_1,\ldots,z_k, y_t(k+1),\ldots, y_t(K);\mu^{(s)},\Sigma^{(s)})dz_1\ldots dz_k$$

- M-step
  - Computation of $E[Z_t|Y_t=y_t,S_t=s;\Theta]$ and $cov(Z_t|Y_t=y_t,S_t=s;\Theta)$
  - …integral expression

HMM with truncated Gaussian fields

Parameter estimation

- Monte Carlo integration
  - If $y_t=(0,\ldots,0,y_t(k+1),\ldots y_t(K))$, then
    - Simulate N samples from the multivariate Gaussian distribution

      $$P(Z(1),\ldots,Z(k)|Z(k+1)=y_t(k+1),\ldots,Z(K)=y_t(K);\mu^{(s)},\Sigma^{(s)})$$

    - Deduce approximate values for the integrals. For ex.:

$$p(Y_t = y \mid S_t = s;\Theta)$$

$$\approx \frac{Nb\, of\ samp.with\, all\, comp. \leq 0}{N} \times p(y_t(k+1),\ldots,y_t(K);\mu^{(s)},\Sigma^{(s)})$$

- MCEM algorithm

HMM with truncated Gaussian fields

	HMMCI (4 states)	HMMdist (4 states)	HMMloc (4 states)	HMMfull (4 states)
loglik	-6837	-6441	-6383	-6311
numpar	68	72	96	152
AIC	13811	13029	12959	**12928**
BIC	14130	**13367**	13409	13641

$$\Sigma^{(s)} = diag((\sigma_1^{(s)})^2, \ldots, (\sigma_K^{(s)})^2)$$

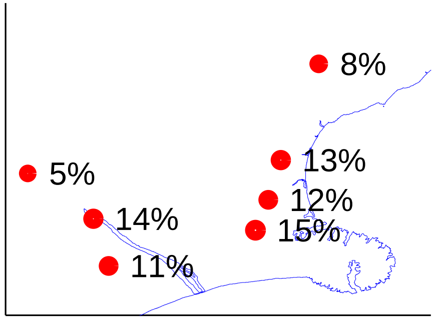
$$\Sigma^{(s)}(i,j) = \sigma_i^{(s)} \sigma_j^{(s)} \exp(-\lambda^{(s)}\, dist(x_i, x_j))$$

$$\Sigma^{(s)}(i,j) = \sigma_i^{(s)} \sigma_j^{(s)} \exp(-\lambda_i^{(s)} \lambda_j^{(s)}\, dist(x_i, x_j))$$

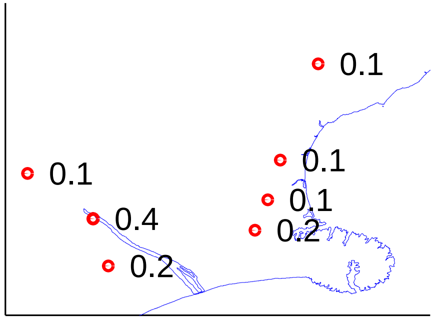
HMM with truncated Gaussian fields

Meteorological interpretability (HMMloc)

