Correlation and dependence

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Dependence

Pairwise dependence of variates is a main issue in statistical analysis and various measures have been proposed.

<u>Measures of dependence</u>

- Rank based measure (Kendall's tau, Spearman's rho)

  + Robustness, and close link to *copula*

  + Difficult to understand the exact meaning

- Second moment measure (Pearson's correlation coefficient)

  + Normalized inner product of two variates

  → Mathematically simple, and easier to understand the meaning

  + Possible to be extended to measures of conditional dependence

  → Partial correlation, conditional correlation

Our interest

For a random vector distributed the multivariate Normal, the following two propositions hold true.

P1: Zero correlation between two variables is equivalent to their independence.

→ very restrictive to the Normal and its neighbors

P2: Partial correlation coefficient is equal to conditional correlation coefficient.

→ not restrictive to the Normal and its neighbors

↓

Question:

Do these propositions hold true when we depart from the normal?

Outline of this presentation

§1 Zero Correlation and Independence

About P1

§2 Partial Correlation and Conditional Correlation

- A necessary and sufficient condition for equivalence of partial and conditional covariance
- A sufficient condition (Condition C) for P2.

§3 Multiplicative Correlations

A key to Condition C

§ 1. Zero Correlation and Independence

There is a case where zero correlation of two variables is equivalent to their independence other than the normal.

Theorem 1. If (X_1, X_2) has a bivariate normal distribution, then

$$\rho\,(\psi_1(X_1), \psi_2(X_2)) = 0 \iff \psi_1(X_1) \perp\!\!\!\perp \psi_2(X_2)$$

for any monotone increasing (decreasing) transforms ψ_1 and ψ_2.

Theorem 1 was directly proved in Baba, Shibata and Sibuya (2004), but it is essentially known from the following properties of a normal copula for increasing transformations:

- A bivariate normal copula with correlation **ρ** is equal to independent copula if and only if **ρ**=0.
- A copula of (X, Y) is invariant under strictly increasing transformations of X and Y.

(see, for example, Nelsen, R. B., 1999)

Example of Theorem 1. If $(X_1, X_2) \sim$ Log-normal, then $\rho(X_1, X_2) = 0 \Longleftrightarrow X_1 \perp\!\!\!\perp X_2$.

§ 2. Partial Correlation and Conditional Correlation

When we think of conditional independence of variates, two typical measures are proposed: Partial correlation and Conditional Correlation.

Notation and Dedinitions

$$\left.\begin{array}{ll} \boldsymbol{X} = (X_1, \ldots, X_p) & (p \geq 2) \\ \boldsymbol{Y} = (Y_1, \ldots, Y_q) & (q \geq 1) \end{array}\right\} : \text{ random vectors}$$

$$\mathsf{V}\left(\begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix}\right) = \begin{bmatrix} \Sigma_{\boldsymbol{XX}} & \Sigma_{\boldsymbol{XY}} \\ \Sigma_{\boldsymbol{YX}} & \Sigma_{\boldsymbol{YY}} \end{bmatrix}$$: Partitioned variance-covariance matrix of $(\boldsymbol{X}, \boldsymbol{Y})$.

Partial correlation

partial covariance matrix of $\boldsymbol{X}$ given $\boldsymbol{Y}$:

$$\Sigma_{XX\cdot Y} = (\sigma_{ij\cdot Y})_{i,j=1,\dots,p} = \Sigma_{XX} - \Sigma_{XY}{\Sigma_{YY}}^{-1}\Sigma_{YX}$$

partial correlation matrix of $\boldsymbol{X}$ given $\boldsymbol{Y}$:

$$R_{XX\cdot Y} = (\rho_{ij\cdot Y})_{i,j=1,\dots,p},$$

$$\begin{aligned} \rho_{ij\cdot Y} &= \frac{\sigma_{ij\cdot Y}}{\sqrt{\sigma_{ii\cdot Y}\sigma_{jj\cdot Y}}} \\ &= \rho\left(X_i - \widehat{X}_i(\boldsymbol{Y}),\ X_j - \widehat{X}_j(\boldsymbol{Y})\right) \text{ for } i,j = 1,\dots,p. \\ &\quad \text{where } \ \widehat{\boldsymbol{X}}(\boldsymbol{Y}) = \mathsf{E}(\boldsymbol{X}) + \Sigma_{XY}\Sigma_{YY}^{-1}(\boldsymbol{Y} - \mathsf{E}(\boldsymbol{Y})) \end{aligned}$$

Conditional correlation

conditional covariance matrix of $\boldsymbol{X}$ given $\boldsymbol{Y}$:

$$\Sigma_{XX|Y} = (\sigma_{ij|Y})_{i,j=1,\ldots,p}$$

$$\text{where} \quad \sigma_{ij|Y} \;=\; \mathsf{Cov}(X_i, X_j|\boldsymbol{Y}) \;=\; \mathsf{E}\left((X_i - \mathsf{E}(X_i|\boldsymbol{Y}))\,(X_j - \mathsf{E}(X_j|\boldsymbol{Y}))\,|\boldsymbol{Y}\right)$$

conditional correlation matrix of $\boldsymbol{X}$ given $\boldsymbol{Y}$:

$$R_{XX|Y} = (\rho_{ij|Y})_{i,j=1,\ldots,p} \quad \text{where} \quad \rho_{ij|Y} = \frac{\sigma_{ij|Y}}{\sqrt{\sigma_{ii|Y}\sigma_{jj|Y}}}.$$

Theorem 2. For any random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, the following two conditions are equivalent.

(i) $\mathsf{E}(\boldsymbol{X}|\boldsymbol{Y}) = \boldsymbol{\alpha} + B\boldsymbol{Y}$ for a vector $\boldsymbol{\alpha}$ and a matrix B

(ii) $\Sigma_{XX\cdot Y} = \mathsf{E}\left(\Sigma_{XX|Y}\right)$.

$$\downarrow$$

Corollary 1. For any $\boldsymbol{X}$ and $\boldsymbol{Y}$, the following two conditions are equivalent.

(i) $\mathsf{E}(\boldsymbol{X}|\boldsymbol{Y}) = \boldsymbol{\alpha} + B\boldsymbol{Y}$ and $\Sigma_{XX|Y}$ is independent of $\boldsymbol{Y}$.

(ii) $\Sigma_{XX\cdot Y} = \Sigma_{XX|Y}$ $a.s.$

Corollary 2. For any $\boldsymbol{X}$ and $\boldsymbol{Y}$, if it holds true

$\mathsf{E}(\boldsymbol{X}|\boldsymbol{Y}) = \boldsymbol{\alpha} + B\boldsymbol{Y}$ and $R_{XX|Y}$ is independent of $\boldsymbol{Y}$,

then $R_{XX\cdot Y} = R_{XX|Y}$ $a.s.$

Condition C

Example I. Elliptical Distribution

$\mathsf{EC}(\boldsymbol{\mu}, \Sigma, \phi)$ is a family of distributions whose c.f. takes the form

$$\Psi(\boldsymbol{t}) = \exp\left(i\boldsymbol{t}^\mathsf{T}\boldsymbol{\mu}\right)\phi\left(\boldsymbol{t}^\mathsf{T}\Sigma\boldsymbol{t}\right) \quad \text{with scalar function } \phi.$$

If $(\boldsymbol{X}, \boldsymbol{Y}) \sim \mathsf{EC}(\boldsymbol{\mu}, \Sigma, \phi)$, then

$$\mathsf{E}(\boldsymbol{X}|\boldsymbol{Y}) = \mathsf{E}(\boldsymbol{X}) + \Sigma_{XY}{\Sigma_{YY}}^{-1}(\boldsymbol{Y} - \mathsf{E}(\boldsymbol{Y}))$$

and

$$\Sigma_{XX|Y} = s(\boldsymbol{Y})\Sigma^*,$$

where s is a function and the matrix $\boldsymbol{\Sigma}^*$ is independent of $\boldsymbol{Y}$.

