Hempel's raven paradox, Hume's problem of induction, Goodman's grue paradox, Peirce's abduction, Flagpole problem are clarified in quantum language

by

Shiro Ishikawa

Shiro Ishikawa

Department of Mathematics

Keio University

Department of Mathematics

Faculty of Science and Technology

Keio University

Hempel's raven paradox, Hume's problem of induction, Goodman's grue paradox, Peirce's abduction, Flagpole problem are clarified in quantum language

Shiro Ishikawa [*1]

Abstract

Recently we proposed "quantum language" (or, "the linguistic Copenhagen interpretation of quantum mechanics", "measurement theory") as the language of science. This theory asserts the probabilistic interpretation of science (= the linguistic quantum mechanical worldview), which is a kind of mathematical generalization of Born's probabilistic interpretation of quantum mechanics. In this preprint, we consider the most fundamental problems in philosophy of science such as Hempel's raven paradox, Hume's problem of induction, Goodman's grue paradox, Peirce's abduction, flagpole problem, which are closely related to measurement. We believe that these problems can never be solved without the basic theory of science with axioms. Since our worldview has the axiom concerning measurement, these problems can be solved easily. Hence there is a reason to assert that quantum language gives the mathematical foundations to science. [*2]

Contents

1   Introduction

This preprint is prepared in order to reconsider ref. [19]: S. Ishikawa, "*Philosophy of science for scientists; The probabilistic interpretation of science*", JQIS, Vol. 3, No.3 , 140-154 . In particular, Section 3 (Hempel's raven paradox) will be discussed more right compared to that of ref. [19].

1.1   Philosophy of science is about as useful to scientists as ornithology is to birds

We think that philosophy of science is classified as follows.

(A_1) Criticism about science, The relation between society and science, History of science, etc.
    (Non-scientists may be interested in these mainly, thus it is called "philosophy of science for the general

---

[*1] Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1, Hiyoshi, Kouhoku-ku Yokohama, Japan. E-mail: ishikawa@math.keio.ac.jp

[*2] For the further information of quantum language, see my homepage (http://www.math.keio.ac.jp/~ishikawa/indexe.html)

public" in this paper)

(A_2) Study on mathematical foundations of science, i.e., research on mathematical properties common to various sciences, that is, a kind of unified theory of science

(This is called "philosophy of science for scientists" in this paper)

The part (A_1) is a large part of the philosophy of science. And it is certain that many valuable results have been produced in the study (A_1) (e.g., Popper's falsificationism (*cf.* Remark 28 later)). On the other hand, it is generally considered that the study (A_2) was not so productive in spite that the part (A_2) (e.g., Hempel's scientific explanation (*cf.* refs. [1, 2])) was expected to be the core of philosophy of science. For example, Prof. Richard Feynman, one of the most outstanding physicists in the 20th century, said that "*Philosophy of science is about as useful to scientists as ornithology is to birds*". If he used the word "philosophy of science" to mean "philosophy of science for scientists", we can agree with his claim. That is, we also consider that the part (A_2) is still undeveloped. Thus, our purpose of this paper is to complete "philosophy of science for scientists" in (A_2).

This will be done by using quantum language (mentioned in Section 2), which has two fundamental concepts "measurement" and "causality" (that are common to various sciences). And in Sections 3-6, we show that famous unsolved problems (i.e., Hempel's raven paradox, Hume's problem of induction, Goodman's grue paradox, Peirce's abduction, flagpole problem) in philosophy of science can be solved in quantum language.

2 Review: Quantum language (= Measurement theory (=MT))

2.1 Quantum language is the language to describe science

Recently, in refs. [3]-[18], we proposed quantum language (or, "the linguistic Copenhagen interpretation of quantum mechanics", "measurement theory (=MT)"), which is a kind of the language of science. This is not only characterized as the metaphysical and linguistic turn of quantum mechanics but also the quantitative turn of Descartes=Kant epistemology and the dualistic turn of statistics. Thus, the location of this theory in the history of scientific worldviews is as follows:

(*cf.* refs. Ishikawa 2012a; Ishikawa 2017a Ishikawa 2018b):
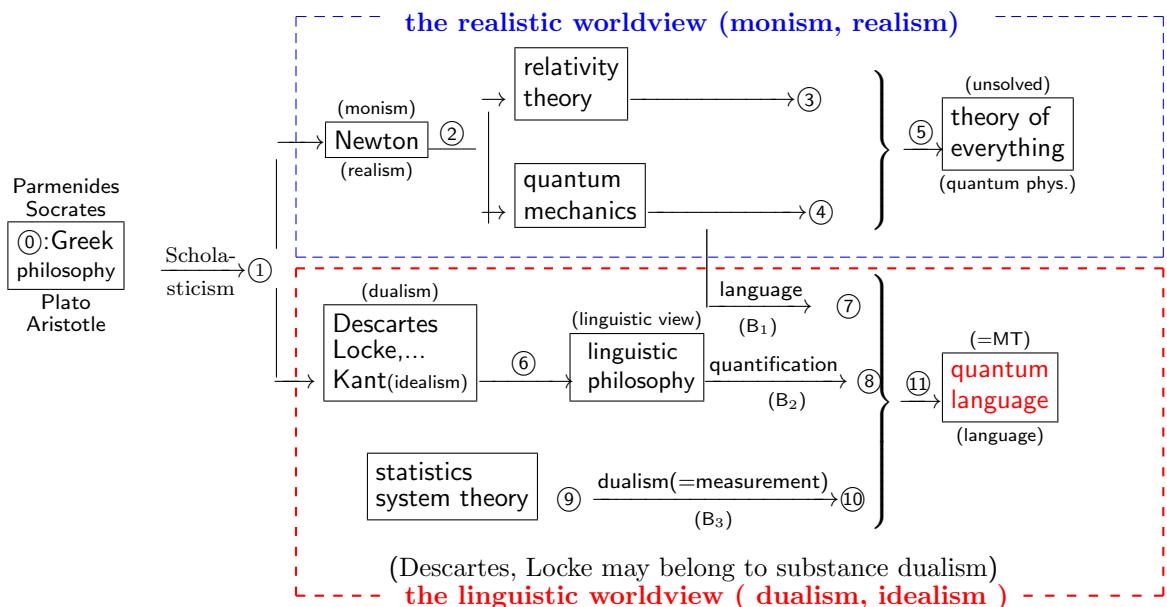
Figure 1: [**The location of quantum language in the history of scientific worldviews**]

Fig. 1 says that quantum language (=the linguistic Copenhagen interpretation) has the following three

aspects

(B$_1$) the linguistic Copenhagen interpretation is the true figure of so-called Copenhagen interpretation (⑦ in Fig. 1), *cf.* refs. [3, 6, 7, 13, 16], particularly, Heisenberg's uncertainty relation was proved in [3], and the von Neumann-Lüders projection postulate (i.e., the meaning of the wave function collapse) was clarified in [13],

(B$_2$) the scientific final goal of dualistic idealism (i.e., Descartes=Kant philosophy) (⑧ in Fig. 1), *cf.* refs. [8, 14, 15, 17], particularly, the mind-body problem was clarified in [15], also, the paradox of brain in bat (*cf.* [20]) was solved in [17].

(B$_3$) statistics (= dynamical system theory) with the concept "measurement" (⑩ in Fig. 1), *cf.* refs. [4, 5, 6, 10, 11]. Thus, quantum language gives the answer of "Why statistics is used in science?"

Hence, it is natural to assume that

(B$_4$) quantum language proposes the probabilistic interpretation of science (= the linguistic quantum mechanical worldview), and thus, it is just the language to describe science. Thus, we assert that science is built on dualism ($\approx$ ["observer"+"matter"]$\approx$ measurement) and idealism ($\approx$ metaphysics $\approx$ language).

which is the most important assertion of quantum language. Also, we assume that to make the language to describe science is one of main purposes of philosophy of science.

As criticism of philosophy of science, there is criticism that scientific philosophy is not very useful for scientists (as mentioned in Section 1). We agree to this criticism. However, as mentioned in the above (B$_1$) and (B$_3$), we say that quantum language is one of the most useful theories in science, and thus it should be regarded as a kind of unified theory of science.

Remark 1. Since space and time are independent in quantum language, the theory of relativity (and further, the theory of everything: ⑤ in Fig. 1) cannot be described by quantum language. We think that the theory of relativity is too special, an exception. It is too optimistic to expect that all scientific propositions can be written in quantum language. However, we want to assert the (B$_4$), that is, quantum language is the most fundamental language for almost all familiar science. We believe that arguments without a worldview do not bring about the success of philosophy of science.

2.2 No scientific argument without scientific worldview

It is well known that the following problems are the most fundamental in philosophy of science:

(C) Hempel's raven paradox, Goodman's grue paradox, Hume's problem of induction, Peirce's abduction, the flagpole problem etc.

In Sections 3~6, we clarify these problems under the linguistic quantum mechanical worldview (B$_4$), since we believe that there is no scientific argument without scientific worldview (or, without scientific language). That is, the above are not problems in mathematics and logic. And we conclude that the reason that these problems are not yet clarified depends on lack of the concept of measurement in philosophy of science.

2.3 Mathematical Preparations

Following refs. [6, 7, 8, 18] (all our results until present are written in ref. [18]), we shall review quantum

language, which has the following form:

$$\underset{(=\text{ measurement theory})}{\boxed{\boxed{\text{Quantum language}}}} = \underset{(\text{Axiom 1})}{\boxed{\text{measurement}}} + \underset{(\text{Axiom 2})}{\boxed{\text{causality}}} + \underset{(\text{how to use Axioms 1 and 2})}{\boxed{\text{linguistic (Copenhagen) interpretation}}} \quad (1)$$

which asserts that "measurement" and "causality" are the most important concepts in science.

Consider an operator algebra $B(H)$ (i.e., an operator algebra composed of all bounded linear operators on a Hilbert space H with the norm $\|F\|_{B(H)} = \sup_{\|u\|_H = 1} \|Fu\|_H$), and consider the triplet $[\mathcal{A} \subseteq \mathcal{N} \subseteq B(H)]$ (or, the pair $[\mathcal{A}, \mathcal{N}]_{B(H)}$), called a *basic structure*. Here, $\mathcal{A}(\subseteq B(H))$ is a C^*-algebra, and $\mathcal{N}$ ($\mathcal{A} \subseteq \mathcal{N} \subseteq B(H)$) is a particular C^*-algebra (called a W^*-algebra) such that $\mathcal{N}$ is the weak closure of $\mathcal{A}$ in $B(H)$.

The measurement theory (= "quantum language" = "the linguistic Copenhagen interpretation") is classified as follows.

$$(D) \quad \underset{(=\text{ quantum language})}{\text{measurement theory}} = \begin{cases} (D_1)\text{: quantum system theory} \quad (\text{when } \mathcal{A} = \mathcal{C}(H)) \\ \\ (D_2)\text{: classical system theory} \quad (\text{when } \mathcal{A} = C_0(\Omega)) \end{cases}$$

The Hilbert space method for the mathematical foundations of quantum mechanics is essentially due to von Neumann (*cf.* ref. [21]). He devoted himself to quantum (D_1). On the other hand, in most cases, we devote ourselves to classical (D_2), and not (D_1). However, the quantum (D_1) is convenient for us, in the sense that the idea in (D_1) is often introduced into classical (D_2).

When $\mathcal{A} = \mathcal{C}(H)$, the C^*-algebra composed of all compact operators on a Hilbert space H, the (D_1) is called quantum measurement theory (or, quantum system theory), which can be regarded as the linguistic aspect of quantum mechanics. Also, when $\mathcal{A}$ is commutative (that is, when $\mathcal{A}$ is characterized by $C_0(\Omega)$, the C^*-algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space Ω (*cf.* refs. [22, 23, 24])), the (D_2) is called classical measurement theory.

Also, note that, when $\mathcal{A} = \mathcal{C}(H)$, i.e., quantum cases,

(E_1) $\mathcal{A}^*(= \mathcal{C}(H)^*) = Tr(H)$ (=trace class), $\mathcal{N} = B(H)$, $\mathcal{N}_* = Tr(H)$ (i.e., pre-dual space), thus, $_{Tr(H)}\big(\rho, T\big)_{B(H)} = \mathrm{Tr}_H(\rho T)$ $(\rho \in Tr(H), T \in B(H))$.

Also, when $\mathcal{A} = C_0(\Omega)$, i.e., classical cases,

(E_2) $\mathcal{A}^*(= C_0(\Omega)^*) = \mathcal{M}(\Omega)$ i.e., "the space of all signed measures on Ω", $\mathcal{N} = L^\infty(\Omega, \nu)(\subseteq B(L^2(\Omega, \nu)))$, $\mathcal{N}_* = L^1(\Omega, \nu)$, where ν is some measure on Ω (with the Borel field $\mathcal{B}_\Omega$, thus, $_{L^1(\Omega,\nu)}\big(\rho, T\big)_{L^\infty(\Omega,\nu)} = \int_\Omega \rho(\omega)T(\omega)\nu(d\omega)$ $(\rho \in L^1(\Omega, \nu), T \in L^\infty(\Omega, \nu))$ (*cf.* ref. [23]).

In Sections 3~6 later, we devote ourselves to a compact space Ω with a probability measure ν (i.e., $\nu(\Omega) = 1$) and thus, $C_0(\Omega)$ is simply denoted by $C(\Omega)$.