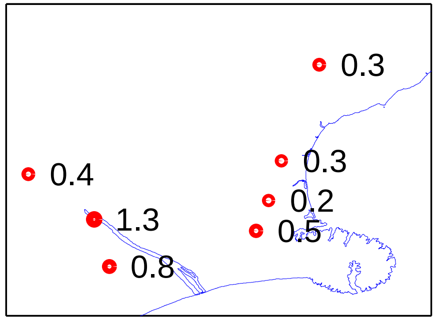
$P[Y_t>0|S_t=1]$



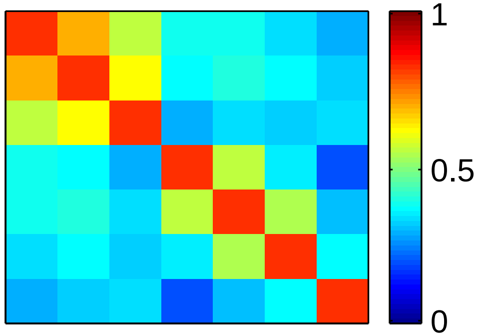
$E[Y_t|S_t=1]$



$\sigma[Y_t|S_t=1]$



$Corr(Y_t|S_t=1)$



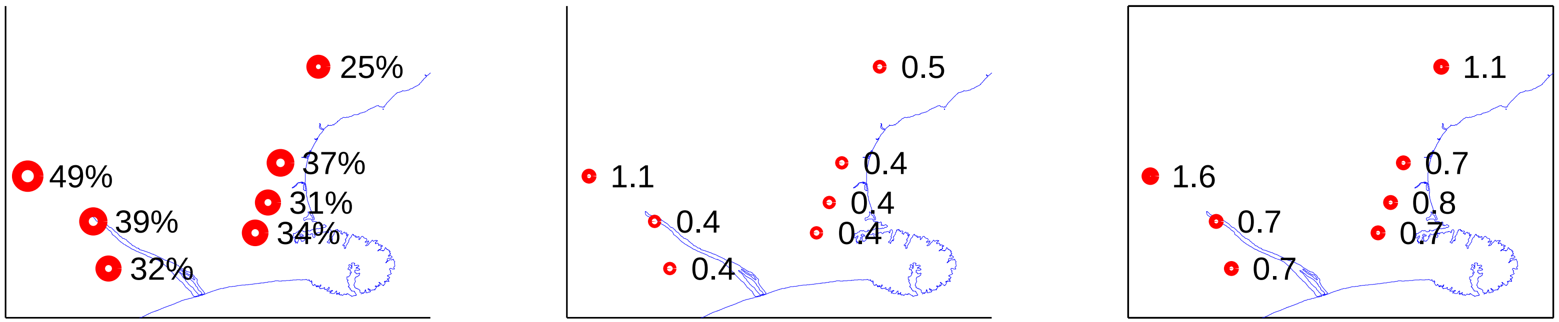
<u>Regime 1</u>

- Low probability of rainfall
- Low amount
- High spatial correlation

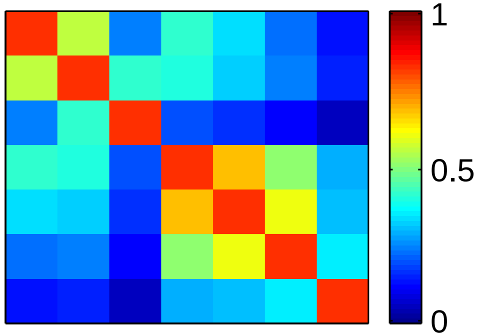
HMM with truncated Gaussian fields

Meteorological interpretability (HMMloc)

$P[Y_t>0|S_t=2]$ $\quad$ $E[Y_t|S_t=2]$ $\quad$ $\sigma[Y_t|S_t=2]$



$Corr(Y_t|S_t=2)$



<u>Regime 2</u>

•Moderate probability of rainfall occurrence, higher at location 3

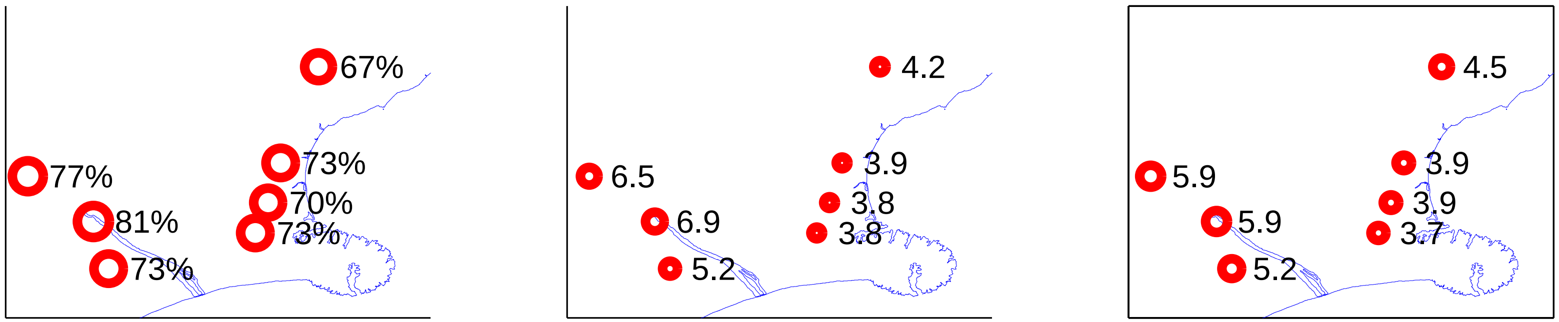
•Low amounts, higher at location 3

•Moderate spatial correlation, low correlation between locations 3 and 7 and other locations

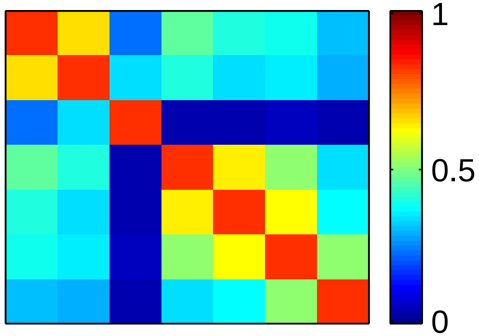
HMM with truncated Gaussian fields

Meteorological interpretability (HMMloc)