Example I. Elliptical Distribution (2)

- If $(\boldsymbol{X}, \boldsymbol{Y}) \sim \mathsf{EC}(\boldsymbol{\mu}, \Sigma, \phi)$, $(\boldsymbol{X}, \boldsymbol{Y})$ satisfies Condition C, so it holds true

$$R_{XX\cdot Y} = R_{XX|Y}.$$

- Kelker (1970) showed that the multivariate Normal is the only one distribution in which it holds ture $\Sigma_{XX\cdot Y} = \Sigma_{XX|Y}$ in the class of $\mathsf{EC}(\boldsymbol{\mu}, \Sigma, \phi)$.

- Assume that $\boldsymbol{X} \sim \mathsf{EC}(\boldsymbol{\mu}, \Sigma, \phi)$. $\quad \boldsymbol{X} \sim \mathsf{N}(\boldsymbol{\mu}, \Sigma)$ if and only if $\rho(X_i, X_j) = 0 \Longleftrightarrow X_i \perp\!\!\!\perp X_j$.

Example II. Distribution generated from generalized Pareto distribution

The generalized Pareto distribution $\mathrm{GPrt}(\gamma, a)$ is defined with the survival function

$$\bar{F}(x;\gamma,a) = \begin{cases} (1+\gamma x/a)^{-1/\gamma} & \text{if } \gamma \neq 0 \\ \exp(-x/a) & \text{if } \gamma = 0 \end{cases} \qquad (a > 0),$$

- Pareto ($\gamma > 0$), exponential ($\gamma = 0$) and uniform ($\gamma = -1$) distributions are members.
- The r-th moment is finite $\iff r < 1/\gamma$.
- If the distribution $\mathrm{GPrt}(\gamma, a)$ is left truncated at u, the truncated distribution is $\mathrm{GPrt}(\gamma, a + \gamma u)$.

Example II. Distribution generated from generalized Pareto distribution (2)

Assume that $\boldsymbol{Z} = (Z_1, \ldots, Z_{p+q})$ is a random sample from GPrt(γ, a), and that $(Z_{(1)}, \ldots, Z_{(p+q)})$ is the order statistics of $\boldsymbol{Z}$ such that $\underbrace{Z_{(1)} \geq \cdots Z_{(p)}}_{\boldsymbol{X}} \geq \underbrace{Z_{(p+1)} \geq Z_{(p+q)}}_{\boldsymbol{Y}}$.

If $\boldsymbol{X} = (Z_{(1)}, \ldots, Z_{(p)})$ and $\boldsymbol{Y} = (Z_{(p+1)}, \ldots, Z_{(p+q)})$, then

$$(\boldsymbol{X}|\boldsymbol{Y} = \boldsymbol{y}) \stackrel{d}{=} y_1 + (1 + \gamma y_1/a)\boldsymbol{X}.$$

Thus it holds true

$$\mathsf{E}(\boldsymbol{X}|\boldsymbol{Y}) = y_1 + (1 + \gamma y_1/a)\mathsf{E}(\boldsymbol{X})$$

and

$$\Sigma_{XX|Y} = (1 + \gamma y_1/a)^2 \mathsf{V}(\boldsymbol{X}).$$

Example II. Distribution generated from generalized Pareto distribution (3)

- If $\boldsymbol{Z} \sim \mathrm{GPrt}(\gamma, a)$ with $\gamma < 1/2$, then $(\boldsymbol{X}, \boldsymbol{Y})$ which is the order statistics of $\boldsymbol{Z}$ in decreasing order satisfies Condition C, so it holds true

$$R_{XX\cdot Y} = R_{XX|Y}.$$

- Especially, if $\boldsymbol{Z}$ has an exponential distribution ($\gamma = 0$), it holds true

$$\Sigma_{XX\cdot Y} = \Sigma_{XX|Y}.$$

- If $\boldsymbol{Z}$ has a geometric distribution (discrete distibution), it holds true

$$\Sigma_{XX\cdot Y} = \Sigma_{XX|Y}.$$

Example III. Distributions in $\mathcal{F}_T$

At first, we prepare the definition of the family of distribution, $\mathcal{F}$.

$\mathcal{F} = \{F_\theta : \theta \in \Theta\}$ is defined as the family of distribution functions which have a semigroup property such that $F_{\theta_1} * F_{\theta_2} = F_{\theta_1+\theta_2} \in \mathcal{F}$ for the convolution of any $F_{\theta_1}, F_{\theta_2} \in \mathcal{F}$. Parameter space Θ is assumed on $(0, \infty)$ or the set of all natural numbers. (If $\Theta = (0, \infty)$, $\mathcal{F}$ is a class of infinitely divisible distributions.)

Ex. N(θ_i, σ^2), Po(θ_i), Bn(θ_i, p), NgBn(θ_i, p), Ga(θ_i, a) are members of $\mathcal{F}$.

Example III. Distributions in $\mathcal{F}_T$ (2)

Assume that random variables $Z_1, \ldots, Z_{p+q}$ are independently distributed as $Z_i \overset{d}{=} F_{\theta_i} \in \mathcal{F}(i = 1, \ldots, p+q)$. $\mathcal{F}_T = \{F_{\boldsymbol{\theta},t} : \boldsymbol{\theta} \in \Theta^{p+q}\}$ is defined as the family of $(p+q)$-dimensional conditional distribution functions of $(Z_1, \ldots, Z_{p+q})$ given $T = \sum_{i=1}^{p+q} Z_i$.

If $(\boldsymbol{X}, \boldsymbol{Y})$ has a distribution in $\mathcal{F}_T$, then $(\boldsymbol{X}, \boldsymbol{Y})$ satisfies Condition C, so it holds true $R_{XX\cdot Y} = R_{XX|Y}$.

Table 1. Distributions in $\mathcal{F}_T$

$Z_i \in \mathcal{F}$	$(\boldsymbol{Z}\|T) \in \mathcal{F}_T$
$\mathrm{N}(\theta_i \mu, \theta)$	$\mathrm{N}\big(t\boldsymbol{\xi},\ \theta(\mathrm{diag}(\boldsymbol{\xi}) - \boldsymbol{\xi}\boldsymbol{\xi}^\top)\big)$
$\mathrm{Po}(\theta_i \lambda)$	$\mathrm{Mn}(t, \boldsymbol{\xi})$
$\mathrm{Bn}(\theta_i, p)$	$\mathrm{MvHg}(t, \boldsymbol{\theta})$
$\mathrm{NgBn}(\theta_i, p)$	$\mathrm{MvNgHg}(t, \boldsymbol{\theta})$
$\mathrm{Ga}(\theta_i, 1/a)$	$\boldsymbol{Z}/t\|t \sim \mathrm{Dir}(\boldsymbol{\theta})$

$$\boldsymbol{\xi} = \boldsymbol{\theta}/\theta, \quad \theta = 1^\top \boldsymbol{\theta}$$

§ 3. Multiplicative Correlations

Introduction I

The conditional covariance matrices of $\boldsymbol{Z}$ given $T = t$ in **Table 1** have a common form

$$\mathsf{V}(\boldsymbol{Z}|T = t) = a\,\mathrm{diag}(\boldsymbol{a}) - \boldsymbol{a}\boldsymbol{a}^{\mathsf{T}} \quad a = \mathbf{1}^{\mathsf{T}}\boldsymbol{a}.$$

This type of covariance matrix is a key to Condition C.

The following theorem gives us other induction into the covariance form.

Theorem 4. Assume that $\boldsymbol{X} = (X_1, \ldots, X_n)$ has a covariance matrix which takes the form

$$\boxed{\mathsf{V}(\boldsymbol{X}) = \mathrm{diag}(\boldsymbol{b}) - \boldsymbol{a}\boldsymbol{a}^{\mathsf{T}}.}$$

Then, $\boldsymbol{b} = (\sum_{i=1}^{n} a_i)\boldsymbol{a}$ if and only if $\sum_{i=1}^{n} X_i$ is almost surely constant.

We start from only assumption of the covariance form

$$\mathsf{V}(\boldsymbol{X}) = \mathrm{diag}(\boldsymbol{b}) - \boldsymbol{a}\boldsymbol{a}^{\mathsf{T}}.$$

Under this covariance form, correlation of (X_i, X_j) takes the form

$$\rho(X_i, X_j) = -\delta_i \delta_j \quad (i \neq j).$$

Introduction II.

We consider **Reduction Method** (for example, Mardia, 1970) which is a natural method by which we define new multivariate distributions as follows:

$$Z_0, Z_1, \ldots, Z_n \sim \mathsf{Ga}(\theta_i) \text{ or } \mathsf{Po}(\theta_i) \text{ indep.}$$

$$\Longrightarrow X_i = Z_0 + Z_i \ (i = 1, \ldots, n)$$

$$\begin{aligned} \rho(X_i, X_j) &= \theta_0 / \sqrt{(\theta_0 + \theta_i)(\theta_0 + \theta_j)} \\ &= \sqrt{\theta_0/(\theta_0 + \theta_i)} \sqrt{\theta_0/(\theta_0 + \theta_j)} \\ &= \delta_i \delta_j \end{aligned}$$

Multiplicative Covariance

$$\begin{aligned} \text{Positive } \Sigma^{+}(\boldsymbol{a},\boldsymbol{b}) &\ :\ \mathsf{V}(\boldsymbol{X}) = \operatorname{diag}(\boldsymbol{b}) + \boldsymbol{a}\boldsymbol{a}^{\mathsf{T}} \\ \text{Negative } \Sigma^{-}(\boldsymbol{a},\boldsymbol{b}) &\ :\ \mathsf{V}(\boldsymbol{X}) = \operatorname{diag}(\boldsymbol{b}) - \boldsymbol{a}\boldsymbol{a}^{\mathsf{T}} \end{aligned}$$

Multiplicative Correlation

$$\begin{aligned} \text{Positive } R^{+}(\boldsymbol{\delta}) &\ :\ \rho(\boldsymbol{X}) = \operatorname{diag}(1-\boldsymbol{\delta}^{2}) + \boldsymbol{\delta}\boldsymbol{\delta}^{\mathsf{T}} \quad (\rho(X_i,X_j) = \delta_i\delta_j) \\ \text{Negative } R^{-}(\boldsymbol{\delta}) &\ :\ \rho(\boldsymbol{X}) = \operatorname{diag}(1+\boldsymbol{\delta}^{2}) - \boldsymbol{\delta}\boldsymbol{\delta}^{\mathsf{T}} \quad (\rho(X_i,X_j) = -\delta_i\delta_j) \end{aligned}$$

Factorization of variables

Theorem 5. For $\boldsymbol{X} = (X_1, \ldots, X_n)$ with $\mathsf{E}(\boldsymbol{X}) = \boldsymbol{0}$, $\boldsymbol{X}$ has a multiplicative covariance $\mathsf{V}(\boldsymbol{X}) = \mathrm{diag}(\boldsymbol{b}) \pm \boldsymbol{a}\boldsymbol{a}^\top$ with $\boldsymbol{b} > \boldsymbol{0}$ if and only if each element of $\boldsymbol{X}$ is written as

$$X_i = a_i Z_0 + \sqrt{b_i} Z_i, \quad i = 1, \ldots, n,$$

where

$$\begin{aligned} \mathsf{E}(Z_i) &= 0 \quad i = 0, 1, \ldots, n \\ \mathrm{var}(Z_i) &= 1 \quad i = 0, 1, \ldots, n \\ \rho(Z_i, Z_j) &= 0 \quad i = 1, \ldots, n \\ \rho(Z_0, Z_i) &= \begin{cases} 0 & \text{for } \Sigma^+(\boldsymbol{a}, \boldsymbol{b}) \\ -a_i/\sqrt{b_i} & \text{for } \Sigma^-(\boldsymbol{a}, \boldsymbol{b}) \end{cases} \qquad i = 1, \ldots, n. \end{aligned}$$

Remark. For $\Sigma^-(\boldsymbol{a}, \boldsymbol{b})$, Z_0 is orthogonal to X_is.

- For $\Sigma^{+}(\boldsymbol{a}, \boldsymbol{b})$, the result is known in **factor analysis**.

- For $R^{\pm}(\boldsymbol{\delta})$ of the **mv. normal distribution**, Gupta (1963) or Six (1981) showed the factorization.

- Kelderman (2004) shows that if $\boldsymbol{X}$ has a **mv. normal distribution**,

$$\mathsf{V}(\boldsymbol{X}) = \mathsf{diag}(\boldsymbol{b}) + \boldsymbol{a}\boldsymbol{a}^{\mathsf{T}} \quad \text{with} \quad \boldsymbol{a} > \boldsymbol{0} \quad \text{and} \quad \boldsymbol{b} > \boldsymbol{0}$$

$$\Updownarrow$$

for any partition $\boldsymbol{Y} = (\boldsymbol{Y}_M, \boldsymbol{Y}_{M^c})$ and any $\boldsymbol{y}^*_{M^c}$ which is permutation of the values $\boldsymbol{y}_{M^c}$,

$$f(\boldsymbol{y}_M | \boldsymbol{Y}_{M^c} = \boldsymbol{y}_{M^c}) = f(\boldsymbol{y}_M | \boldsymbol{Y}_{M^c} = \boldsymbol{y}^*_{M^c})$$

where $\boldsymbol{Y} = \boldsymbol{\alpha} + \mathsf{diag}(\beta)\boldsymbol{X}$ $(\beta > \boldsymbol{0})$.
(*measurement exchangeability*)

Eigen values

Proposition 1. The following inequality holds true for the eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ of $R^+(\boldsymbol{\delta})$.

$$1 - \delta_{k_1}^2 = \lambda_1 = \cdots = \lambda_{n_1-1} < \lambda_{n_1} < 1 - \delta_{k_2}^2 = \lambda_{n_1+1} = \cdots$$

$$\cdots = \lambda_{n_1+n_2-1} < \lambda_{n_1+n_2} < \cdots$$

$$\cdots < \lambda_{n-n_m} < 1 - \delta_{k_m}^2 = \lambda_{n-n_m+1} = \cdots = \lambda_{n-1} < \lambda_n,$$

where $\delta_{k_1}^2 > \delta_{k_2}^2 > \cdots > \delta_{k_m}^2$ are m distinct values in $\delta_1^2, \ldots, \delta_n^2$ and $n_1, \ldots, n_m$ $\left(\sum_{i=1}^m n_i = n\right)$ are the multiplicities of $\delta_{k_1}^2, \ldots, \delta_{k_m}^2$, respectively.

This proposition shows that $R^+(\boldsymbol{\delta})$ **allows only one eigen value larger than 1**, so that it is useful to check if the observed correlation matrix is $R^+(\boldsymbol{\delta})$.

Invariance Properties

Theorem 6. Let $(\boldsymbol{X}, T)$ be a random vector and assume that

$$\mathsf{V}(\boldsymbol{X} |\ T = t) = \sigma(t)\left(\mathsf{diag}(\boldsymbol{b}) \pm \boldsymbol{a}\boldsymbol{a}^{\mathsf{T}}\right)$$

for a $\sigma(t) > 0$. If

$$\mathsf{E}(\boldsymbol{X} |\ T = t) = \mu(t)\boldsymbol{a} + \boldsymbol{c}$$

for a constant vector $\boldsymbol{c}$, then the unconditional covariance is again multiplicative,

$$\mathsf{V}(\boldsymbol{X}) = \mathsf{E}\left(\sigma(T)\right)\mathsf{diag}(\boldsymbol{b}) + \{\mathsf{var}(\mu(T)) \pm \mathsf{E}\left(\sigma(T)\right)\}\,\boldsymbol{a}\boldsymbol{a}^{\mathsf{T}}.$$

Invariance Properties (2)

Theorem 7. Assume that $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2)$ has a $\Sigma^{\pm}(\boldsymbol{a}, \boldsymbol{b})$. If $\mathsf{V}(\boldsymbol{X}_2)$ is non-singular and all elements of $\boldsymbol{b}_2$ are positive, then the **partial covariance of $\boldsymbol{X}_1$ given $\boldsymbol{X}_2$ is also multiplicative** and

$$\mathrm{diag}(\boldsymbol{b}_1) \pm \boldsymbol{a}_1\boldsymbol{a}_1^\top / \{1 \pm \boldsymbol{a}_2^\top \mathrm{diag}(\boldsymbol{b}_2)^{-1}\boldsymbol{a}_2\}.$$

This theorem means that if $\boldsymbol{X}$ has a $\Sigma^{\pm}(\boldsymbol{a}, \boldsymbol{b})$, **zero correlation** is equivalent to **zero partial correlation**.

Multivariate Distributions with Multiplicative Correlations

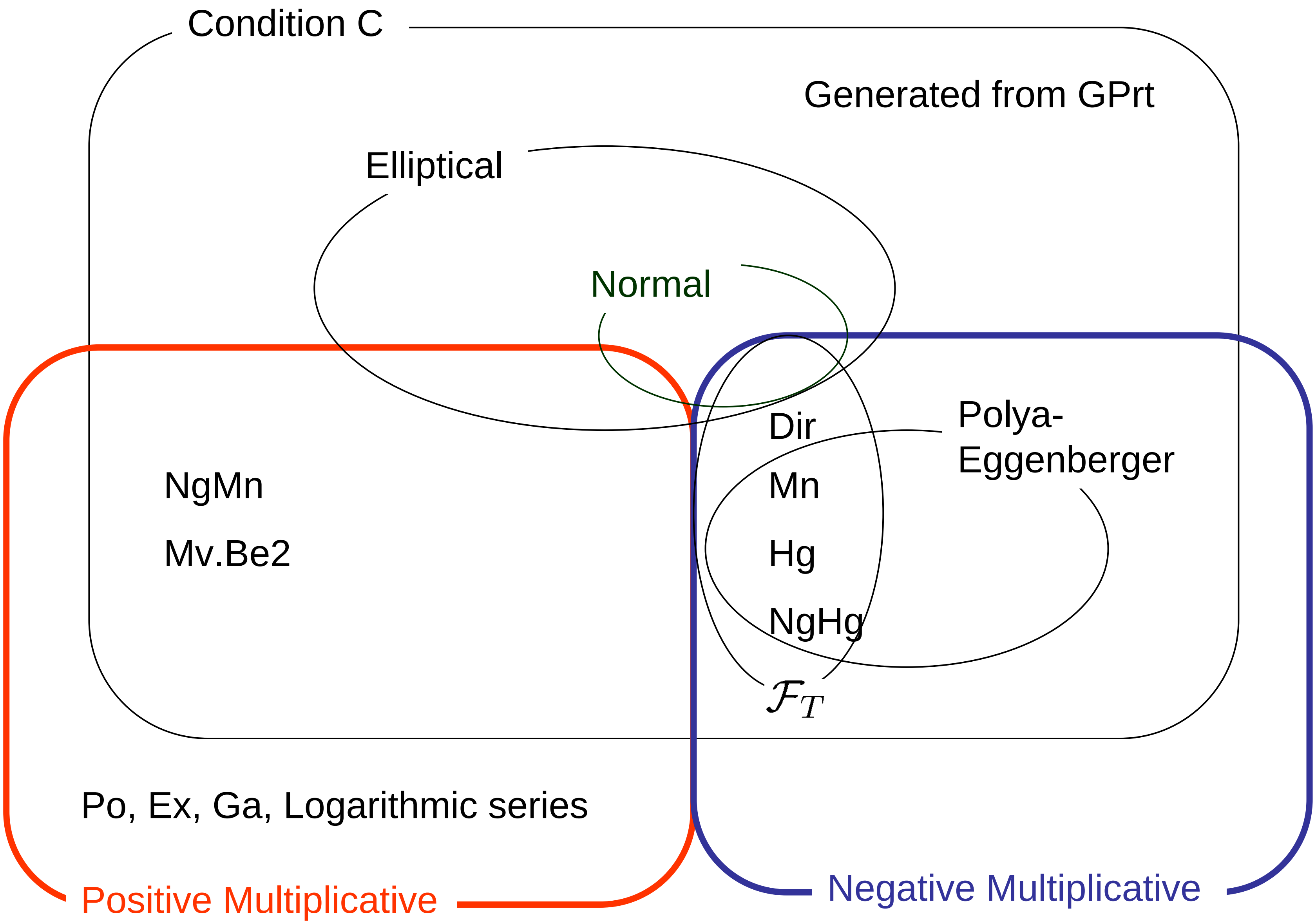
Discrete Multivariate Distributions in Johnson *et al.* (1997).

Family	Subfamily	Positive or Negative
35 Multinomial		Negative
36 Negative multinomial		Positive
37 Poisson		Positive
38 Power series	Logarithmic series	Positive
39 Hypergeometric		Negative
40 Pólya-Eggenberger		Negative
41 Ewens	—	—
42 Distributions of	Negative binomial of order s	Positive
order s	Logarithmic distr. of order s	Negative

Multivariate Distributions with Multiplicative Correlations (2)

Continuous Multivariate Distributions in Kotz *et al.* (2000).

Family		Subfamily	Positive or Negative
45	Normal	Multiplicatively correlated normal	Both
47	Exponential	Moran and Downton's	Positive
48	Gamma	Cheriyan and Ramabhadran's	Positive
49	Dirichlet		Positive
49	Inverted Dirichlet		Negative
50	Liouvill		Both
51	Logistic	Gumbel-Malik-Abraham	Positive
		Farlie-Gumbel-Morgenstern	Negative
52	Pareto	The first kind	Positive
53	Extreme value	—	—


Condition C
Generated from GPrt
Elliptical
Normal
Dir
Mn
Hg
NgHg
Polya-
Eggenberger
$\mathcal{F}_T$
NgMn
Mv.Be2
Po, Ex, Ga, Logarithmic series
Positive Multiplicative
Negative Multiplicative

Concluding Remarks

1. The distributions for which zero correlation is equal to independence are very restrictive to the Normal and its neighbors.

2. Other than the Normal, there are rarely cases in where it hold true $\Sigma_{XX\cdot Y} = \Sigma_{XX|Y}$, and Condition C is satisfied for several distributions (elliptical, generated form GPrt, $\mathcal{F}_T$).

3. The studies for Multiplicative Correlations which is a key to Condition C connect
   - to check if the observed correlation matrices are suitable for one-factor model,
   - to propose a new one-factor model for negative multiplicative correlations.