Let $\mathcal{A}(\subseteq B(H))$ be a C^*-algebra, and let $\mathcal{A}^*$ be the dual Banach space of $\mathcal{A}$. That is, $\mathcal{A}^* = \{\rho \mid \rho$ is a continuous linear functional on $\mathcal{A}$ }, and the norm $\|\rho\|_{\mathcal{A}^*}$ is defined by $\sup\{|\rho(F)| \mid F \in \mathcal{A}$ such that $\|F\|_{\mathcal{A}}(= \|F\|_{B(H)}) \leq 1\}$. Define the *mixed state* ρ $(\in \mathcal{A}^*)$ such that $\|\rho\|_{\mathcal{A}^*} = 1$ and $\rho(F) \geq 0$ for all $F \in \mathcal{A}$ such that $F \geq 0$. And define the mixed state space $\mathfrak{S}^m(\mathcal{A}^*)$ such that

$$\mathfrak{S}^m(\mathcal{A}^*) = \{\rho \in \mathcal{A}^* \mid \rho \text{ is a mixed state}\}.$$

A mixed state $\rho(\in \mathfrak{S}^m(\mathcal{A}^*))$ is called a *pure state* if it satisfies that "$\rho = \theta\rho_1 + (1-\theta)\rho_2$ for some $\rho_1, \rho_2 \in \mathfrak{S}^m(\mathcal{A}^*)$ and $0 < \theta < 1$" implies "$\rho = \rho_1 = \rho_2$". Put

$$\mathfrak{S}^p(\mathcal{A}^*) = \{\rho \in \mathfrak{S}^m(\mathcal{A}^*) \mid \rho \text{ is a pure state}\},$$

which is called a *state space*. It is well known (*cf.* ref. [23]) that $\mathfrak{S}^p(\mathcal{C}(H)^*) = \{|u\rangle\langle u|$ (i.e., the Dirac notation) $\mid \|u\|_H = 1\}$, and $\mathfrak{S}^p(C_0(\Omega)^*) = \{\delta_{\omega_0} \mid \delta_{\omega_0}$ is a point measure at $\omega_0 \in \Omega\}$, where $\int_\Omega f(\omega)\delta_{\omega_0}(d\omega) = f(\omega_0)$ $(\forall f \in C_0(\Omega))$. The latter implies that $\mathfrak{S}^p(C_0(\Omega)^*)$ can be also identified with Ω (called a *spectrum*

space or simply *spectrum*) such as

$$\underset{\text{(state space)}}{\mathfrak{S}^p(C_0(\Omega)^*)} \ni \delta_\omega \leftrightarrow \omega \in \underset{\text{(spectrum)}}{\Omega}$$

In this paper, Ω and $\omega(\in \Omega)$ is respectively called a state space and a state.

In Axiom 1 later, we need the value of $_{\mathcal{A}^*}(\rho, G)_{\mathcal{N}}$, where $\rho \in \mathfrak{S}^p(\mathcal{A}^*)$, $G \in \mathcal{N}$. In quantum cases, we see that $_{\mathcal{A}^*}(\rho, G)_{\mathcal{N}} = {}_{Tr(H)}(\rho, G)_{B(H)}$. Thus, the value of $_{\mathcal{A}^*}(\rho, G)_{\mathcal{N}}$ is clearly determined. However, in classical cases (i.e., $_{\mathcal{A}^*}(\rho, G)_{\mathcal{N}} = {}_{\mathcal{M}(\Omega)}(\rho, G)_{L^\infty(\Omega, \nu)}$), we have to prepare the following definition.

(E_1) [Essentially continuous in general cases]   An element $F(\in \mathcal{N})$ is said to be *essentially continuous at* $\rho_0(\in \mathfrak{S}^p(\mathcal{A}^*))$, if there uniquely exists a complex number α such that
- if ρ ($\in \mathcal{N}_*$, $\|\rho\|_{\mathcal{N}_*} = 1$, $\rho \geq 0$) converges to $\rho_0(\in \mathfrak{S}^p(\mathcal{A}^*))$ in the sense of weak* topology of $\mathcal{A}^*$, that is,

$$\rho(G) \longrightarrow \rho_0(G) \ (\forall G \in \mathcal{A}(\subseteq \mathcal{N})),$$

  then $\rho(F)$ converges to α.

(E_2) [Essentially continuous in quantum cases (i.e., $\mathcal{C}(H) \subseteq B(H) = B(H)$)]   An element $F(\in B(H))$ is said to be *essentially continuous at* $|e\rangle\langle e|$ ($e \in H$, $\|e\|_H = 1$), if there uniquely exists a complex number α such that
- if ρ ($\in Tr(H)$, $\|\rho\|_{Tr(H)} = 1$, $\rho \geq 0$) converges to $|e\rangle\langle e|$ in the sense of weak* topology of $Tr(H)$, that is,

$${}_{Tr(H)}(\rho, G)_{\mathcal{C}(H)} \longrightarrow {}_{Tr(H)}\big(|e\rangle\langle e|, G\big)_{\mathcal{C}(H)}(= \langle e, Ge\rangle_H) \ (\forall G \in \mathcal{C}(H)(\subseteq B(H))),$$

  then $_{Tr(H)}(\rho, F)_{B(H)}$ converges to α_{ω_0}.

(E_3) [Essentially continuous in classical cases]   An element $F(\in L^\infty(\Omega, \nu))$ is said to be *essentially continuous at* $\delta_{\omega_0}(\in \mathfrak{S}^p(\mathcal{M}(\Omega)))$ (i.e., at $\omega_0(\in \Omega)$), if there uniquely exists a complex number α_{ω_0} such that
- if ρ ($\in L^1(\Omega, \nu)$, $\|\rho\|_{L^1(\Omega, \nu)} = 1$, $\rho \geq 0$) converges to $\delta_{\omega_0}(\in \mathfrak{S}^p(\mathcal{M}(\Omega)))$ in the sense of weak* topology of $\mathcal{M}(\Omega)$, that is,

$$\rho(G)(= \int_\Omega G(\omega)\rho(\omega)\nu(d\omega)) \longrightarrow \int_\Omega G(\omega)\delta_{\omega_0}(d\omega)(= G(\omega_0)) \quad (\forall G \in C_0(\Omega)(\subseteq L^\infty(\Omega, \nu))),$$

(2)

  then $\rho(F)(= \int_\Omega F(\omega)\rho(\omega)\nu(d\omega))$ converges to α_{ω_0}.
  Define $\widehat{F}$ ($\in L^\infty(\Omega)$) such that

$$\widehat{F}(\omega) = \begin{cases} \alpha_\omega & (\text{ if } F \text{ is essentially continuos at } \omega) \\ \\ F(\omega) & (\text{otherwise}) \end{cases}$$

Let $D \in \mathcal{B}_\Omega$. Define $\chi_D \in L^\infty(\Omega, \nu)$ such that

$$\chi_D(\omega) = \begin{cases} 1 & (\text{ if } \omega \in D) \\ \\ 0 & (\text{otherwise}) \end{cases}$$

Then define $\widehat{D}$ such that $\{\omega \in \Omega \mid \widehat{\chi}_D(\omega) = 1\}$

Assumption 2. (i): *When we consider* $F(\in L^\infty(\Omega, \nu))$, *we always assume that* F *satisfies that* $F = \widehat{F}$.
(ii): *When we consider* $D(\in \mathcal{B}_\Omega)$, *we always assume that* D *satisfies that* $D = \widehat{D}$.

The following definition is due to E.B. Davies (*cf.* ref. [25]).

Definition 3. [Observable] An *observable* $\mathsf{O} = (X, \mathcal{F}, F)$ in $\mathcal{N}$ is defined as follows:

(i) [σ-field] X is a set, $\mathcal{F}(\subseteq 2^X \equiv \mathcal{P}(X)$, the power set of X) is a σ-field of X, that is, "$\Xi_1, \Xi_2, ... \in \mathcal{F} \Rightarrow \cup_{n=1}^{\infty} \Xi_n \in \mathcal{F}$", "$\Xi \in \mathcal{F} \Rightarrow X \setminus \Xi (\equiv \{x \mid x \in X, x \notin \Xi\} \equiv \Xi^c$, i.e., the complement of $\Xi) \in \mathcal{F}$".

(ii) [Countable additivity] F is a mapping from $\mathcal{F}$ to $\mathcal{N}$ satisfying: (a): for every $\Xi \in \mathcal{F}$, $F(\Xi)$ is a non-negative element in $\mathcal{N}$ such that $0 \leq F(\Xi) \leq I$, (b): $F(\emptyset) = 0$ and $F(X) = I$, where 0 and I is the 0-element and the identity in $\mathcal{N}$ respectively. (c): for any countable decomposition $\{\Xi_1, \Xi_2, \ldots, \Xi_n, ...\}$ of Ξ (i.e., $\Xi, \Xi_n \in \mathcal{F}$ ($n = 1, 2, 3, ...$), $\cup_{n=1}^{\infty} \Xi_n = \Xi$, $\Xi_i \cap \Xi_j = \emptyset$ ($i \neq j$)), it holds that $F(\Xi) = \sum_{n=1}^{\infty} F(\Xi_n)$ in the sense of weak* topology in $\mathcal{N}$.

In classical case, an observable $(X, \mathcal{F}, F)$ in $L^{\infty}(\Omega, \nu)$ is said to be *continuous observable*, if $F(\Xi)$ is essentially continuous on Ω ($\forall \Xi \in \mathcal{F}$).

2.4 Axiom 1 [Measurement] and Axiom 2 [Causality]

Quantum language (1) is composed of two axioms (i.e., Axioms 1 and 2) as follows. With any *system S*, a basic structure $[\mathcal{A} \subseteq \mathcal{N} \subseteq B(H)]$ can be associated in which the measurement theory (A) of that system can be formulated. A *state* of the system S is represented by an element $\rho(\in \mathfrak{S}^p(\mathcal{A}^*))$ and an *observable* is represented by an observable $\mathsf{O} = (X, \mathcal{F}, F)$ in $\mathcal{N}$. Also, the *measurement of the observable O for the system S with the state ρ* is denoted by $\mathsf{M}_{\mathcal{N}}(\mathsf{O}, S_{[\rho]})$ (or more precisely, $\mathsf{M}_{\mathcal{N}}(\mathsf{O} := (X, \mathcal{F}, F), S_{[\rho]})$). An observer can obtain a measured value x ($\in X$) by the measurement $\mathsf{M}_{\mathcal{N}}(\mathsf{O}, S_{[\rho]})$.

The Axiom 1 presented below is a kind of mathematical generalization of Born's probabilistic interpretation of quantum mechanics. And thus, it is a statement without reality.

Now we can present Axiom 1 in the W^*-algebraic formulation as follows.

Axiom 1 [Measurement, the probabilistic interpretation of science]. *The probability that a measured value x ($\in X$) obtained by the measurement $\mathsf{M}_{\mathcal{N}}(\mathsf{O} := (X, \mathcal{F}, F), S_{[\rho]})$ belongs to a set $\Xi (\in \mathcal{F})$ is given by $\rho(F(\Xi))$ if $F(\Xi)$ is essentially continuous at $\rho(\in \mathfrak{S}^p(\mathcal{A}^*))$.*

This axiom gives answers to "What is probability?" and "What is measurement?".

Remark 4. (i): In quantum cases (i.e, the cases that $\rho \in \mathfrak{S}^p(Tr(H)) \subseteq Tr(H)$, $F(\Xi) \in B(H)$), the probability $\rho(F(\Xi))(= {}_{Tr(H)}(\rho, T)_{B(H)} = \text{Tr}_H(\rho T))$ is always defined (*cf.* (E$_1$)). That is, the $F(\Xi)$ is always essentially continuous at any $\rho \in \mathfrak{S}^p(Tr(H))$. On the other hand, in the classical cases (i.e., the cases that $\omega \in \Omega$, $F(\Xi) \in L^{\infty}(\Omega, \nu)$), it is not guaranteed that $F(\Xi)$ is essentially continuous at ω ($\in \Omega$)). Thus, put $\mathcal{F}_{\omega_0} = \{\Xi \in \mathcal{F} : F(\Xi)$ is essentially continuous at $\omega_0\}$. If $\mathcal{F}_{\omega_0} = \mathcal{F}$, the measurement $\mathsf{M}_{L^{\infty}(\Omega, \nu)}(\mathsf{O} := (X, \mathcal{F}, F), S_{[\omega_0]})$ makes the sample probability space $(X, \mathcal{F}, [F(\cdot)](\omega_0))$, which is usual in statistics. Thus, roughly speaking, statistics starts from "sample probability space", on the other hand, quantum language starts from "measurement".

(ii): In quantum cases, Axiom 1 is well known as Born's probabilistic interpretation of quantum mechanics. In classical case, it is natural to ask "Why does probability appear?". For the answer of this question, see Chapter 1 (Section 1.3) page 12 in ref. [18]: *Example: measurement of "Cold or Hot"*.

(iii): Axiom 1 is the quantitative realization of the spirit: "there is no science without measurements". And, we think that Axiom 1 means the probabilistic interpretation of science since it is a kind of mathematical generalization of Born's probability interpretation of quantum mechanics.

Example 5. [Exact measurement, exact observable] Consider a basic structure $[C(\Omega) \subseteq L^{\infty}(\Omega, \nu) \subseteq$

$B(L^2(\Omega, \mathcal{B}_\Omega, \nu))$]. Define the exact observable $\mathsf{O}_e = (X(=\Omega), \mathcal{F}(=\mathcal{B}_\Omega), F_e)$ in $L^\infty(\Omega, \nu)$ such that

$$[F_e(\Xi)](\omega) = 1 \;\; (\omega \in \Xi \in \mathcal{F}), \quad = 0 \;\; (\omega \notin \Xi \in \mathcal{F}).$$

Let $\omega^0 \in \Omega$. Thus, we have the exact measurement $\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}_e = (X, \mathcal{F}, F_e), S_{[\omega^0]})$. Then we have the following statement

- Let $D(\subseteq X = \Omega)$ be any open set such that $\omega^0 \in D$. Then Axiom 1 says that the probability that the measured value $x(\in X)$ obtained by the measurement $\mathsf{M}_{L^\infty(\Omega,\nu)}\;(\mathsf{O}_e = (X, \mathcal{F}, F_e), S_{[\omega^0]})$ belongs to D is given by 1.

This implies that $x = \omega^0$, since D is arbitrary open set such that $\omega^0 \in D$. Also, it should be noted that $F_e(\Xi)$ is not essentially continuous at ω^0 if $\omega^0 \in \partial(\Xi)$, i.e., the boundary of Ξ. Thus, $(X, \mathcal{F}, [F_e(\cdot)](\omega^0))$ is not always a probability space. However, note that there exists a probability space $(X, \mathcal{F}, \mu)$ such that $[F_e(\Xi)](\omega^0) = \mu(\Xi)$ ($\forall \Xi (\in \mathcal{F})$ such that $\omega^0 \notin \partial(\Xi)$), though the uniqueness is not guaranteed. The probability space $(X, \mathcal{F}, \mu)$ is often called the sample space of the measurement $\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}_e = (X, \mathcal{F}, F_e), S_{[\omega^0]})$

Next, we explain Axiom 2. Let $[\mathcal{A}_1, \mathcal{N}_1]_{B(H_1)}$ and $[\mathcal{A}_2, \mathcal{N}_2]_{B(H_2)}$ be basic structures. A continuous linear operator $\Phi_{1,2} : \mathcal{N}_2$ (with weak* topology) $\to \mathcal{N}_1$(with weak* topology) is called a *Markov operator*, if it satisfies that (i): $\Phi_{1,2}(F_2) \geq 0$ for any non-negative element F_2 in $\mathcal{N}_2$, (ii): $\Phi_{1,2}(I_2) = I_1$, where I_k is the identity in $\mathcal{N}_k$, ($k = 1, 2$). In addition to the above (i) and (ii), we assume that $\Phi_{1,2}(\mathcal{A}_2) \subseteq \mathcal{A}_1$ and $\sup\{\|\Phi_{1,2}(F_2)\|_{\mathcal{A}_1} \mid F_2 \in \mathcal{A}_2 \text{ such that } \|F_2\|_{\mathcal{A}_2} \leq 1\} = 1$.

It is clear that the dual operator $\Phi_{1,2}^* : \mathcal{A}_1^* \to \mathcal{A}_2^*$ satisfies that $\Phi_{1,2}^*(\mathfrak{S}^m(\mathcal{A}_1^*)) \subseteq \mathfrak{S}^m(\mathcal{A}_2^*)$. If it holds that $\Phi_{1,2}^*(\mathfrak{S}^p(\mathcal{A}_1^*)) \subseteq \mathfrak{S}^p(\mathcal{A}_2^*)$, the $\Phi_{1,2}$ is said to be deterministic. If it is not deterministic, it is said to be non-deterministic. Also note that, for any observable $\mathsf{O}_2 := (X, \mathcal{F}, F_2)$ in $\mathcal{N}_2$, the $(X, \mathcal{F}, \Phi_{1,2}F_2)$ is an observable in $\mathcal{N}_1$.

Definition 6. [Sequential causal operator; Heisenberg picture of causality]   Let $(T, \leq)$ be a tree like semi-ordered set such that "$t_1 \leq t_3$ and $t_2 \leq t_3$" implies "$t_1 \leq t_2$ or $t_2 \leq t_1$". The family $\{\Phi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}\}_{(t_1,t_2) \in T_\leqq^2}$ is called a **sequential causal operator**, if it satisfies that

(i) For each t ($\in T$), a basic structure $[\mathcal{A}_t \subseteq \mathcal{N}_t \subseteq B(H_t)]$ is determined.
(ii) For each $(t_1, t_2) \in T_\leqq^2$, a causal operator $\Phi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}$ is defined such as $\Phi_{t_1,t_2}\Phi_{t_2,t_3} = \Phi_{t_1,t_3}$ ($\forall(t_1, t_2), \forall(t_2, t_3) \in T_\leqq^2$). Here, $\Phi_{t,t} : \mathcal{N}_t \to \mathcal{N}_t$ is the identity operator.

Now we can propose Axiom 2 (i.e., causality). (For details, see ref. [18].)

Axiom 2[Causality]; For each $t(\in T =$"tree like semi-ordered set")), consider the basic structure:

$$[\mathcal{A}_t \subseteq \mathcal{N}_t \subseteq B(H_t)]$$

Then, the **chain of causalities** is represented by a **sequential causal operator** $\{\Phi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}\}_{(t_1,t_2) \in T_\leqq^2}$.

When parameters t_1, t_2 ($t_1 < t_2$) are regarded as time, we usually consider that a causal operator $\Phi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}$ represents "causality". Thus, this axiom gives an answer to "What is causality?". That is, we consider that, if Axiom 2 is used in the quantum linguistic representation of a phenomenon, causality exists in the phenomenon.

2.5   The linguistic Copenhagen interpretation (= the manual to use Axioms 1 and 2)

It is well known (*cf.* ref. [26]) that the Copenhagen interpretation of quantum mechanics has not been established yet. For example, about the right or wrong of the wave function collapse, opinions are divided

in the Copenhagen interpretation. Thus, the Copenhagen interpretation is often called "so-call Copenhagen interpretation". However, we believe that the linguistic Copenhagen interpretation of quantum language (B) (i.e., both quantum (B_1') and classical (B_2')) is uniquely determined. For example, for the quantum linguistic opinion about the wave function collapse, see ref. [13] or §11.2 in ref. [18]. As mentioned in (B_1), we believe that the linguistic Copenhagen interpretation is the true figure of so called Copenhagen interpretation.

Now we explain the linguistic Copenhagen interpretation in what follows. In the above, Axioms 1 and 2 are kinds of spells, (i.e., incantation, magic words, metaphysical statements), and thus, it is nonsense to verify them experimentally. Therefore, what we should do is not "to understand" but "to use". After learning Axioms 1 and 2 by rote, we have to improve how to use them through trial and error. We may do well even if we do not know the linguistic Copenhagen interpretation (= the manual to use Axioms 1 and 2). However, it is better to know the linguistic Copenhagen interpretation, if we would like to make progress quantum language early. We believe that the linguistic Copenhagen interpretation is the true Copenhagen interpretation, which does not belong to physics.
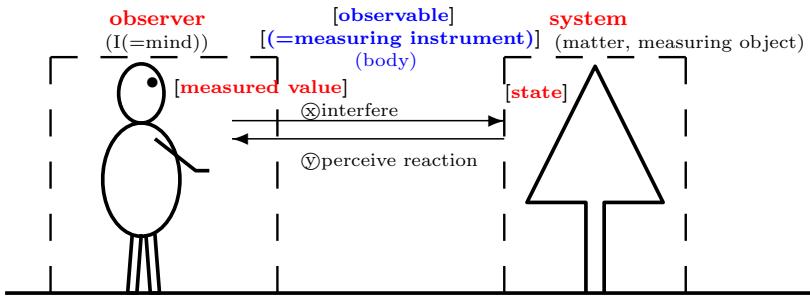

Figure 2: [Descartes Figure]: Image of "measurement($=$ⓧ$+$ⓨ)" in mind-matter dualism

The essence of the manual is as follows: In Fig. 2, we remark:

(F_1) ⓧ: it suffices to understand that "interfere" is, for example, "apply light".
      ⓨ: perceive the reaction.

That is, "measurement" is characterized as the interaction between "observer" and "measuring object (= matter)". However,

(F_2) in measurement theory (=quantum language), "interaction" must not be emphasized.

Therefore, in order to avoid confusion, it might better to omit the interaction "ⓧ and ⓨ" in Fig. 2.
  After all, we think that:

(F_3) it is clear that there is no measured value without observer (i.e., "I", "mind"). Thus, we consider that measurement theory is composed of three key-words: "measured value","observable","state" (*cf.* § 3.1(p.63) in [18]).

$$\boxed{\text{measured value}}_{\text{(I, observer, mind)}}, \quad \boxed{\text{observable (= measuring instrument)}}_{\text{(body(= sensory organ), thermometer, eye, ear, compass)}} \quad , \quad \boxed{\text{state}}_{\text{(matter)}},$$

Hence, quantum language is based on dualism, i.e., a kind of mind-matter dualism.

The linguistic Copenhagen interpretation says that

(G_1) **Only one measurement is permitted**. And therefore, the state after a measurement is meaningless since it cannot be measured any longer. Thus, the collapse of the wavefunction is prohibited (*cf.* ref. [13]; projection postulate). We are not concerned with anything after measurement. Strictly speaking, the phrase "*after the measurement*" should not be used. Also, the causality should be assumed only in the side of system, however, a state never moves. Thus, the Heisenberg picture should be adopted, and thus, the Schrödinger picture should be prohibited.

(G$_2$) "Observer"(="I") and "system" are completely separated in order not to make self-reference propositions appear. Hence, the measurement $\mathsf{M}_{\mathcal{N}}(\mathsf{O}:=(X,\mathcal{F},F),\ S_{[\rho]})$ does not depend on the choice of observers. That is, any proposition (except Axiom 1) in quantum language is not related to "observer"(="I"), therefore, there is no "observer's space and time" in quantum language. And thus, it does not have tense (i.e., past, present, future).

(G$_3$) there is no probability without measurements (*cf.* Bertrand's paradox (in § 9.12 of ref.[18]))

(G$_4$) Leibniz's relationalism concerning space-time (e.g, time should be regarded as a parameter), (*cf.* ref. [17]).

(G$_5$) A family of measurements $\{\ \mathsf{M}_{\mathcal{N}_i}(\mathsf{O}_i:=(X_1,\mathcal{F}_i,F_i),\ S_{[\rho_i]}):\quad i=1,2,3,...\}$ is realized as the paralell measurement $\mathsf{M}_{\bigotimes_{i=1}^{\infty}\mathcal{N}_i}(\ \bigotimes_{i=1}^{\infty}\mathsf{O}_i:=(\times_{i=1}^{\infty}X_1,\boxtimes_{i=1}^{\infty}\mathcal{F}_i,\ \bigotimes_{i=1}^{\infty}F_i),\ S_{[\ \bigotimes_{i=1}^{\infty}\rho_i]})$ (*cf.* Definition 8 later). For details about the tensor product "$\bigotimes_{i=1}^{\infty}$", see ref. [18].

and so on.

Remark 7. (i): In ref. [3] (1991), we proposed the mathematical formulation of Heisenberg's uncertainty relation. However, so-called Copenhagen interpretation is not firm (*cf.* ref. [26]). Thus, in order to understand our work (i.e., Heisenberg's uncertainty relation) deeply, we proposed the linguistic Copenhagen interpretation (= quantum language) as the true Copenhagen interpretation. For details of Heisenberg's uncertain relation, see §4.3 in ref. [18].

(ii): We consider that the above (G$_1$) is closely related to Kolmogorov's extension theorem (*cf.* ref. [27]), which says that only one probability space is permitted. For details, see §4.1 in ref. [18].

(iii): The formula (1) says that scientific explanation is to explain phenomena in terms of "measurement"(Axiom 1) and "causality" (Axiom 2). If we are allowed to use the famous metaphor of Kant's Copernican revolution, to do familiar sciences is to see this world through colored glasses of measurement and causality (*cf.* [17]), or to use the metaphor of Wittgenstein's saying, the limits of quantum language are the limits of familiar science. Therefore, the explanation problem of scientific philosophy is automatically clarified in quantum language.

(iv): Violating the linguistic Copenhagen interpretation (G$_2$), we have many paradoxes of self-reference type such as "brain in a vat", "five-minute hypothesis", "I think, therefore I am", "McTaggart's paradox". Cf. ref. [17] or §10.8 in ref. [18].

(v): We want to understand that Zeno's paradox is not a problem concerning geometric series or spatial division, but the problem concerning the worldview. That is, "Propose a certain scientific worldview, in which Zeno's paradox should be studied !" That is because we think that there is no scientific argument without scientific language ($\approx$ scientific worldview). And our answer (*cf.* § 14.4 in ref. [18]) is "If Zeno's paradox is a problem in science, it should be studied in quantum language". That is because our assertion is "Quantum language is the language of science". Also, Monty Hall problem, two envelope problem, three prisoners problem etc. are not only mathematical puzzles but also profound problems in quantum language (*cf.* refs. [11, 18]).

As the further explanation of parallel measurement in the linguistic Copenhagen interpretation (G$_5$), we have to add the following definition.

Definition 8. [Parallel measurement (*cf.* [18])] Though the parallel measurement can be defined in both classical and quantum systems, we, for simplicity, devote ourselves to classical systems as follows. Let $[C(\Omega)\subseteq L^\infty(\Omega.\nu)\subseteq[B(L^2(\Omega,\nu))]$ be a classical basic structure, where we assume, for simplicity, that Ω is compact space and ν is a measure such that $\nu(\Omega)=1$ and $\nu(D)>0$ ($\forall$open set $D\subseteq\Omega$). Consider a family of measurements $\{\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}_i:=(X_i,\mathcal{F}_i,F_i),S_{[\omega_i]})\mid i=1,2,...,N\}$. However, the linguistic Copenhagen interpretation (G$_1$) says "Only one measurement is permitted". Therefore, instead of the family of measurements, we consider the parallel measurement $\bigotimes_{i=1}^{N}\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}_i:=(X_i,\mathcal{F}_i,F_i),S_{[\omega_i]})$, which is

9

defined by

$$\bigotimes_{i=1}^{N} \mathsf{M}_{L^{\infty}(\Omega,\nu)}(\mathsf{O}_i := (X_i, \mathcal{F}_i, F_i), S_{[\omega_i]})$$

$$= \mathsf{M}_{L^{\infty}(\times_{i=1}^{N}\Omega, \bigotimes_{i=1}^{N}\nu)}\left(\otimes_{i=1}^{N}\mathsf{O}_i := (\underset{i=1}{\overset{N}{\times}} X_i, \boxtimes_{i=1}^{N}\mathcal{F}_i, \otimes_{i=1}^{N}F_i), S_{[(\omega_1,\omega_2,...,\omega_N)]}\right)$$

where $\times_{i=1}^{N}\Omega$ is the finite product compact space of Ωs, $\bigotimes_{i=1}^{N}\nu$ is the finite product probability of νs. Also, $\boxtimes_{i=1}^{N}\mathcal{F}_i$ ($\subseteq \mathcal{P}(\times_{i=1}^{N} X_i)$) is the infinite product σ-field, i.e., the smallest σ-field that includes

$$\left\{ \underset{i=1}{\overset{N}{\times}} \Xi_i \mid \quad \Xi_i \in \mathcal{F}_i \right\}$$

And further, define the observable $\bigotimes_{i=1}^{N} F_i$ in $L^{\infty}(\times_{i=1}^{N}\Omega, \bigotimes_{i=1}^{N}\nu)$ which satisfies that

$$[(\otimes_{i=1}^{N}F_i)((\underset{i=1}{\overset{N}{\times}}\Xi_i)](\omega_1,\omega_2,...,\omega_N) = \underset{i=1}{\overset{N}{\times}}[F_i(\Xi_i)](\omega_i) \quad \forall\Xi_i \in \mathcal{F}_i, \omega_i \in \Omega$$

Then, Axiom 1 [measurement] says that

(H) the probability that a measured value obtained by the parallel measurement $\bigotimes_{i=1}^{N} \mathsf{M}_{L^{\infty}(\Omega,\nu)}(\mathsf{O}_i := (X_i, \mathcal{F}_i, F_i), S_{[\omega_i]})$ belongs to $\times_{i=1}^{N}\Xi_i$ is given by $\times_{i=1}^{N}[F_i(\Xi_i)](\omega_i)$, if $F_i(\Xi_i)$ is essentially continuous at ω_i ($\forall i = 1, 2, ..., N$).

Remark 9. The above finite parallel measurement can be generalized to the case that the index set Λ is infinite. That is,

$$\bigotimes_{\lambda \in \Lambda} \mathsf{M}_{L^{\infty}(\Omega,\nu)}(\mathsf{O}_\lambda := (X_\lambda, \mathcal{F}_\lambda, F_\lambda), S_{[\omega_\lambda]})$$

$$= \mathsf{M}_{L^{\infty}(\times_{\lambda\in\Lambda}\Omega, \bigotimes_{\lambda\in\Lambda}\nu)}\left(\otimes_{\lambda\in\Lambda}\mathsf{O}_\lambda := (\underset{\lambda\in\Lambda}{\times} X_\lambda, \boxtimes_{\lambda\in\Lambda}\mathcal{F}_\lambda, \otimes_{\lambda\in\Lambda}F_\lambda), S_{[(\omega_\lambda)_{\lambda\in\Lambda}]}\right)$$

The existence of the parallel measurement is guaranteed in both classical and quantum systems. Cf. § 4.2 in ref. [18]. It is not so difficult to extend the above finite parallel measurements to infinite parallel measurements for mathematicians. However, in this paper, we are not concerned with the infinite parallel measurement. That is because our concern is not mathematics but foundations of philosophy of science.

Here we add the following definition.

Definition 10. [Implication: *cf.* refs. [4, 18]] Consider a basic structure:

$$[\mathcal{A} \subseteq \mathcal{N} \subseteq B(H)]$$

Let $\mathsf{O}_1 = (X_1, \mathcal{F}_1, F_1)$ and $\mathsf{O}_2 = (X_2, \mathcal{F}_2, F_2)$ be observables in $\mathcal{N}$. Let $\mathsf{O}_{12} = (X_1 \times X_2, \mathcal{F}_1 \boxtimes \mathcal{F}_2, F_{12})$ be an observable such that $F_1(\Xi_1) = F_{12}(\Xi_1 \times X_2)$ and $F_2(\Xi_2) = F_{12}(X_1 \times \Xi_2)$ ($\forall\Xi_1 \in \mathcal{F}_1, \Xi_2 \in \mathcal{F}_2$). Let $\rho \in \mathfrak{S}^p(\mathcal{A}^*)$, $\Gamma_1 \in \mathcal{F}_1$, $\Gamma_2 \in \mathcal{F}_2$. Then, if it holds that

$$\rho(F_{12}(\Gamma_1 \times (X_2 \setminus \Gamma_2))) = 0$$

this is denoted by

$$[\mathsf{O}_1; \Gamma_1] \underset{\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{12}, S_{[\rho]})}{\Longrightarrow} [\mathsf{O}_2; \Gamma_2] \quad \text{or, equivalently,} \quad [\mathsf{O}_1; X_1 \setminus \Gamma_1] \underset{\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{12}, S_{[\rho]})}{\Longleftarrow} [\mathsf{O}_2; X_2 \setminus \Gamma_2]$$

That is because the probability that a measured value $(x_1.x_2)$ obtained by the measurement $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{12} := (X_1 \times X_2, \mathcal{F}_1 \boxtimes \mathcal{F}_2, F_{12}), S_{[\rho]})$ belongs to $\Gamma_1 \times (X_2 \setminus \Gamma_2)$ is equal to 0.

Remark 11. [Syllogism] Using Definition 10, we showed that

- Syllogism always holds in classical systems (*cf.* ref. [4])
- Syllogism does not always hold in quantum systems (*cf.* ref. [9], § 8.7 in ref. [18])

2.6   Inference; Fisher's maximum likelihood method

We begin with the following notation:

Notation 12. $[\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}, S_{[*]})]$:   Consider a measurement $\mathsf{M}_{L^\infty(\Omega,\nu)}$ $(\mathsf{O}{:=}(X,\mathcal{F},F),\ S_{[\omega_0]})$ formulated in the basic structure $[C(\Omega) \subseteq L^\infty(\Omega,\nu) \subseteq B(L^2(\Omega,\nu))]$. Here, note that

(I_1) in most cases that the measurement $\mathsf{M}_{L^\infty(\Omega,\nu)}$ $(\mathsf{O}{:=}(X,\mathcal{F},F),\ S_{[\omega_0]})$ is taken, it is usual to think that the state $\omega_0(\in \Omega)$ is unknown.

That is because

(I_2) the measurement $\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}, S_{[\omega_0]})$ may be taken in order to know the state ω_0.

Therefore, when we want to stress that we do not know the state ω_0, the measurement $\mathsf{M}_{L^\infty(\Omega,\nu)}$ $(\mathsf{O}{:=}(X,\mathcal{F},F),\ S_{[\omega_0]})$ is often denoted by $\mathsf{M}_{L^\infty(\Omega,\nu)}$ $(\mathsf{O}{:=}(X,\mathcal{F},F),\ S_{[*]})$

Theorem 13. [Inference; Fisher's maximum likelihood method (*cf.* ref. [5] or §5.2 in ref. [18]] For simplicity, assume that X is finite set. Assume that the measured value $x(\in X)$ is obtained by the measurement $\mathsf{M}_{L^\infty(\Omega,\nu)}$ $(\mathsf{O}{:=}(X,2^X,F),\ S_{[*]})$. Then, the unknown state $[*]$ can be inferred to be $\omega_0(\in \Omega)$ such that

$$[F(\{x\})](\omega_0) = \max_{\omega\in\Omega}[F(\{x\})](\omega)$$

Proof. It is an easy consequence of Axiom 1 (*cf.* §5.2 in ref. [18]). $\qquad\qquad\square$

Remark 14. [Inference and Control *cf.* §5.2 in ref. [18]]   The inference problem is characterized as the reverse problem of measurements. That is, we consider that

(J_1) (state ω_0, observable O) $\dfrac{\mathsf{M}_{L^\infty(\Omega,\nu)}\ (\mathsf{O}{:=}(X,2^X,F),\ S_{[\omega_0]})}{\text{measurement (Axiom 1)}}$ measured value x_0

On the other hand

(J_2) (measured value x_0, observable O) $\dfrac{\mathsf{M}_{L^\infty(\Omega,\nu)}\ (\mathsf{O}{:=}(X,2^X,F),\ S_{[*]})}{\text{inference (reverse Axiom 1)}}$ state ω_0

Thus, (J_1) and (J_2) are in reverse problem.
Also, we note, from the mathematical point of view, that inference problem (J_3) and control problem (J_4) are essentially the same as follows.

(J_3) [Inference problem; statistics]: when measured value x_0 is obtained, infer the unknown state ω_0!

and

(J_4) [Control problem; dynamical system theory]: Settle the state ω_0 such that measured value x_0 will be obtained!

Thus, we think, from the theoretical point of view, that statistics and dynamical system theory are essentially the same. Thus, we consider that statistics (= dynamical system theory) is the mathematical representation of classical mechanical worldview. On the other hand, quantum language is regarded as the mathematical representation of quantum mechanical worldview.

3 Hempel's raven paradox in the linguistic quantum mechanical worldview

3.1 What is Hempel's raven paradox?

Although all results mentioned in this paper hold in both classical and quantum systems, we, for simplicity, devote ourselves to classical systems.

In this section we discuss Hempel's raven paradox (*cf.* ref. [1]) in the linguistic quantum mechanical worldview. There may be no consensus among philosophers on the problem "What is Hempel's raven paradox?". Some people may even think there is no paradox in Hempel's raven problem. Thus, we mention our opinion about "What is Hempel's raven paradox?" in what follows.

Let U be the universal set of all birds. Let $B(\subseteq U)$ be a set of all black birds. Let $R(\subseteq U)$ be a set of all ravens. Thus, the statement: "any raven is black" is logically denoted by

(K$_1$) "Any raven is black" : $(\forall x\ )[x \in R \longrightarrow x \in B]$ *i.e.*, $R \subseteq B \subseteq U$,

This is logically equivalent to the followings:

(K$_2$) "Every non-black bird is a nonraven" : $(\forall x\ )[x \in U \setminus B \longrightarrow x \in U \setminus R]$ *i.e.*, $U \setminus B \subseteq U \setminus R$
(K$_3$) "Non-black raven does not exist": $(\neg \exists x)\ [x \in R \cap (U \setminus B)]$

However, if the three are equivalent, then we have the following questions (i.e., raven problem):

(K$_4$) Why is the actual verification of (K$_2$) much more difficult than the actual verification of (K$_1$)?
(K$_5$) Why can the truth of "(K$_1$): any raven is black" be known by (K$_2$), i.e., without seeing a raven also at once?
(K$_6$) Is it possible to experimentally verify "Any raven is black"?

These may be so called Hempel's raven problem. However, now we think that the most essential problem concerning Hempel's raven paradox is as follows:

(K$_7$) What is the true meaning of "Any raven is black"?

In order to consider this problem, we must prepare the measurement theoretical formulation of ornithology, under which the meaning of "Any raven is black" will be clarified in the following sections.

Remark 15. (i): In Section 3 of ref. [19]: S. Ishikawa, *Philosophy of science for scientists; The probabilistic interpretation of science*, JQIS, Vol. 3, No.3 , 140-154 , we discussed Hempel's raven problem (K$_4$)-(K$_6$). However, now we consider that the arguments about (K$_4$)-(K$_6$) in ref. [19] were not sufficient since the argument concerning (K$_7$) was not sufficient. Hence, in this paper, we are eager to investigate the problem (K$_7$) sufficiently.
(ii): Also, in this paper we consider that

$$\text{"any raven is black"} = \text{"every raven is black"} = \text{"all ravens are black".}$$

following the rule of mathematics.

3.2 The measurement theoretical formulation of ornithology

We think that Hempel's raven paradox raises the question of "What is 'Any raven is black'?". For this, we must prepare the measurement theoretical formulation of ornithology

Since a state is a quantity expression of a property of an object, we have the following definitions.

Definition 16. Let U be a (finite) set of all birds. Put $B = \{u \in U \mid u$ is a black bird$\}$, and put $R = \{u \in U \mid u$ is a raven$\}$. Further put

$$U = \{u_1, u_2, ..., u_{\sharp[U]}\}, \qquad B = \{b_1, b_2, ..., b_{\sharp[B]}\}, \qquad R = \{r_1, r_2, ..., r_{\sharp[R]}\}.$$

And define $\overline{B}(\supseteq B)$ by $\overline{B} \setminus B = \{u \in U \mid u$ is located between the black birds and the non-black birds$\}$, and define $\overline{R}(\supseteq R)$ by $\overline{R} \setminus R = \{u \in U \mid u$ is located between the ravens and the non-ravens$\}$. Thus, it holds that $B \subseteq \overline{B} \subseteq U$ and $R \subseteq \overline{R} \subseteq U$.

Definition 17. [(i): Universal state space]: Consider the basic structure:

$$[C_0(\Omega) \subseteq L^\infty(\Omega, \nu) \subseteq B(L^2(\Omega, \nu))] \tag{3}$$

where Ω (called a *universal state space* or in short, *state space*) is assumed to be the state space that includes the states of all the considered objects. In this paper it suffices to consider that Ω is the state space in which the states of all birds are expressed. Here Ω is a locally compact space with the Borel field $\mathcal{B}_\Omega$, ν is measure such that $\nu(D) > 0(\forall$ open set $D \subseteq \Omega)$ and $\nu(\{\omega\}) = 0(\forall \omega \in \Omega)$.
[(ii): State map]: Any bird u $(\in U)$ has a state $\widetilde{\omega}(u)(\in \Omega)$. Here the state map $\widetilde{\omega} : U \to \Omega$ is injection (i.e., one to one). Note that $\widetilde{\omega}(U)$ is not necessarily dense in Ω.
[(iii): Raven observable, Black bird observable]: Here we consider two continuous observables (i.e., Black bird observable, Raven observable) $\mathsf{O}_B = (\{1, 0\}, 2^{\{1,0\}}, F_B)$, $\mathsf{O}_R = (\{1, 0\}, 2^{\{1.0\}}, F_R)$ in $L^\infty(\Omega, \nu)$. Also, put

$$\Omega_B := \{\omega \in \Omega \mid [F_B(\{1\})](\omega) = 1\}, \qquad \Omega_R := \{\omega \in \Omega \mid [F_R(\{1\})](\omega) = 1\}$$

which is respectively called a *black bird state class* and a *raven state class* . Also, put

$$\Omega_{\overline{B} \setminus B} := \{\omega \in \Omega \mid 0 < [F_B(\{1\})](\omega) < 1\}, \qquad \Omega_{U \setminus \overline{B}} := \{\omega \in \Omega \mid [F_B(\{1\})](\omega) = 0\},$$
$$\Omega_{\overline{R} \setminus R} := \{\omega \in \Omega \mid 0 < [F_R(\{1\})](\omega) < 1\}, \qquad \Omega_{U \setminus \overline{R}} := \{\omega \in \Omega \mid [F_R(\{1\})](\omega) = 0\},$$

Thus, $\mathsf{O}_B = (\{1, 0\}, 2^{\{1,0\}}, F_B)$ has the following form.

$$[F_B(\{1\})](\omega) = 1 \ (\text{ if } \omega \in \Omega_B), \quad 0 < [F_B(\{1\})](\omega) < 1 \ (\text{ if } \omega \in \Omega_{\overline{B} \setminus B}),$$
$$[F_B(\{1\})](\omega) = 0 \ (\text{ if } \omega \in \Omega_{U \setminus \overline{B}}), \quad [F(\{0\})](\omega) = 1 - [F_B(\{1\})](\omega) \ (\forall \omega \in \Omega) \tag{4}$$

Also, $\mathsf{O}_R = (\{1, 0\}, 2^{\{1,0\}}, F_R)$ is similarly defined (see Fig. 3 below).
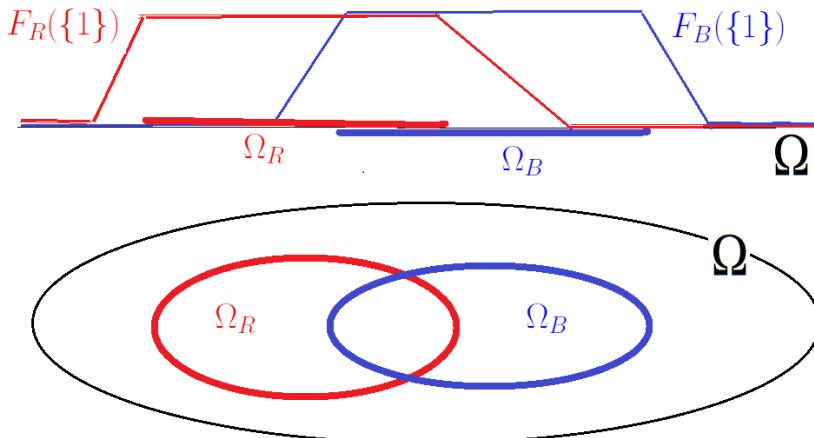
Figure 3: [Raven state class Ω_R, black bird state class Ω_B]

Now we propose the following declaration.

Declaration 18. [The mathematical formulation of ornithology]: *We assume that the state map $\widehat{\omega} : U \to \Omega$ satisfies*

$$\text{``}\widetilde{\omega}(u) \in \Omega_B\text{''} \Leftrightarrow \text{``}u \in B\text{''}, \quad \text{``}\widetilde{\omega}(u) \in \Omega_{\overline{B} \backslash B}\text{''} \Leftrightarrow \text{``}u \in \overline{B} \backslash B\text{''}, \quad \text{``}\widetilde{\omega}(u) \in \Omega_{U \backslash \overline{B}}\text{''} \Leftrightarrow \text{''}u \in U \backslash \overline{B}\text{''},$$

$$\text{``}\widetilde{\omega}(u) \in \Omega_R\text{''} \Leftrightarrow \text{``}u \in R\text{''}, \quad \text{``}\widetilde{\omega}(u) \in \Omega_{\overline{R} \backslash R}\text{''} \Leftrightarrow \text{``}u \in \overline{R} \backslash R\text{''}, \quad \text{``}\widetilde{\omega}(u) \in \Omega_{U \backslash \overline{R}}\text{''} \Leftrightarrow \text{''}u \in U \backslash \overline{R}\text{''} \tag{5}$$

Remark 19. (i): There may be several opinions about Definition 16. That is, "U is the set of all birds" may be as ambiguous as "U_D is the set of all dinosaurs". In order to read this paper, it suffices to consider that the U may be a gathering of birds as far as we know. Also, upper argument may include contradiction. For example, "$\widetilde{\omega}(u) \in \Omega_R$" ⇔ "$u \in R$" in Declaration 18 should be regarded as the definition of "raven". However, the word :"raven" is already used in Definition 16. In order to resolve this contradiction, names such as "the set of all birds" etc. should be given after Declaration 18.

(ii): Everything's nature of birds is expressed in state space Ω. Therefore, it is natural to consider that the above definitions (i.e., O_B, O_R, etc.) were decided by specialists of ornithology. In fact the classification of birds is one of their jobs. Also, the characterization of O_B, O_R, etc. may be changed by the development of ornithology. That is, we consider that progress of ornithology is equal to precise-izing of the declaration. This will be shown in Declarations 22 and 27 later.

Exercise 20. Under the Declaration 18, Axiom 1 says:

(L_1) for each $u_i \in U$, the probability that the measured value 1 [resp. 0] is obtained by the measurement $\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}_R = (\{1,0\}, 2^{\{1,0\}}, F_R), S_{[\widetilde{\omega}(u_i)]})$ is equal to $[F_R(\{1\})](\widetilde{\omega}(u_i))$ [resp. $[F_R(\{0\})](\widetilde{\omega}(u_i))$].

(L_2) for each $u_i \in U$, the probability that the measured value 1 [resp. 0] is obtained by the measurement $\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}_B = (\{1,0\}, 2^{\{1,0\}}, F_B), S_{[\widetilde{\omega}(u_i)]})$ is equal to $[F_B(\{1\})](\widetilde{\omega}(u_i))$ [resp. $[F_B(\{0\})](\widetilde{\omega}(u_i))$].

Let $\mathsf{O}_{RB} = (\{1,0\} \times \{(0,1)\}, 2^{\{\{1,0\} \times \{(0,1)\}\}}, F_R \times F_B)$ be the product observable of $\mathsf{O}_R = (\{1,0\}, 2^{\{1,0\}}, F_R)$ and $\mathsf{O}_B = (\{1,0\}, 2^{\{1,0\}}, F_B)$. Then we see:

(L_3) for each $u_i \in U$, the probability that the measured value $(1,1)$ [resp. $(1,0), (0,1).(0.0)$] is obtained by the measurement $\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}_{RB} = (\{1,0\} \times \{(0,1)\}, 2^{\{\{1,0\} \times \{(0,1)\}\}}, F_R \times F_B), S_{[\widetilde{\omega}(u_i)]})$ is equal to $[F_R(\{1\})](\widetilde{\omega}(u_i)) \cdot [F_B(\{1\})](\widetilde{\omega}(u_i))$ [resp. $[F_R(\{1\})](\widetilde{\omega}(u_i)) \cdot [F_B(\{0\})](\widetilde{\omega}(u_i))$, $[F_R(\{0\})](\widetilde{\omega}(u_i)) \cdot [F_B(\{1\})](\widetilde{\omega}(u_i))$, $[F_R(\{0\})](\widetilde{\omega}(u_i)) \cdot [F_B(\{0\})](\widetilde{\omega}(u_i))$].

The exercise of the representation using parallel measurements is left to readers.

3.3 A priori proposition: "Any small black bird is black"

Next consider the following figure:
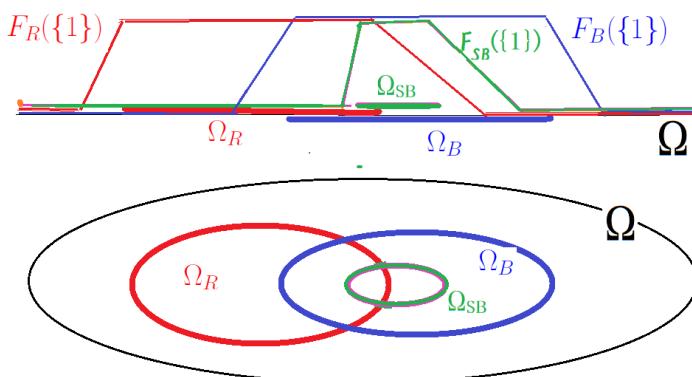
Figure 4: [Raven state class Ω_R, black bird state class Ω_B,small black bird state class Ω_{SB}]

That is, we add the small black bird observable:

Definition 21. Define SB by the set of all small black birds. Of course it is clear that $SB \subseteq B$. And further consider the observable $O_{SB} = (\{0,1\}, 2^{\{0,1\}}, F_{SB})$ (called the small black bird observable) in $L^\infty(\Omega, \nu)$. Also, put

$$\Omega_{SB} := \{\omega \in \Omega \mid [F_{SB}(\{1\})](\omega) = 1\}, \qquad \Omega_{\overline{SB} \backslash SB} := \{\omega \in \Omega \mid 0 < [F_{SB}(\{1\})](\omega) < 1\},$$
$$\Omega_{U \backslash \overline{SB}} := \{\omega \in \Omega \mid [F_{SB}(\{1\})](\omega) = 0\},$$

Now we can propose the following declaration, which is more general than Declaration 18.

Declaration 22. [The mathematical formulation of ornithology (II)]: *Assume Declaration 18 and Definition 21. And further, we assume that the state map $\widehat{\omega} : U \to \Omega$ satisfies the following:*

$$\text{``}\widetilde{\omega}(u) \in \Omega_{SB}\text{''} \Leftrightarrow \text{``}u \in SB\text{''}, \quad \text{``}\widetilde{\omega}(u) \in \Omega_{\overline{SB} \backslash SB}\text{''} \Leftrightarrow \text{``}u \in \overline{SB} \backslash SB\text{''} \quad \text{``}\widetilde{\omega}(u) \in \Omega_{U \backslash \overline{SB}}\text{''} \Leftrightarrow \text{``}u \in U \backslash \overline{SB}\text{''},$$

Also, note that "Any small black bird is black" is a priori proposition (i.e., this proposition is self-evident, and thus, it does not need to be experimentally verified), which has the measurement theoretical representation as follows:

$$\Omega_{SB} \subseteq \Omega_B \tag{6}$$

Exercise 23. It is easy to see the followings:

(M$_1$) for each $u_i \in SB$, the probability that the measured value 1 is obtained by the measurement $\mathsf{M}_{L^\infty(\Omega,\nu)}(O_B = (\{1,0\}, 2^{\{1,0\}}, F_B), S_{[\widetilde{\omega}(u_i)]})$ is equal to 1.

Or formally (*cf.* (G$_1$): Only one measurement is permitted),

(M$_2$) the probability that the measured value $\underbrace{(1,1,1...,1)}_{\sharp[SB]}$ is obtained by the parallel measurement $\bigotimes_{u_i \in SB} \mathsf{M}_{L^\infty(\Omega,\nu)} (O_B = (\{1,0\}, 2^{\{1,0\}}, F_B), S_{[\widetilde{\omega}(u_i)]})$ is equal to 1.

The two say *"When all the small black birds were observed, they were all black"*.

Remark 24. It should be noted that we have three "Any small black bird is black"s, that is,

(N$_1$) $\Omega_{SB} \subseteq \Omega_B$
(N$_2$) $SB \subseteq B$
(N$_3$) *"When all the small black birds were observed, they were all black"* in (M$_2$)

We believe that (N$_1$) is the true "Any small black bird is black". That is because we think that we should not depend on U so much by the reason that it was written in Remark 19. Thus, we assert that

$$\text{``Any small black bird is black''} :\overset{\text{def}}{\Longleftrightarrow} (\text{N}_1){:}\Omega_{SB} \subseteq \Omega_B$$

3.4 A posteriori proposition: "Any raven is black"

Now we consider the proposition:*"Any raven is black"*. As mentioned in the above, we have three "Any small black bird is black"s, that is,

(N$_1'$) $\Omega_R \subseteq \Omega_B$
(N$_2'$) $R \subseteq B$
(N$_3'$) *"When all the ravens were observed, they were all black"*

By the same reason mentioned in the above Remark 24, we think that

$$\text{"Any raven is black"} :\stackrel{\text{def}}{\Longleftrightarrow} (N_1'){:}\Omega_R \subseteq \Omega_B$$

though it is a posteriori proposition (i.e., this proposition is not self-evident, and thus, it needs to be experimentally verified).

Now we have the following theorem:

Theorem 25. *It holds that*

$$(N_1'){:}\Omega_R \subseteq \Omega_B \implies (N_2'){:}\ R \subseteq B$$
$$\implies (N_3'){:}\text{"When all the ravens were observed, they were all black"}$$

Proof. Assume that $\Omega_R \subseteq \Omega_B$ and $R \cap B^c \neq \emptyset$ (i.e., $R \subseteq B$ does not hold). If $u_0 \in R \cap B^c$, then $\widetilde{\omega}(u_0) \in R$ and $\widetilde{\omega}(u_0) \in B^c$. This implies that $\Omega_R \cap \Omega_{B^c} \neq \emptyset$. This contradicts $\Omega_R \subseteq \Omega_B$. Thus, we see that $(N_1') \implies (N_2')$.

Assume that $:\Omega_R \subseteq \Omega_B$ and $R \subseteq B$. Then we see

(N_3'') the probability that the measured value $(\underbrace{1,1,1,...,1}_{\sharp[R]})$ is obtained by the parallel measurement

$$\bigotimes_{u_i \in R} \mathsf{M}_{L^\infty(\Omega,\nu)}\ (\mathsf{O}_B = (\{1,0\}, 2^{\{1,0\}}, F_B), S_{[\widetilde{\omega}(u_i)]})\ \text{is equal to 1.}$$

which is the same as (N_3'). $\qquad\qquad\square$

Example 26. Here we show that

(O_1) there exists the case such that (N_1') does not hold (e.g., it holds that $\Omega_R \cap \Omega_B \neq \emptyset$, $\Omega_B \setminus \Omega_R \neq \emptyset$, $\Omega_R \setminus \Omega_B \neq \emptyset$) but it holds that $(N_2'){:}R \subseteq B$ and $(N_3'){:}$"When all the ravens were observed, they were all black".

Consider the state map $\widetilde{\omega} : U \to \Omega$ such that

$$\widetilde{\omega}(R) \subseteq \Omega_R \cap \Omega_B \quad \text{and} \quad \widetilde{\omega}(B \setminus R) \subseteq \Omega_B \setminus \Omega_R$$

Then we see

(O_2) the probability that the measured value $(\underbrace{1,1,1...,1}_{\sharp[R]})$ is obtained by the parallel measurement

$$\bigotimes_{u_i \in R} \mathsf{M}_{L^\infty(\Omega,\nu)}\ (\mathsf{O}_B = (\{1,0\}, 2^{\{1,0\}}, F_B), S_{[\widetilde{\omega}(u_i)]})\ \text{is equal to 1.}$$

which is the same as (N_3').
This example says that

(P_1) "When all the ravens were observed, they were all black" does not guarantee "All ravens are black".

Or equivalently

(P_2) we can not experimentally verify "$\Omega_R \subseteq \Omega_B$(i.e., any raven is black)".

Thus, our present problem is as follows.

(P_3) What should we do to be sure of "$\Omega_R \subseteq \Omega_B$"?

This will be answered in the following section 3.5.

3.5   Falsification test

In this section, we study the problem (P_3), i.e., What should we do to be sure of "$\Omega_R \subseteq \Omega_B$"?. In order to do it, we can use popper's falsifiability ($\approx$ statistical hypothesis testing) as follows.

Assume the following fact (which is regarded as one of the consequences of "All ravens are black (i.e., $\Omega_R \subseteq \Omega_B$)"):

(Q$_1$) It was found that one hundred ravens were black continuously.

What we can do is to reject the null hypothesis of the (Q$_1$).
For instance, it is usual to assume the following null hypothesis:

(Q$_2$) Non-black ravens can be observed at 3 time of a rate to 100 times.

Example (I): pure measurement
For example, this null hypothesis (Q$_2$) is represented in quantum language as the measurement $\mathsf{M}_{L^\infty(\Omega,\nu)}$ $(\mathsf{O}'_B := (X, 2^X, F'_B), S_{[\widetilde{\omega}(r)]}), (\forall r \in R))$ where

$$X = \{1, 0\}, \quad [F'_B(\{1\})](\omega) = \frac{97}{100} \quad (\text{if } \omega \in \Omega_R), \qquad [F'_B(\{1\})](\omega) = 0 \quad (\text{if } \omega \in \Omega \setminus \Omega_R),$$
$$[F'_B(\{0\})](\omega) = 1 - [F'_B(\{1\})](\omega) \quad (\forall \omega \in \Omega)$$

which is a slight modification of the formula (4). And thus, we calculate, by Axiom 1 [measurement], that

(Q$_3$) the probability that a measured value $\underbrace{(1, 1, 1..., 1)}_{100}$ is obtained by the parallel measurement

$\bigotimes_{r \in \{r_1, r_2, ..., r_{100}\} \subseteq R} \mathsf{M}_{L^\infty(\Omega,\nu)}$ $(\mathsf{O}'_B := (X, 2^X, F'_B), S_{[\widetilde{\omega}(r)]})$ is given by $(97/100)^{100}(< 0.048)$. That is, the probability that (Q$_1$) is realized (i.e., we meet one hundred black ravens continuously) is less than 0.048 $(> (97/100)^{100})$.

Thus, we may reject the null hypothesis (Q$_2$) since probability 0.048 is quite rare. That is, we couldn't deny "$\Omega_R \subseteq \Omega_B$ (i.e., any raven is black)".
Example (II): mixed measurement (*cf.* Chapter 9 in ref. [18])
Also the null hypothesis (Q$_2$) is represented in quantum language as the nmixed measurement $\mathsf{M}_{L^\infty(\Omega,\nu)}$ $(\mathsf{O}_B := (X, 2^X, F_B), S_{[*]}(\nu_0))$, where the mixed state ρ_0 is defined by

$$\rho_0(D) = \frac{3\nu((\Omega_R \setminus \Omega_B) \cap D)}{100\nu(\Omega_R \setminus \Omega_B)} + \frac{97\nu((\Omega_R \cap \Omega_B) \cap D)}{100\nu(\Omega_R \cap \Omega_B)} \qquad (\forall D \in \mathcal{B}_\Omega)$$

And thus, we calculate, by Axiom$^{(m)}$ 1 [in Chapter 9 of ref. [18]: mixed measurement], that

(Q$_4$) the probability that a measured value $\underbrace{(1, 1, 1..., 1)}_{100}$ is obtained by the parallel mixed measurement

$\bigotimes_{i=1}^{100} \mathsf{M}_{L^\infty(\Omega,\nu)}$ $(\mathsf{O}_B := (X, 2^X, F_B), S_{[*]}(\rho_0))$ is given by $(97/100)^{100}(< 0.048)$. That is, the probability that (Q$_1$) is realized (i.e., we meet one hundred black ravens continuously) is less than 0.048 $(> (97/100)^{100})$.

Thus, we may reject the null hypothesis (Q$_2$) since probability 0.048 is quite rare. That is, we couldn't deny "$\Omega_R \subseteq \Omega_B$ (i.e., any raven is black)".

If many statistical hypothesis tests are cleared, we may believe in "$\Omega_R \subseteq \Omega_B$ (i.e., any raven is black)". If so, we can propose the following new declaration:

Declaration 27. [The mathematical formulation of ornithology (III)]: *Assume Declaration 22. Further, we add "$\Omega_R \subseteq \Omega_B$ (i.e., any raven is black)".*

This is progress of ornithology (i.e., Declaration 18 $\overset{\text{progress}}{\Longrightarrow}$ Declaration 22 $\overset{\text{progress}}{\Longrightarrow}$ Declaration 27).
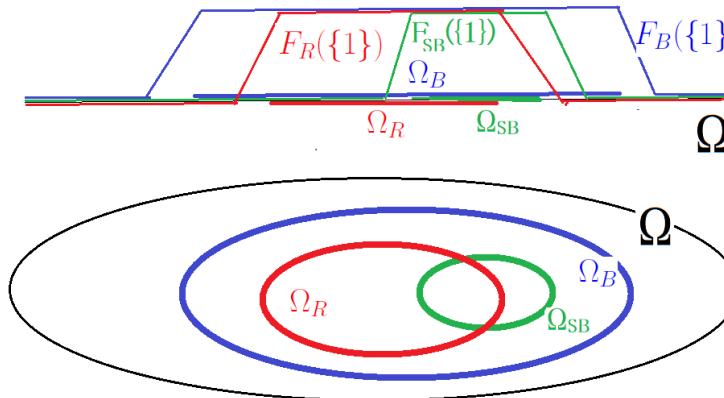
Figure 5: [All ravens are black: Fig. 3 $\overset{\text{progress}}{\Longrightarrow}$ Fig. 4 $\overset{\text{progress}}{\Longrightarrow}$ Fig. 5]

Remark 28. Note that the above argument is popular as statistical hypothesis testing in statistics, though statistics does not have the concept of "measurement". In our worldview (i.e., linguistic quantum mechanical worldview), we consider that Popper's falsificationism (*cf,* ref. [28]) and statistical hypothesis testing are almost the same. However his theory was supported by philosophers rather than scientists since his proposal was not proposed under a certain scientific worldview. That is, Popper's falsificationism belongs to (A_1): "philosophy of science for the general public".

4 Hume's problem of induction

4.1 Problem of Induction in the linguistic quantum mechanical world view

Although David Hume (1711-1776), British experimentalist, suspected the justification of induction, in this section we show that the justification is easily solved in our worldview. If we expect a scientific answer to Hume's problem, we must start with the scientific definition of "the uniformity principle of nature", i.e., the following Definition 29 [The uniformity principle of nature]. Some may feel that the uniformity principle of nature (i.e., the condition in Definition 29) is too strong. However, we think that it is impossible to propose the different quantitative definition of the uniformity principle of nature that leads to a result like Theorem 30 [Inductive reasoning] (i.e., *If similar measurements are performed, the similar measured values are obtained*).

Definition 29. [The uniformity principle of nature] Let $[C(\Omega) \subseteq L^\infty(\Omega, \nu) \subseteq B(L^2(\Omega, \nu))]$ be a classical basic structure such that Ω is compact and $\nu(\Omega) = 1$. A family of measurements $\{\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}_i := (X, \mathcal{F}, F_i), S_{[\omega_i]}) \mid i = -n, -n+1, ..., -1, 0, 1, 2, ..., N\}$ is said to satisfy the *uniformity principle of nature* (*concerning* μ), if there exists a probability space $(X, \mathcal{F}, \mu)$ such that

$$[F_i(\Xi)](\omega_i) = \mu(\Xi) \qquad \forall \Xi \in \mathcal{F}, \forall i = -n, -n+1, ..., -1, 0, 1, 2, ..., N$$

Under this definition, we assert the following theorem, which should be regarded as the fundamental theorem in philosophy of science.

Theorem 30. [Inductive reasoning, the quantum linguistic solution of Hume's problem of induction]. Let $[C(\Omega) \subseteq L^\infty(\Omega, \nu) \subseteq B(L^2(\Omega, \nu))]$ be a basic structure such that Ω is compact and $\nu(\Omega) = 1$. Assume that a family of measurements $\{\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}_i := (X, \mathcal{F}, F_i), S_{[\omega_i]}) \mid i = -n, -n+1, ..., -1, 0, 1, 2, ..., N\}$ satisfies the uniformity principle of nature (concerning μ). Let $(x_{-n}, x_{-n+1}, ..., x_{-1}, x_0, x_1, ..., x_N) \in \times_{i=-n}^N X$ be a measured value by the parallel measurement $\bigotimes_{i=-n}^N \mathsf{M}_{L^\infty(\Omega,\nu)}$ ($\mathsf{O}_i := (X, \mathcal{F}, F_i)$, $S_{[\omega_i]}$). Then, we see that

$$\frac{\sharp\{k \mid x_k \in \Xi, k = -n, -n+1, ..., -1, 0\}}{n} \approx \mu(\Xi)(= [F_i(\Xi)](\omega_i))$$
$$(\Xi \in \mathcal{F}, i = -n, -n+1, ..., -1, 0, 1, 2, ..., N) \tag{7}$$

where n is sufficiently large. Here $\sharp[\Theta]$ is the number of elements in a set Θ.

Proof. Let $\Xi_i \in \mathcal{F}$ ($i = -n, -n+1, ..., -1, 0, 1, ..., N$). Axiom 1 [measurement] says that the probability that a measured value $(x_{-n}, x_{-n+1}, ..., x_{-1}, x_0, x_1, ..., x_N)$ obtained by the parallel measurement $\bigotimes_{i=-n}^N \mathsf{M}_{L^\infty(\Omega,\nu)}$ ($\mathsf{O}_i := (X, \mathcal{F}, F_i)$, $S_{[\omega_i]}$) belongs to $\times_{i=-n}^N \Xi_i$ is given by $\times_{i=-n}^N [F_i(\Xi_i)](\omega_i) = \times_{i=-n}^N \mu(\Xi_i)$. Thus, the sequence $\{x_i\}_{i=-n}^N$ can be regarded as independent random variables with the identical distribution μ. Hence, using the law of large numbers, we can immediately get the formula (7). Also, this theorem is a direct consequence of the law of large numbers for parallel measurements (*cf.* refs. [5], or § 4.2 in ref.[18]). □

Remark 31. (i): Recall that the law of large numbers (which is almost equivalent to Theorem 30) says that

"frequency probability" = "the probability in Axiom 1" (*cf.* ref. [5])

though the probability in Axiom 1 has the several aspects. Also, note that the law of large numbers in statistics (*cf.* ref. [27]) has already been accepted as the fundamental theorem in science. Therefore, even if Theorem 30 ([Inductive reasoning]+(7)) is called the fundamental theorem in philosophy of science, we don't think it's exaggerated. We believe that our proposal (i.e., Theorem 30) is completely true in our worldview. Thus, we think that the solution of Hume's problem of induction was practically already found as the law

of large numbers. In the framework of our worldview, we are convinced that the above is the definitive solution to Hume's problem. However, there may be another idea if some start from another worldview. Hence, as described at the end of this paper, we hope that many philosophers propose various mathematical foundations of scientific philosophy, in which Hume's problem of induction are discussed from the various viewpoints.

(ii): In Definition 29 [The uniformity principle of nature] and Theorem 30 [Inductive reasoning], we consider the family of measurements $\{\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}_i := (X, \mathcal{F}, F_i), S_{[\omega_i]}) \mid i = -n, -n+1, ..., -1, 0, 1, 2, ..., N\}$. This may be too general. Usually, it suffices to consider that $\{\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}_i := (X, \mathcal{F}, F), S_{[\omega_i]}) \mid i = -n, -n+1, ..., -1, 0, 1, 2, ..., N\}$, i.e., $F = F_i$ $(-n \leq \forall i \leq N)$.

(iii): It may be understandable to consider two measurements: $\bigotimes_{i=-n}^{0} \mathsf{M}_{L^\infty(\Omega,\nu)}$ $(\mathsf{O} := (X, \mathcal{F}, F_i), S_{[\omega_i]})$ and $\bigotimes_{i=1}^{N} \mathsf{M}_{L^\infty(\Omega,\nu)}$ $(\mathsf{O} := (X, \mathcal{F}, F_i), S_{[\omega_i]})$. The reason that we do not consider two measurements is due to the linguistic Copenhagen interpretation (G_1), i.e., *only one measurement is permitted.*

Example 32. [Coin tossing]. Let us discuss the unfair coin tossing as the most understandable example of Theorem 30 [Inductive reasoning]. Consider a basic structure $[C(\Omega) \subseteq L^\infty(\Omega,\nu) \subseteq B(L^2(\Omega,\nu))]$. Let $\{\omega_i\}_{i=-n}^{N}$ be a sequence in Ω, where ω_i is the state of i-th coin tossing $(i = -n, -n+1, ..., 0, 1, 2, 3, ..., N)$. Let $\mathsf{O} = (X, 2^X, F)$ be an observable in $L^\infty(\Omega,\nu)$ such that

$$X = \{H, T\}, (\text{ where } H\text{: head, } U\text{: tail}),$$
$$[F(\{H\})](\omega_i) = \mu(\{H\}) = 2/3, \quad [F(\{T\})](\omega_i) = \mu(\{T\}) = 1/3 \quad (\forall i = -n, -n+1, ..., -1, 0, 1, 2, ..., N) \tag{8}$$

That is, a family of measurements $\{\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O} := (X, 2^X, F), S_{[\omega_i]}) \mid i = -n, -n+1, ..., -1, 0, 1, 2, ..., N\}$ satisfies the uniformity principle of nature (concerning μ). Let $(x_{-n}, x_{-n+1}, ..., x_{-1}, x_0, x_1, ..., N) \in \times_{i=-n}^{N} X$ be a measured value obtained by the parallel measurement $\bigotimes_{i=-n}^{N} \mathsf{M}_{L^\infty(\Omega,\nu)}$ $(\mathsf{O} := (X, 2^X, F), S_{[\omega_i]})$, i.e., infinite coin throws. Here, Theorem 30 [Inductive reasoning] say that it is natural to assume that, for sufficiently large n,

$$(x_{-n}, x_{-n+1}, ..., x_{-1}, x_0) = (\underbrace{\text{T H H T H H H T T ..... T H H}}_{n+1}) \tag{9}$$
(where the number of Hs $\approx 2n/3$, Ts $\approx n/3$)

Then we can believe that we see that $x_i = H$ with probability $2/3$ [resp. $x_i = T$ with probability $1/3$] for each $i = 1, 2, ..., N$. It should be noted that even without knowing (8), we can conclude that if we know (9).

Remark 33. It should be noted that the above example shows that Theorem 30 [Inductive reasoning] (or equivalently, the law of large numbers), like Newton's kinetic equation, has the power to predict the future. This is the reason that Hume's problem of induction keeps attracting much researcher's interest for a long time.

Example 34. [Induction concerning raven problem]. Let $R \subseteq B \subseteq U$ be as in Section 3.2. Consider a basic structure $[C(\Omega) \subseteq L^\infty(\Omega,\nu) \subseteq B(L^2(\Omega,\nu))]$. Let $\mathsf{O}_B = (X, 2^X, F_B)$ be an observable in $L^\infty(\Omega,\nu)$ such as defined in the formula (4), that is,

$$X = \{1, 0\}, \quad [F_B(\{1\})](\omega) = 1 \quad (\text{if } \omega \in \Omega_B), \qquad [F_B(\{0\})](\omega) = 0 \quad (\text{if } \omega \in \Omega \setminus \Omega_B),$$
$$[F_B(\{0\})](\omega) = 1 - [F_B(\{1\})](\omega) \quad (\forall \omega \in \Omega)$$

Let $\{\omega_i\}_{i=-n}^{N}$ be a sequence in $\Omega_R(\subseteq \Omega)$. Clearly, a family of measurements $\{\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}_B := (X, 2^X, F_B), S_{[\omega_i]}) \mid i = -n, -n+1, ..., -1, 0, 1, 2, ..., N\}$ satisfies the uniformity principle of nature (concerning μ) such that

$$[F_B(\{1\})](\omega_i) = \mu(\{1\}) = 1, \quad [F_B(\{0\})](\omega_i) = \mu(\{0\}) = 0, \qquad (i = -n, -n+1, ..., 0, 1, 2, 3, ..., N)$$

Let $(x_{-n}, x_{-n+1}, ..., x_{-1}, x_0, x_1, ..., x_N) \in \times_{i=-n}^{N} X$ be a measured value obtained by the parallel measurement $\bigotimes_{i=-n}^{N} \mathsf{M}_{L^\infty(\Omega,\nu)} (\mathsf{O}_B := (X, 2^X, F_B), S_{[\omega_i]})$, We see, of course, that $x_i = b$ $(i = -n, -n+1, .., -1, 0)$. And thus, we can believe, by Theorem 30 [Inductive reasoning], that $x_1 = x_2 = ... = x_N = b$.

4.2 Grue paradox cannot be represented in quantum language

If our understanding of inductive reasoning (mentioned in the above) is true, we can solve the grue paradox (*cf.* ref. [29]). Let us mention it as follows.

Consider a basic structure $[C(\Omega) \subseteq L^\infty(\Omega, \nu) \subseteq B(L^2(\Omega, \nu))]$. Let Ω_g, Ω_b be the open subsets of the state space Ω such that $\Omega_g \cap \Omega_b = \emptyset$. And put $\Omega_o = \Omega \setminus (\Omega_g \cup \Omega_b)$. Let $\mathsf{O} = (X \equiv \{g, b, o\}, 2^X, F)$ be the observable in $L^\infty(\Omega, \nu)$ such that

$$[F(\{g\})](\omega) = 1 \ (\omega \in \Omega_g), \quad = 0 \ (\omega \in \Omega \setminus \Omega_g) \qquad [F(\{b\})](\omega) = 1 \ (\omega \in \Omega_b), \quad = 0 \ (\omega \in \Omega \setminus \Omega_b)$$
$$[F(\{o\})](\omega) = 1 - [F(\{g\})](\omega) - [F(\{b\})](\omega) \ (\omega \in \Omega) \tag{10}$$

where "g", "b", "o" respectively means "green", "blue", "others".

Let $\{e_{-n}, e_{-n+1}, ..., e_{-1}, e_0, e_1, e_2, ..., e_N\}$ be the set of (green) emeralds. And assume that $\omega_i (\in \Omega_g^\circ)$ is the state of emerald e_i $(i = -n, -n+1, ..., -1, 0, 1, 2, ..., N)$.

A family of measurements $\{\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}_i := (X, 2^X, F), S_{[\omega_i]}) \mid i = -n, -n+1, ..., -1, 0, 1, 2, ..., N\}$ clearly satisfies the uniformity principle of nature, that is, there exists an probability space $(X, 2^X, \mu)$ such that

$$[F(\Xi)](\omega_i) = \mu(\Xi) \quad \forall \Xi \in 2^X, \forall i = -n, -n+1, ..., -1, 0, 1, 2, ..., N$$

where $\mu(\{g\}) = 1$, $\mu(\{b, o\}) = 0$.

Let $(x_{-n}, x_{-n+1}, ..., x_{-1}, x_0, x_1, ..., x_N) \in \times_{i=-n}^{N} X$ be a measured value obtained by the parallel measurement $\bigotimes_{i=-n}^{N} \mathsf{M}_{L^\infty(\Omega,\nu)} (\mathsf{O} := (X, 2^X, F), S_{[\omega_i]})$. We see, of course, that $x_i = g$ $(i = -n, -n+1, .., -1, 0)$. And thus, we can believe, by Theorem 30 [Inductive reasoning], that $x_1 = x_2 = ... = x_N = g$. For the sake of completeness, note that we can predict $x_1 = x_2 = ... = x_N = g$ only by the data $x_{-n} = x_{-n+1} = ... = x_0 = g$. This is usual arguments concerning Theorem 30 [Inductive reasoning].

On the other hand, Goodman's grue paradox is as follows (*cf.* ref. [29]).

(R$_1$) Define that Y has a grue property iff Y is green at time i such that $i \leq 0$ and Y is blue at time i such that $0 < i$. Suppose that we have examined the emeralds at $-n, -n+1, ... - 1, 0$, and found them to all be green (and hence also grue). Then, "so-called inductive reasoning" says that emeralds at $1, 2, ..., N$ have the grue property (and hence blue) as well as green. Thus, a contradiction is gotten.

However, we think that this (R$_1$) cannot be described in quantum language. If we try to describe the (R$_1$), we may consider as follows.

(R$_2$) Let $\{e_{-n}, e_{-n+1}, ..., e_{-1}, e_0, e_1, e_2, ..., e_N\}$ be the set of emeralds. Let $\omega_i (\in \Omega_g^\circ)$ be the state of emerald e_i $(i = -n, -n+1, ..., -1, 0)$, and let $\omega_i (\in \Omega_b^\circ)$ be the state of emerald e_i $(i = 1, 2, ..., N)$. However, it should be noted that a family of measurements $\{\mathsf{M}_{L^\infty(\Omega,\nu)}(\mathsf{O}_i := (X, 2^X, F), S_{[\omega_i]}) \mid i = -n, -n+1, ..., -1, 0, 1, 2, ..., N\}$ does not satisfy the uniformity principle of nature. That is because

$$[F(\{g\})](\omega_i) = 1 \ (i = -n, -n+1, ..., 0), \qquad [F(\{g\})](\omega_i) = 0 \ (i = 1, 2, ..., N)$$

Hence Theorem 30 [Inductive reasoning] cannot be applied.

Or,

(R$_3$) Let $\{e_{-n}, e_{-n+1}, ..., e_{-1}, e_0, e_1, e_2, ..., e_N\}$ be the set of emeralds. And let $\omega_i (\in \Omega_g^\circ)$ is the state of emerald e_i such that $\omega = \omega_i$ $(i = -n, -n+1, ..., -1, 0, 1, 2, ..., N)$. Let $\mathsf{O}_i = (X, 2^X, F_i)$ be the

observable ($i = -n, -n+1, ..., -1, 0, 1, 2, ..., N$) such that O_i is the same as $\mathsf{O}(= (X \equiv \{g, b, o\}, 2^X, F))$ in (10) (if $i = -n, -n + 1, ..., -1, 0$), and $\mathsf{O}_i = (X, 2^X, F_i)$ (if $0, 1, 2, ..., N$) is defined by $F_i(\{g\}) = F(\{b\})$, $F_i(\{b\}) = F(\{g\})$, $F_i(\{o\}) = F(\{o\})$. However, in this case, it should be noted that a family of measurements $\{\mathsf{M}_{L^\infty(\Omega, \nu)}(\mathsf{O}_i := (X, 2^X, F_i), S_{[\omega_i]}) \mid i = -n, -n + 1, ..., -1, 0, 1, 2, ..., N\}$ does not satisfy the uniformity principle of nature. That is because

$$[F_i(\{g\})](\omega_i) = [F(\{g\})](\omega_i) = 1 \qquad (i = -n, -n + 1, ..., 0),$$
$$[F_i(\{g\})](\omega_i) = [F(\{b\})](\omega_i) = 0 \qquad (i = 1, 2, ..., N)$$

Hence Theorem 30 [Inductive reasoning] cannot be applied.

Therefore Goodman's grue paradox (R_1) cannot be described in quantum language.

Remark 35. We believe that there is no scientific argument without scientific worldview. Thus, we can immediately conclude that Goodman's discussion (R_1) is doubtful since his argument is not based on any scientific worldview. In this sense, the above arguments (R_2) and (R_3) may not be needed. That is, the confusion of grue paradox is due to lack of the understanding of Hume's problem of induction in the linguistic quantum mechanical worldview, and not lack of the term "grue" is non-projectible (*cf.* ref. [29]). Thus, we think that to solve Goodman's grue paradox is to answer the following:

(S) Propose a worldview! And further formulate Hume's induction as the fundamental theorem in the worldview! In this formulation, confirm that Goodman's paradox is eliminated naturally.

What we did is this.

5 The measurement theoretical representation of abduction

5.1 Deduction, abduction and abduction in "logic"

In the conventional world view (i.e., the "logical" world view (*cf.* the formula (12) later)), the relation among deduction, abduction and abduction is shown like the following figure.
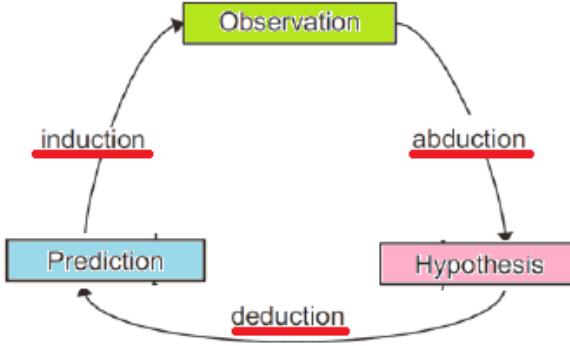

Figure 6: [Deduction, abduction and abduction in "logic"]

Let us explain this figure as follows.

A typical example of deduction is as follows: $\Big($ In the following, (S_1') and $(\widehat{S}_1')$ are often omitted. $\Big)$

(S_1) All the beans in this bag B_1 are white: [bag $B_1 \longrightarrow$ "w" ($\approx$ white)]

(S_1') All the beans in that bag B_2 are white or black fifty-fifty (or generally, the ratio of white beans to black beans is $p/(1-p)$ where $0 < p < 1$): [bag $B_2 \longrightarrow$ "w" ($\approx$ white) or "b" ($\approx$ black)]

(S_2) This bean is from this bag B_1: [bag B_1]

(S_3) Therefore, this bean is white: ["w" ($\approx$ white)]

It is, of course, obvious and ordinary.

On the other hand, C.S, Peirce (*cf.* ref. [30]) proposed abduction. The example of abduction is as follows:

($\widehat{S}_1$) All the beans in this bag B_1 are white: [bag $B_1 \longrightarrow$ "w" ($\approx$ white)]

($\widehat{S}_1'$) All the beans in that bag B_2 are white and black fifty-fifty (or generally, the ratio of white beans to black beans is $p/(1-p)$): [bag $B_2 \longrightarrow$ "w" ($\approx$ white) or "b" ($\approx$ black)]

($\widehat{S}_2$) This bean (from B_1 or B_2) is white: ["w" ($\approx$ white)]

($\widehat{S}_3$) Therefore, this bean is from this bag B_1 : [bag B_1]

This is wrong from the logical point of view. However, the abduction (abductive reasoning) is known as one of useful tools to find a best solution. Also, note that $[(S_2)\longrightarrow(S_3)]$ and $[(\widehat{S}_2)\longrightarrow(\widehat{S}_3)]$ are in reverse relation.

Further, induction (inductive reasoning) is as follows.

($\widetilde{S}_1$) $1000p$ white beans and $1000(1-p)$ black beans are mixed well in this bag B_3 (here, $0 < p < 1$). Assume that we do not know the value p $(0 < p < 1)$.

($\widetilde{S}_2$) When we took 20 beans out of this bag B_3, every bean was white.

($\widetilde{S}_3$) Therefore, the bean picked out from this bag B_3 next can be presumed to be white.

5.2   The measurement theoretical representation of deduction, abduction and induction

In our world view (i.e., the quantum mechanical world view $\approx$ the measurement theoretical world view (*cf.* the formula (12) later)), the relation among deduction, abduction and abduction is characterized as follows. First, we will show that the abduction $[(\widehat{S}_1)\text{-}(\widehat{S}_3)]$ can be justified in quantum language. Consider the state space $\Theta = \{\theta_1, \theta_2\}$ with the discrete topology, and the classical basic structure $[C(\Theta) \subseteq L^\infty(\Theta, \nu) \subseteq B(L^2(\Theta, \nu))]$, where $\nu(\{\theta_1\}) = \nu(\{\theta_2\}) = 1/2$. Assume that

$$\theta_1 \approx \text{the state of the bag } B_1, \quad \theta_2 \approx \text{the state of the bag } B_2,$$

Assume that 1000 white beans belong to bag B_1, and further, $1000p$ white beans and $1000(1-p)$ black beans belong to the bag B_2 (where $0 < p < 1$). Thus we have the observable $O = (\{w, b\}, 2^{\{w,b\}}, F)$ in $L^\infty(\Theta, \nu)$ such that

$$[F(\{w\})](\theta_1) = 1 \qquad [F(\{b\})](\theta_1) = 0$$
$$[F(\{w\})](\theta_2) = p \qquad [F(\{b\})](\theta_2) = 1 - p \qquad (0 < p < 1)$$

where "w" and "b" means "white" and "black" respectively.

Thus, we have the measurement $\mathsf{M}_{L^\infty(\Theta,\nu)}(O := (\{w, b\}, 2^{\{w,b\}}, F), S_{[\theta_i]})$, $i = 1, 2$. For example, Axiom 1 [measurement] says that

(U$_1$) [measurement]:   The probability that the measured value w is obtained by $\mathsf{M}_{L^\infty(\Theta,\nu)}(O := (\{w, b\}, 2^{\{w,b\}}, F), S_{[\theta_1]})$ is equal to 1

This is the same as the deduction (i.e., (S$_1$)–(S$_3$)).

Next, under the circumstance that bags B_1 and B_2 cannot be distinguished, we consider the following inference problem:

($\widehat{U}_2$) [inference problem]:   When the measured value w is obtained by the measurement $\mathsf{M}_{L^\infty(\Theta,\nu)}(O := (\{w, b\}, 2^{\{w,b\}}, F), S_{[*]})$, which do you infer, $[*] = \theta_1$ or $[*] = \theta_2$?
Theorem 13 [Fisher's maximum likelihood method] says that $[*] = \theta_1$, since

$$\max\{F(\{w\})](\theta_1), F(\{w\})](\theta_2)\} = \max\{1, p\} = 1 = [F(\{w\})](\theta_1)$$

This implies $(\widehat{S}_3)$.

Therefore, the above $(\widehat{U}_2)$ is the measurement theoretical representation of abduction (i.e., $(\widehat{S}_1)$–$(\widehat{S}_3)$). For the sake of completeness, note that (U$_1$) and $(\widehat{U}_2)$ are in reverse problem (*cf.* Remark 14). That is, we have the following correspondence:

$$
\begin{array}{ccc}
\underset{\text{deduction}}{[(S_1),(S_1'),(S_2) \longrightarrow (S_3)]} & \xleftarrow{\text{simplified form}} & \underset{(\text{Axiom 1})}{(U_1):\ \text{measurement}} \\
\Big\downarrow \text{reverse} & & \Big\downarrow \text{reverse} \qquad (11) \\
\underset{\text{abduction}}{[(\widehat{S}_1),(\widehat{S}_1'),(\widehat{S}_2) \longrightarrow (\widehat{S}_3)]} & \xleftarrow{\text{simplified form}} & \underset{(\text{Fisher's maximum likelihood method})}{(\widehat{U}_2):\ \text{inference}}
\end{array}
$$

Thus, the scientific meaning of abduction can be completely clarified in the translation from logic to quantum language.

Lastly we should mention that

($\widetilde{U}_3$)   the above $(\widetilde{S}_1)$-$(\widetilde{S}_3)$ (i.e., inductive reasoning) are already discussed in quantum language (*cf.* Section 4).

6 Flagpole problem
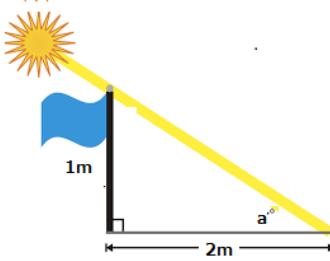
Figure 7: [Flagpole problem]

Let us explain the flagpole problem as follows. Suppose that the sun is at an elevation angle $\alpha°$ in the sky. Assume that $\tan \alpha° = 1/2$. There is a flagpole which is ω_0^0 meters tall. The flagpole casts a shadow ω_1^0 meters long. Suppose that we want to explain the length of the flagpole's shadow. On Hempel's model, the following explanation is sufficient.

(V_1) 1. The sun is at an elevation angle $\alpha°$ in the sky.
     2. Light propagates linearly.
     3. The flagpole is ω_0^0 meters high.
     Then,
     4. The length of the shadow is $\omega_1^0 = \omega_0^0 / \tan \alpha° = 2\omega_0^0$

This is a good explanation of "Why is that shadow $2\omega_0^0$ meters long?"
Similarly, we may consider as follows.

(V_2) 1. The sun is at an elevation angle $\alpha°$ in the sky.
     2. Light propagates linearly.
     3. The length of the shadow is ω_1^0
     Then,
     4. The flagpole is $\omega_0^0 (= (\tan \alpha°)\omega_1^0 = \omega_1^0/2)$ meters tall.

However, this is not sufficient as the explanation of "Why is the flagpole $\omega_0^0(= \omega_1^0/2)$ meters tall?"

The confusion between (V_1) and (V_2) is due to the lack of measurement. In what follows, we discuss it. For each time $t = 0, 1$, consider a basic structure $[C(\Omega_t) \subseteq L^\infty(\Omega_t, \nu_t) \subseteq B(L^2(\Omega_t, \nu_t))]$, where $\Omega_0 = [0, 1]$ is the state space (in which the length of the flagpole is represented) at time 0 (where the closed interval in the real line $\mathbb{R}$), $\Omega_1 = [0, 2]$ is the state space (in which the length of the shadow is represented) at time 1 and the ν_t is the Lebesgue measure. Since the sun is at an elevation angle $\alpha°$ in the sky, it suffices to consider to the map $\phi_{0,1} : \Omega_0 \to \Omega_1$ such that $\phi_{0,1}(\omega_0) = 2\omega_0$ $(\forall \omega_0 \in \Omega_0)$. Thus, we can define the causal operator $\Phi_{0,1} : L^\infty(\Omega_1) \to L^\infty(\Omega_0)$ such that $(\Phi_{0,1}f_1)(\omega_0) = f_1(\phi(\omega_0))$ $(\forall f_1 \in L^\infty(\Omega_1), \omega_0 \in \Omega_0)$.

Let $O_e = (X, \mathcal{F}, F_e)$ be the exact observable in $L^\infty(\Omega_1, \nu_1)$ (*cf.* Example 5). That is, it satisfies that $X = \Omega_1, \mathcal{F} = \mathcal{B}_{\Omega_1}$ (i.e., the Borel field in Ω), $[F_e(\Xi)](\omega_1) = 1$ (if $\omega_1 \in \Xi$), $= 0$ (otherwise).

Thus, we have the measurement $\mathsf{M}_{L^\infty(\Omega_0, \nu_0)}(\Phi_{0,1}O_e = (X, \mathcal{F}, \Phi_{0,1}F_e), S_{[\omega_0^0]})$. Then we have the following statement

(W_1) [Measurement]; the probability that the measured value $x(\in X)$ obtained by the measurement $\mathsf{M}_{L^\infty(\Omega_0, \nu_0)}(\Phi_{0,1}O_e = (X, \mathcal{F}, \Phi_{0,1}F_e), S_{[\omega_0^0]})$ is equal to $2\omega_0^0$ is given by 1.

which is the measurement theoretical representation of (V_1). That is, we consider that the (V_1) is the simplified form (or, the rough representation) of (W_1). Also,

(W$_2$) [Inference]; Assume that the measured value $\omega_1^0 (\in X)$ is obtained by the measurement $\mathsf{M}_{L^\infty(\Omega_0,\nu_0)}(\Phi_{0,1}\mathsf{O}_e = (X, \mathcal{F}, \Phi_{0,1}F_e), S_{[*]})$. Then, we can infer that $[*] = \omega_1^0/2$

which is the measurement theoretical representation of (V$_2$). That is, we consider that the (V$_2$) is the simplified form (or, the rough representation) of (W$_2$).

Thus, we conclude that "scientific explanation" is to describe by quantum language. Also, we have to add that the flagpole problem is not trivial but significant, since this is never solved without Axiom 1 (measurement) and Axiom 2 (causality) (i.e., the answers to the problems "What is measurement ?" and "What is causality ?").

7 Conclusion; To do science is to describe phenomena by quantum language

7.1 Summary of comparison between logic (in ordinary language), statistics and quantum language

"What is science ?" is the main question of philosophy of science. There may be the following three answers:

($\sharp_1$) Science is to describe phenomena by logic (i.e., the logical world view)

($\sharp_2$) Science is to describe phenomena by statistics (i.e., the classical mechanical world view)

($\sharp_3$) Science is to describe phenomena by quantum language (i.e., the quantum mechanical world view)

In this paper, we asserted that ($\sharp_3$), rather than ($\sharp_1$) and ($\sharp_2$)], more essential. In what follow, again let us examine this:

[($\sharp_1$):**Logic**]: Some may say "Science is to describe phenomena by logic", which may be due to the logical positivism (or, the tradition of Aristotle's syllogism). However, as seen in Sections 3∼6, Hempel's raven paradox, Hume's problem of induction, Goodman's grue paradox , Peirce's abduction and flagpole problem are related to the concept of measurement (= inference), and thus, these problems cannot be adequately handled by logic alone. Thus, we think that logic is the language of mathematics, and not the language of science. Mathematical logic (i.e., the language of mathematics) should not be confused with usual logic. As seen throughout this paper, we believe that the representation using "logic" is rough in most cases. So-called logic plays an essential role in everyday conversation (e.g., trial, business negotiations, politics, romance, etc.). On the other hand, science requires quantitative discussion, and thus, science may choose statistics (or, quantum language) rather than logic. It should be noted that

[($\sharp_2$): **Statistics; the classical mechanical world view**]: Statistics are used everywhere in science, and thus, statistics may be the principle of science. Therefore some may say "Science is to describe phenomena in the classical mechanical worldview ($\approx$ statistics $\approx$ dynamical system theory)". This answer may be somewhat better as follows.

(X_1) economics is to describe economical phenomena by statistics (it is usual to regard economics as the application of dynamical system theory ($\approx$ statistics))

(X_2) psychology is to describe psychological phenomena by statistics

(X_3) biology is to describe biological phenomena by statistics

(X_4) medicine is to describe medical phenomena by statistics (i.e., medical statistics)

Also, since dynamical system theory is considered as a kind of mathematical generalization of Newtonian mechanics, we may be allowed to say:

(X_5) Newtonian mechanics is to describe classical mechanical phenomena by statistics (= dynamical system theory). Also, it is clear that dynamical system theory plays a central role in engineering.

though Newtonian mechanics is physics, and thus, it belongs to the realistic worldview in Fig. 1.
However, statistics ($\approx$ dynamical system theory (*cf.* Remark 14)) is too mathematical. Hence, "Science is to describe phenomena in the classical mechanical worldview ($\approx$ statistics $\approx$ dynamical system theory)" is almost the same as "Science is to describe phenomena using the mathematical theories of probability and differential equation". And thus, the framework of the classical mechanical worldview is ambiguous. Since statistics ($\approx$ dynamical system theory) does not have clear axioms, we think that it is a little unreasonable to say that statistics is the language of science.

For example, we don't know how to clarify Hempel's raven problem (K_4)-(K_7) from the statistical point of view, since statistics does not have the concept of measurement. As seen below, the relationship between

science and statistics is revealed by quantum language (*cf.* ⑩ in Fig. 1).

[(♯₃): **Quantum language; the linguistic quantum mechanical worldview**]: We choose quantum language (i.e., the linguistic quantum mechanical worldview, or the probabilistic interpretation of science), and we assert "Science is to describe phenomena by quantum language". That is, in a similar sense of (X_1)-(X_5), we say that

(Y_1) economics is to describe economical phenomena by quantum language

(Y_2) psychology is to describe psychological phenomena by quantum language (*cf.* Chapter 18 in ref [18])

(Y_3) biology is to describe biological phenomena by quantum language

(Y_4) medicine is to describe medical phenomena by quantum language

(Y_5) Newtonian mechanics is to describe classical mechanical phenomena by quantum language (in the same meaning as the (X_5), also recall the history: ② ