Dependence order of multivariate extreme value distributions

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Abstract

Dependence among components of the multivariate extreme value distributions MvEV is characterized by its spectrum measure S or by its exponent measure μ. In this paper, a dependence order based on stable tail dependence function, related to μ, is introduced. The implication of the order in the risk theory is discussed.

Introduction

Let $(X_{1n}, \ldots, X_{dn})_{n=1}^{\infty}$ be a sequence of i.i.d. multivariate random variables with d.f. $F(\mathsf{x})$, $\mathsf{x} \in \mathcal{R}^d$. Its componentwise maximum

$$\mathsf{M}_n = \left(\bigvee_{i=1}^{n} X_{1i}, \ldots, \bigvee_{i=1}^{n} X_{di} \right)$$

has the d.f. $F^n(\mathsf{x})$. Assume that there exist sequences of normalizing $\mathcal{R}^d$-sequences $(\mathsf{a}_n)_n$, $\mathsf{a}_n > \mathsf{0}$, and $(\mathsf{b}_n)_n$, and a d.f. $G(\mathsf{x})$ with nondegenerate margins such that (in componentwise algebraic notations)

$$P\{(\mathsf{M}_n - \mathsf{b}_n)/\mathsf{a}_n \le \mathsf{y}\} = F^n(\mathsf{a}_n \mathsf{y} + \mathsf{b}_n) \quad \xrightarrow{D} \quad G(\mathsf{y}), \quad n \to \infty. \tag{1}$$

The d.f. G is then called a *multivariate extreme value distribution* (MvEV), F is said to belong the domain of attraction.

From (1), all the marginal distributions of a MvEV are, if not degenerated, again MvEV of lower dimensions, and univariate margins are *generalized extreme value distributions*, GEV$(\eta, \zeta; \gamma)$, with d.f. defined by

$$G_0(y; \eta, \zeta, \gamma) = \exp\left(- \left(1 + \gamma \frac{y - \eta}{\zeta} \right)_{+}^{-1/\gamma} \right), \quad y \in \mathcal{R};$$

$$\zeta \in \mathcal{R}_+, \ \eta \in \mathcal{R}, \ \gamma \in \mathcal{R}; \quad (z)_+ = \begin{cases} z, & if \quad z > 0, \\ 0, & otherwise. \end{cases}$$

To obtain a standard expression, transform all the marginal GEV's to the standard Fréchet distribution, $\mathsf{GEV}(1,1,1)$ or the standard reciprocal exponential distribution, and

$$G(\mathsf{y}) = \exp\left(- \int_{\mathcal{S}} \bigvee_{j=1}^{d} \frac{w_j}{y_j} \, \mathrm{d}S(\mathsf{w}) \right), \quad \mathsf{y} \in \mathcal{E}, \tag{2}$$

with

$$\int_{\mathcal{S}} w_j \, \mathrm{d}S(\mathsf{w}) = 1, \quad 1 \le j \le d, \tag{3}$$

where

$$\mathcal{S} = \{\mathsf{y} \in \mathcal{E} : \ ||\mathsf{y}|| = 1\}, \quad \mathcal{E} = [0,\infty]^d \setminus \{\mathsf{0}\},$$

$||\cdot||$ is a norm on $\mathcal{R}^d$ and S is a finite measure on $\mathcal{S}$, called spectrum measure of G. In the following $||\cdot||$ is 1-norm and $\mathcal{S}$ is the unit simplex. The condition (3) is necessary since, for example,

$$G(y_1, \infty, \ldots, \infty) = \exp\left(- \int_{\mathcal{S}} (w_1/y_1) \, \mathrm{d}S(\mathsf{w}) \right) = \exp(-1/y_1).$$

The spectrum measure S is related to another measure μ defined by Borel sets in $\mathcal{E}$;

$$\mu\left(\mathsf{y} \in \mathcal{E} : \|\mathsf{y}\| > r, \ \frac{\mathsf{y}}{\|\mathsf{y}\|} \in A \right) = \frac{S(A)}{r}, \quad A \in \mathcal{S}, \tag{4}$$

and

$$G(\mathsf{y}) = \exp(-\mu((\mathsf{0}, \mathsf{y}]^c)). \tag{5}$$

μ is called the *exponent measure*. The expression (5) implies that there exists a nonhomogeneous Poisson process $(T_k, \mathsf{Q}_k)_k$ on $[0,\infty) \times \mathcal{E}$ with intensity measure Λ, such that, for a Borel set $B \subset \mathcal{E}$,

$$\Lambda([0,t] \times B) = t\mu(B), \quad \text{and} \quad G(\mathsf{y}) = P\left\{ \bigvee_{T_k \le 1} \mathsf{Q}_k \le \mathsf{y} \right\}.$$

The exponent measure satisfies further, by (4), for any Borel set $B \subset \mathcal{E}$

$$\mu(t \cdot B) = t^{-1}\mu(B). \tag{6}$$

MvEV is discussed by Joe (1997), Fougères (2004) and Beirlant, Goegebeur, Segers and Teugels (2004) among many others.

Independence and dependence

If MvEV is independent, its d.f. is

$$G(\mathsf{y}) = \exp\left(- \sum_{j=1}^{d} \frac{1}{y_j} \right), \quad \mathsf{y} \in \mathcal{E},$$

and the spectrum measure $\mathcal{S}$ has unit point mass at the vertices $\mathsf{e}_1, \ldots, \mathsf{e}_d$. This is true for any marginals with 2 or more variates. Hence, it has a peculiar property:

Proposition 1. *If a MvEV is pairwise independent, it is independent. Further, if there exists* $\mathsf{y} \in \mathcal{R}^d$ *with* $0 < G_j(y_j) < 1,\ j = 1, \ldots, d$, *for the marginal distributions* G_j *of* G, *such that*

$$G_{ij}(y_i, y_j) = G_i(y_i) G_j(y_j), \quad 1 \le i < j \le d,$$

for the bivariate marginal distributions G_{ij} *of* G, G *is independent.*

□

If MvEV is completely dependent, it is degenerated on the line $\{\mathsf{x} \in \mathcal{R}^d : x_1 = \cdots = x_d\}$,

$$G(\mathsf{y}) = \exp\left(-\bigwedge_{j=1}^{d} \frac{1}{y_j}\right), \quad \mathsf{y} \in \mathcal{E},$$

and spectrum measure S is degenerated to the center $\mathsf{y} = (1/d, \ldots, 1/d)$.

A wide class of distributions on the unit simplex has negative dependence. For example, the uniform density on $d-1$ dimensional unit simplex has partial correlations $-1/d$. The uniform mass at vertexes has partial correlations $-1/(d-1)$. $G(\mathsf{y})$ is positively independent nevertheless: the dependence of spectrum measure S and that of G is closely related but different one.

Dependence order

Let the exponent measure be denoted in another form

$$G(\mathsf{y}) = \exp(-V(\mathsf{y})), \quad V(\mathsf{y}) = \mu((\mathsf{0}, \mathsf{y}]^c),$$

and define *stable tail dependence function* φ by

$$\varphi(\mathsf{v}) = V(1/v_1, \ldots, 1/v_d) = -\log G(1/v_1, \ldots, 1/v_d), \quad \mathsf{v} \in [\mathsf{0}, \boldsymbol{\infty}].$$

This function has the following properties

1. $\varphi(t\mathsf{v}) = t\varphi(\mathsf{v}), \quad 0 < t < \infty,$
2. $\varphi(\mathsf{e}_j) = 1, \quad j = 1, \ldots, d,$ where e_j is the j-th unit vector in $\mathcal{R}^d$.
3. $\bigvee_{j=1}^{d} v_j \le \varphi(\mathsf{v}_j) \le \sum_{j=1}^{d} v_j,$
   and the upper bound is attained if G is independent, and the lower bound if G is completely dependent. See, Fig. 1.
4. $\varphi(\mathsf{v})$ is convex.

Definition

Suppose $G_1(\mathsf{y})$ and $G_2(\mathsf{y})$ be MvEV of d-dimension, defined by (2) and (3), and let $\varphi_1(\mathsf{v})$ and $\varphi_2(\mathsf{v})$ denote their stable tail dependence function, respectively.

If $\varphi_1(\mathsf{v}) \le \varphi_2(\mathsf{v})$, G_1 is more *extreme dependent* than G_2, and the relation is written as

$$G_1 \succ G_2 \quad \text{(extDep)}$$

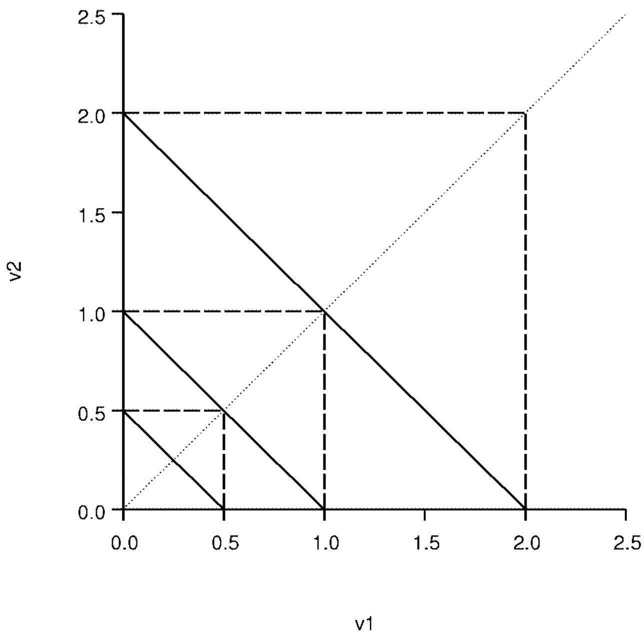

Figure 1: Contours of upper bound (solid line) and lower bound (broken line). The level of contours are equal to the coordinate values of their end points at axes. Those for $\varphi(\mathsf{v})$ are concave lines between them.

$\square$

This is a partial order among MvEV's with a fixed marginal distributions, representing the dependence among components. Note that the order applies to all the marginals of bivariate or more variate.

A way to understand the definition is to compare copulas of G_1 and G_2. The quantile function of the marginals of MvEV is $y = -1/\log u,\ 0 < u < 1$, and copula $C(\mathsf{u})$ of MvEV $G(\mathsf{y})$ is obtained by replacing $y_j = -1/\log u_j,\ j = 1, \ldots, d$.

Proposition 2. *Let* $C_k(\mathsf{u}),\ \mathsf{u} \in [0,1]$, *denote the copula of* MvEV $G_k(\mathsf{y}),\ k = 1, 2,$

$$C_1(\mathsf{u}) \geq C_2(\mathsf{u}) \quad \textit{iff} \quad G_1 \succ G_2 \quad \textit{(extDep).}$$

Proof. Using componentwise algebraic symbolism

$$\begin{aligned} C_k(\mathsf{u}) &= G_k(-1/\log \mathsf{u}) = \exp(-V_k(-1/\log \mathsf{u})) \\ &= \exp(-\varphi_k(-\log \mathsf{u})), \end{aligned}$$

and $C_1(\mathsf{u}) \geq C_2(\mathsf{u})$ if $\varphi_1(\mathsf{v}) \leq \varphi_2(\mathsf{v})$.

□

Note that for positively dependent copula

$$\prod_{i=1}^{d} u_i \leq C(\mathsf{u}) \leq \bigwedge_{i=1}^{d} u_i,$$

and the upper bound is attained if $C(\mathsf{u})$ is completely dependent and the lower bound is attained if $C(\mathsf{u})$ is independent.

In general, $C_1(\mathsf{u}) \geq C_2(\mathsf{u})$ defines *concordance order* and its properties are somehow known. Especially in bivariate case it is a rather strong order. See, for example, Müller and Stoyan (2002) and Embrechts, McNeil and Straumann (2002). In the univariate case larger d.f. means stochastically copulas in general, since univariate marginals are fixed, the order compares *concordance*, that is, a sort of dependence.

If a copula satisfies

$$C(u_1, \ldots, u_d) \geq \prod_{j=1}^{d} u_j, \quad 0 \leq u_j \leq 1, \quad j = 1, \ldots, d.$$

C is said *concordant* (otherwise *disconcordant*). MvEV distributions are all concordant unless they are independent, and concordance is $\succ$ (*extDep*) than independence. Hence, the study on the difference between concordance and independence can be extended to that on the effect of order.

Proposition 3. *Suppose G_1 and G_2 be* MvEV *such that*

$$G_1 \succ G_2 \quad (extDep),$$

and define

$$G_0 = G_1^{\alpha_1} G_2^{\alpha_2}, \quad \alpha_1 > 0, \quad \alpha_2 > 0, \quad \alpha_1 + \alpha_2 = 1.$$

G_0 is also MvEV *and*

$$G_1 \succ G_0 \succ G_2 \quad (extDep)$$

Proof. Check that stable tail dependence function φ_0 of G_0 satisfies $\varphi_0 = \alpha_1\varphi_1 + \alpha_2\varphi_2$. Actually spectral measure of G_0 is a mixture of those of G_1 and G_2.

□

An alternative dependence function

For bivariate MvEV distributions, stable tail dependence function can be expressed as

$$\varphi(v_1, v_2) = -\log G(1/v_1, 1/v_2) = (v_1 + v_2) A\left(\frac{v_2}{v_1 + v_2}\right),$$

The function A, called *Pickands dependence function*, has the following properties

1. $(1-t)\bigvee t \le A(t) \le 1,\ t \in [0,1]$,

   The lower bound is attained if G is completely dependent, and the upper bound if G is independent.

2. A is convex.

   Hence, as the definition of A shows, the order in terms of A is equivalent to that by φ.

Proposition 4. *Suppose G_1 and G_2 be bivariate* MvEV *distribution functions, and let A_1 and A_2 be their Pickands dependence function.*

$$G_1 \succ G_2 \quad (extDep) \quad \textit{iff} \quad A_1(t) \le A_2(t), \quad t \in [0,1].$$

A use of dependence order

Definition of extreme dependence order is to compare MvEV distributions. However, it compares also dependence among components, especially dependence of pairs of components, (Y_{i1}, Y_{i2}) and (Y_{j1}, Y_{j2}). All four components are different, or one of first and second pairs can be common.
For applications, a measure (index) for dependence, or linear ordering, is required, and a popular one is defined by

$$\lambda = \lim_{u \to 1} \frac{\overline{C}(u,u)}{1-u} = 2 - \lim_{u \to 1} \frac{\log C(u,u)}{\log u},$$

where $\overline{C}$ is the survival copula. The measure satisfies $0 \le \lambda \le 1$ and $\lambda = 0$ means independence and $\lambda = 1$ complete dependence. Our partial order implies this total order.

Another measure for dependence in bivariate MvEV is defined using Pickands dependence function as $\theta = 2A(1/2)$. Actually $\theta = 2 - \lambda$. It satisfies $1 \le \theta \le 2$, and $\theta = 2$ corresponds to independence and $\theta = 1$ to complete dependence.

A parametric family

A popular parametric subfamily of MvEV distributions is *asymmetric logistic distributions* introduced by Coles and Tawn (1991). It is popular since it has explicit formulas of d.f. and marginals for any dimension, as well as the densities of spectral measure. For $d = 3$ its exponent measure is as follows:

$$\begin{aligned} V(\mathsf{x}) = -\log G(\mathsf{x}) = {} & \frac{\theta_{11}}{x_1} + \frac{\theta_{12}}{x_2} + \frac{\theta_{13}}{x_3} + \left(\left(\frac{\theta_{(21)1}}{x_1} \right)^{r_2} + \left(\frac{\theta_{(12)2}}{x_2} \right)^{r_2} \right)^{1/r_2} \\ & + \left(\left(\frac{\theta_{(31)1}}{x_1} \right)^{r_2} + \left(\frac{\theta_{(13)3}}{x_3} \right)^{r_2} \right)^{1/r_2} + \left(\left(\frac{\theta_{(32)2}}{x_2} \right)^{r_2} + \left(\frac{\theta_{(23)3}}{x_3} \right)^{r_2} \right)^{1/r_2} \\ & + \left(\left(\frac{\theta_{31}}{x_1} \right)^{r_3} + \left(\frac{\theta_{32}}{x_2} \right)^{r_3} + \left(\frac{\theta_{33}}{x_3} \right)^{r_3} \right)^{1/r_3}, \quad \mathsf{x} \in \mathcal{R}_+^3, \quad r_2 \ge 1, \quad r_3 \ge 1, \quad \theta_{\cdot} \ge 0, \end{aligned}$$

$$\theta_{11} + \theta_{(21)1} + \theta_{(31)1} + \theta_{31} = 1, \quad \theta_{12} + \theta_{(12)2} + \theta_{(32)2} + \theta_{32} = 1,$$

$$\text{and} \quad \theta_{13} + \theta_{(13)3} + \theta_{(23)3} + \theta_{33} = 1.$$

There are 11 parameters. If $r_2 = r_3 = 1$ $G(\mathsf{x})$ is independent. If r_2 or r_3 is increased with other parameters are fixed, $G(\mathsf{x})$ becomes more dependent.

For the marginals, for example

$$\begin{aligned} V(x_1, x_2, \infty) &= -\log G(x_1, x_2, \infty) \\ &= \frac{\theta_{11} + \theta_{(31)1}}{x_1} + \frac{\theta_{12} + \theta_{(32)2}}{x_2} \\ &+ \left(\left(\frac{\theta_{(21)1}}{x_1} \right)^{r_2} + \left(\frac{\theta_{(12)2}}{x_2} \right)^{r_2} \right)^{1/r_2} + \left(\left(\frac{\theta_{31}}{x_1} \right)^{r_3} + \left(\frac{\theta_{32}}{x_2} \right)^{r_3} \right)^{1/r_3} \end{aligned}$$

Further comparison can be made computing $\varphi(\mathsf{v}) = V(1/\mathsf{v})$.

Discussion on risks

Risks are related to extremes and dependence. Disasters happen from extreme value of loss, and they worsen when loss values are positively dependent. Hence dependence among extreme values are important in many situations.

For applications, asymptotic dependence among larger components of multivariate random variables will be realistic. The approach in this paper focuses on one aspect of problems, but the result is clear so far as MvEV distributions are used.

Data analysis in real situations will follow.

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