$P[Y_t>0|S_t=3]$ $E[Y_t|S_t=3]$ $\sigma[Y_t|S_t=3]$



$Corr(Y_t|S_t=3)$



<u>Regime 3</u>

•Moderate probability of rainfall occurrence

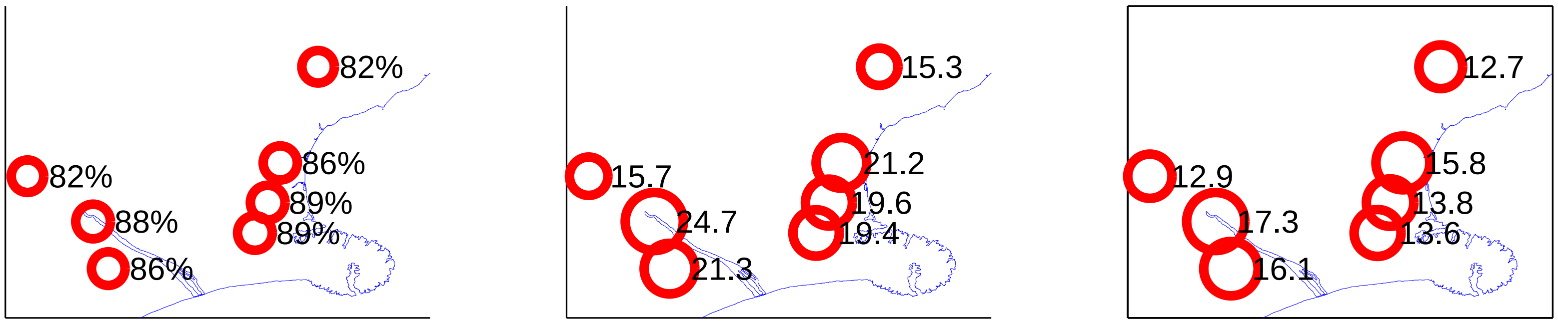
•Moderate amounts, higher in the west

•Moderate spatial correlation, low correlation between location 3 and other locations

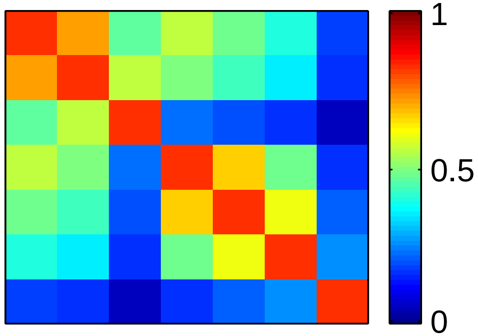
HMM with truncated Gaussian fields

Meteorological interpretability (HMMloc)

$P[Y_t>0|S_t=4]$ $E[Y_t|S_t=4]$ $\sigma[Y_t|S_t=4]$



$Corr(Y_t|S_t=4)$



<u>Regime 4</u>

- •High probability of rainfall occurrence
- •High amount
- •High spatial correlation, except between
  - •location 3 and locations in the east
  - •location 7 and other locations

HMM with truncated Gaussian fields

Meteorological interpretability (HMMloc)

- Transition matrix, stationary distribution , mean durations

0.73	0.18	0.06	0.01
0.29	0.42	0.27	0.03
0.23	0.32	0.36	0.08
0.06	0.42	0.23	0.28

0.48
0.29
0.18
0.05

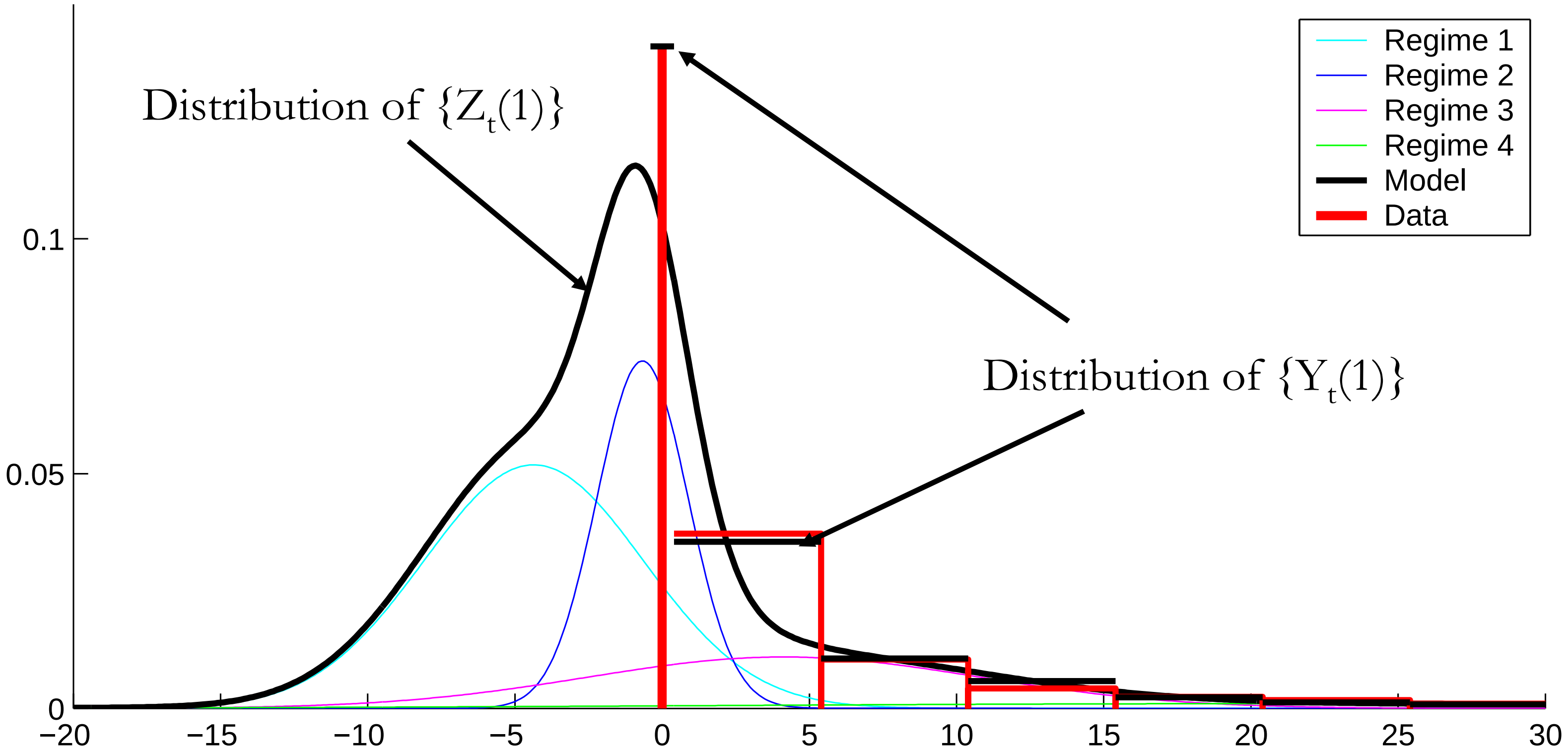
3.78
1.71
1.55
1.40

- Summary:
  - **Regime 1:** low rainfall
  - **Regime 2 and 3:** intermediate patterns, regional differences, higher rainfall in regime 3, short persistence
  - **Regime 4:** high rainfall
  - Location 3 and 7 have specific behaviors

HMM with truncated Gaussian fields

Realism of simulated sequences (HMMloc)

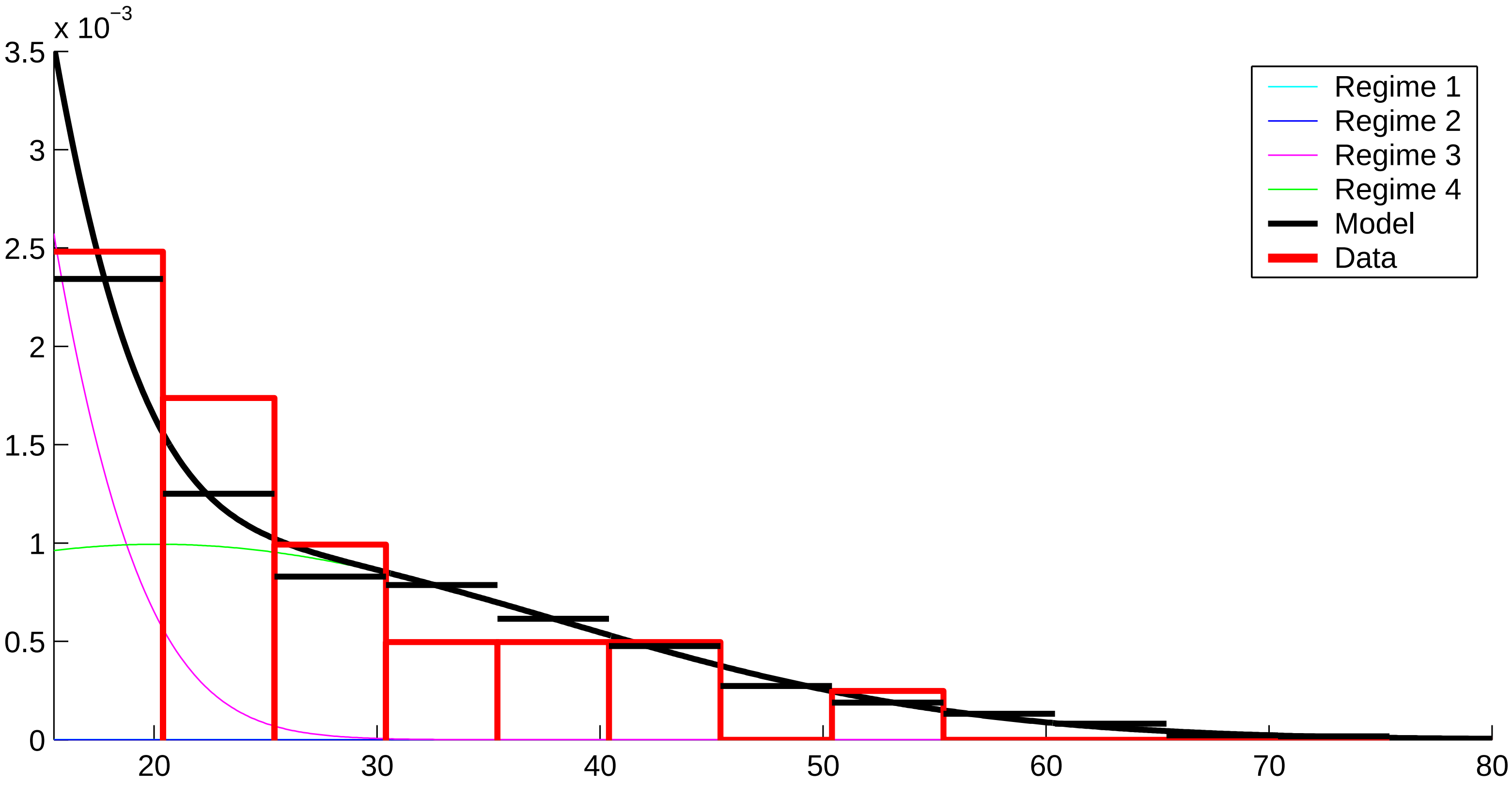
- Marginal distribution (location 1, Winchmore)



HMM with truncated Gaussian fields

Realism of simulated sequences (HMMloc)

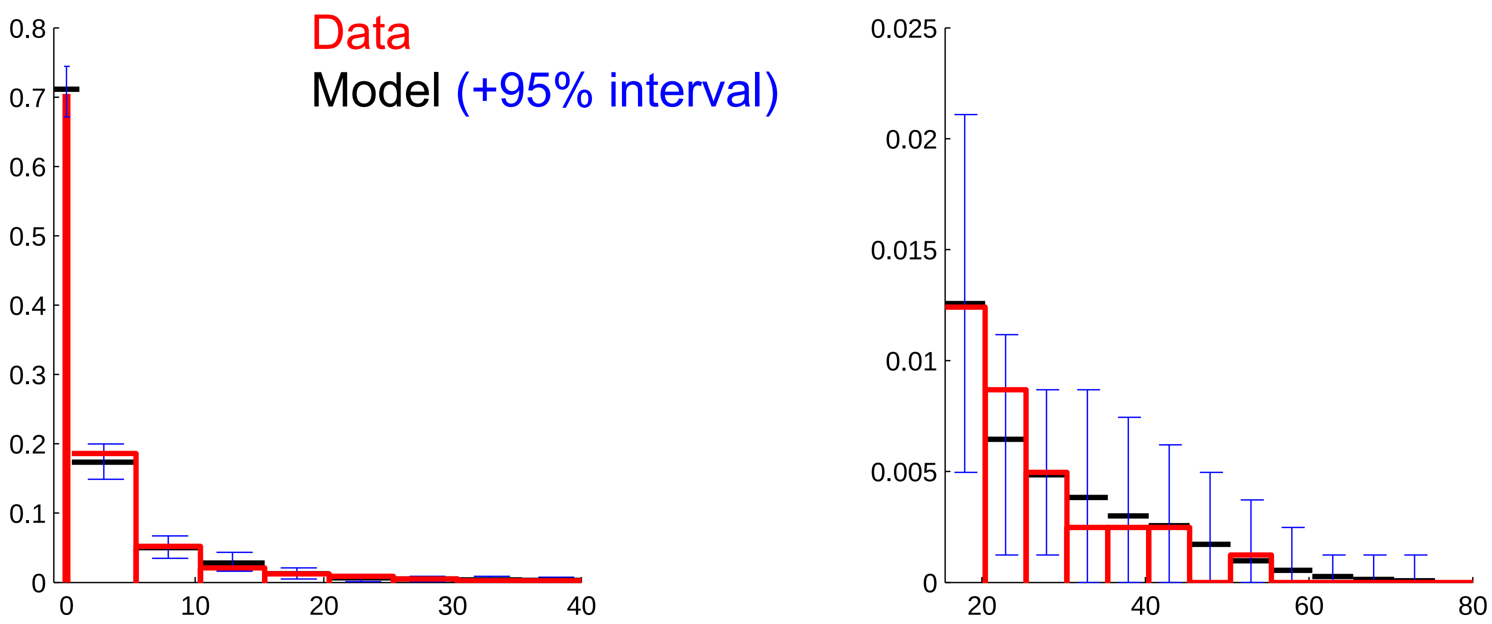
- Marginal distribution (location 1, Winchmore)



HMM with truncated Gaussian fields

Realism of simulated sequences (HMMloc)

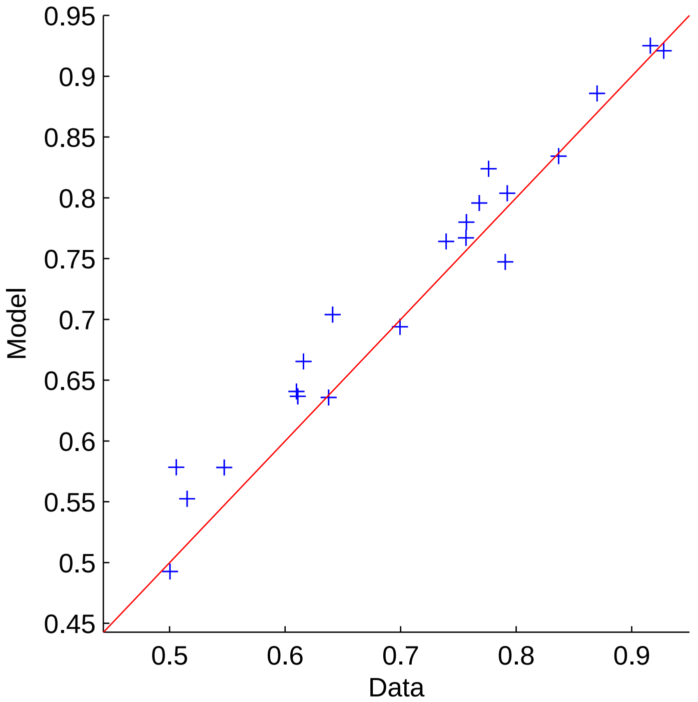
- Marginal distribution (location 1, Winchmore)



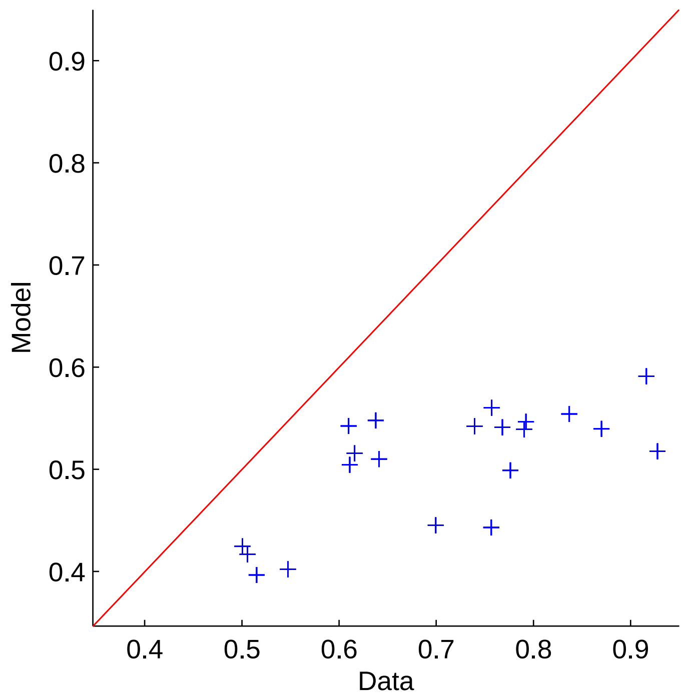
HMM with truncated Gaussian fields

Realism of simulated sequences (HMMloc)

- Pair-wise spatial correlations (amounts)



Model with spatial dependence

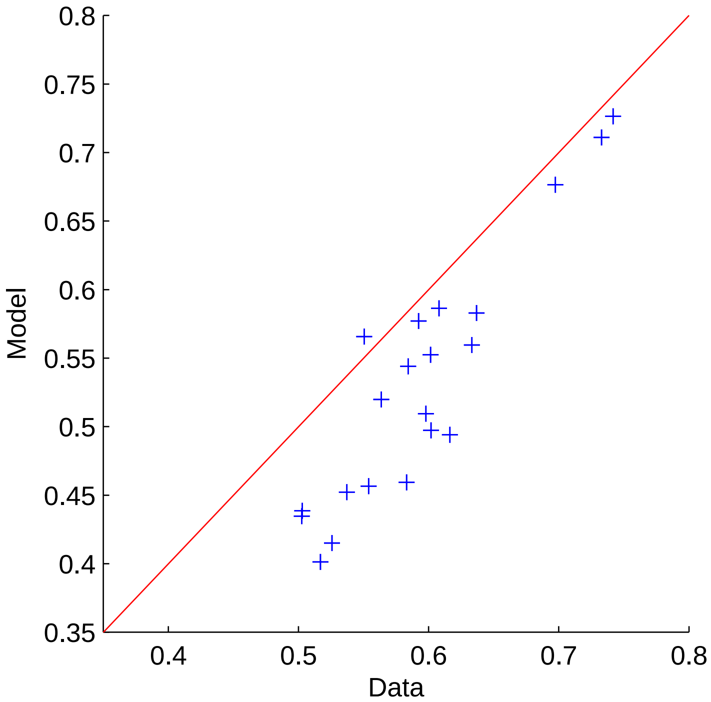


Conditional spatial independence

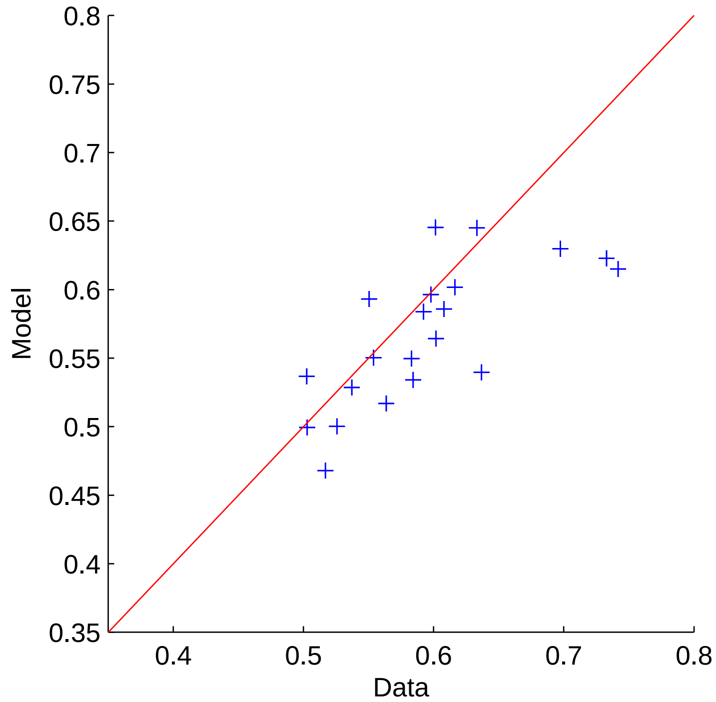
HMM with truncated Gaussian fields

Realism of simulated sequences (HMMloc)

- Pair-wise spatial correlations (occurrence)



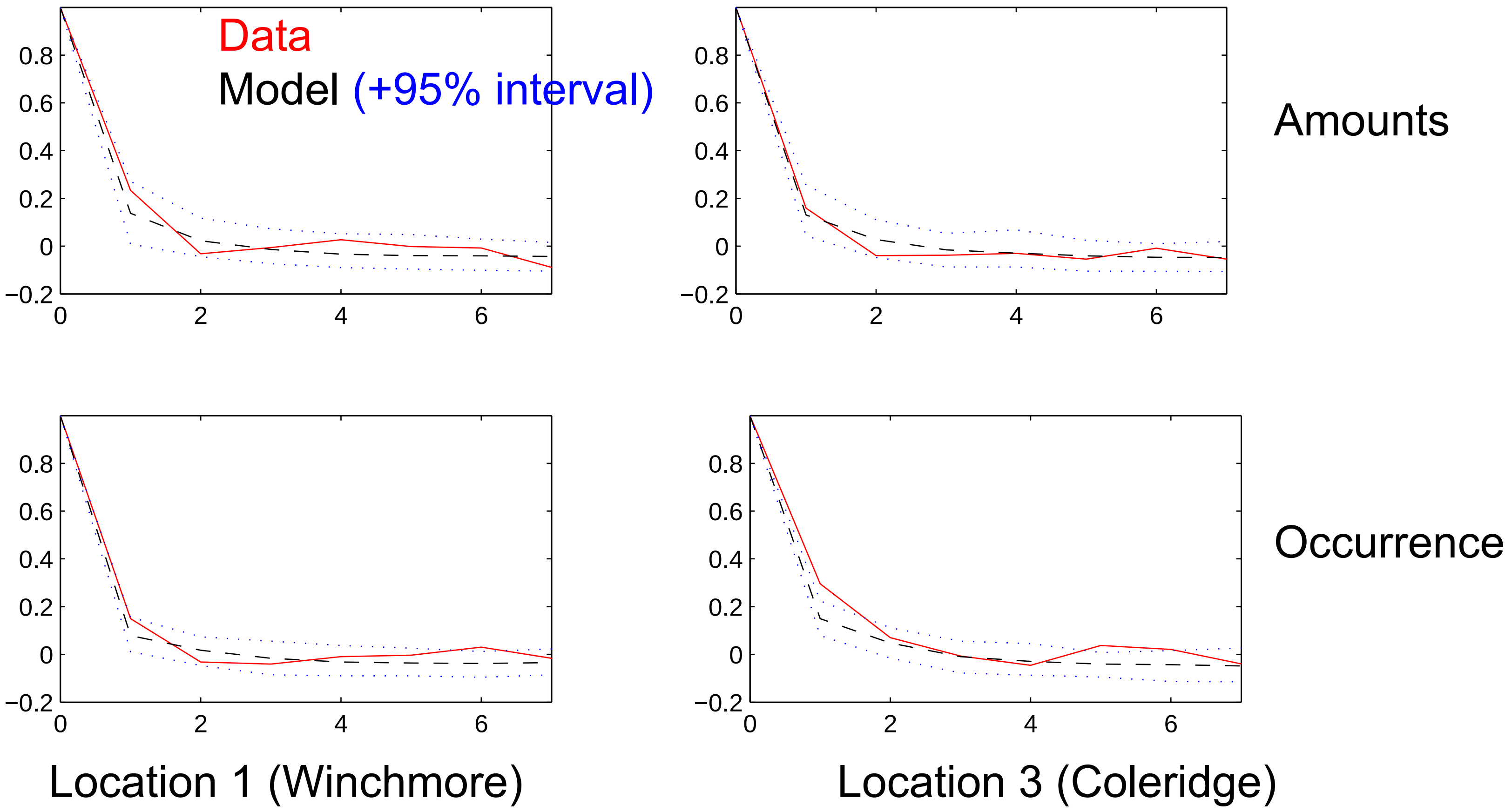
Model with spatial dependence



Conditional spatial independence

HMM with truncated Gaussian fields

Realism of simulated sequences (HMMloc)



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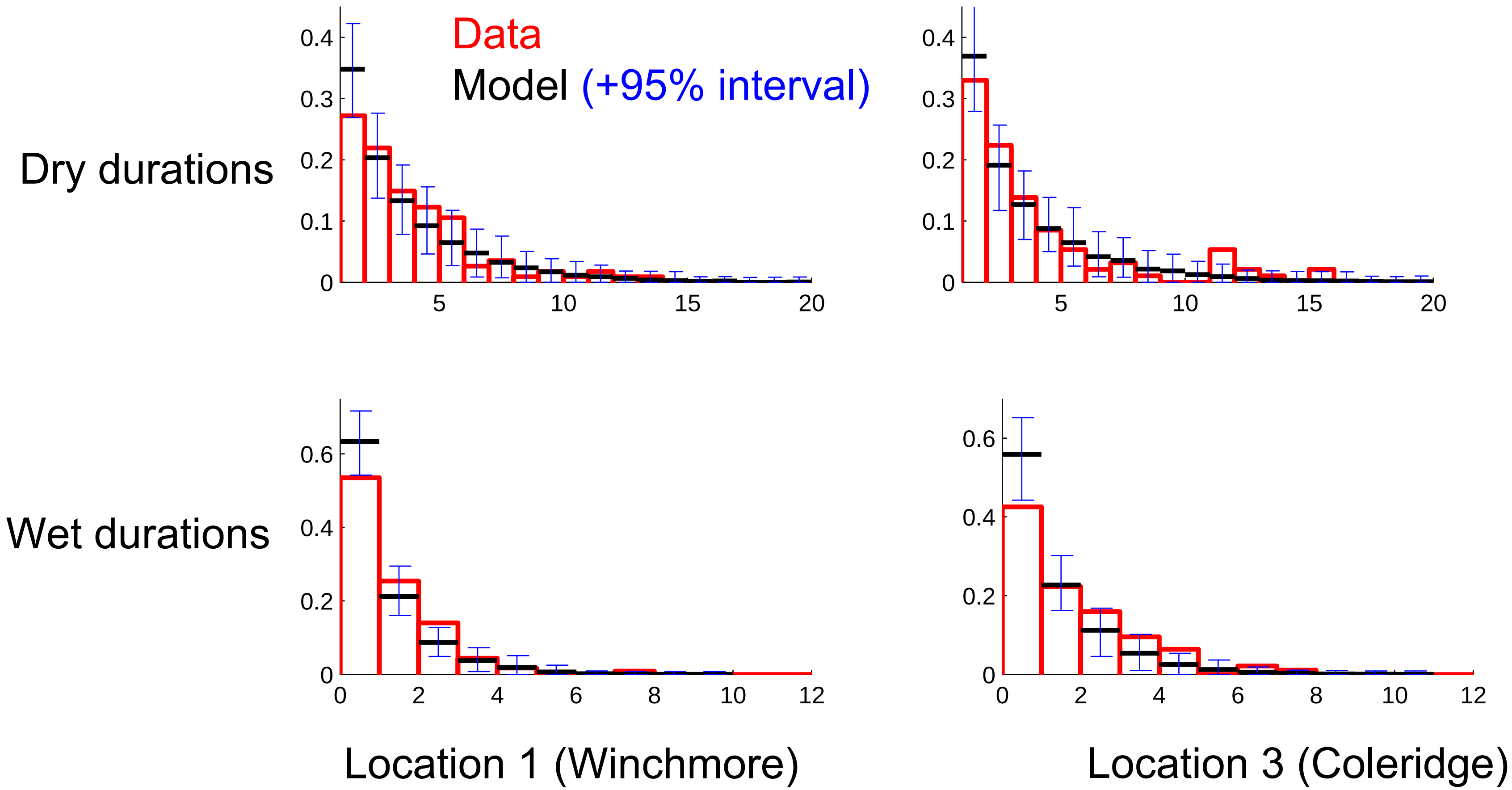

Conclusion & perspectives

- HMM with conditional spatial independence assumption cannot reproduce the spatial structure
- HMM with truncated correlated Gaussian distribution better reproduces the spatial structure
- Dynamics still not well reproduced
  - Add an autoregressive part?
- Explore other possibilities
  - Markov random fields, local weather types,…
- Seasonality, inter-annual variability….

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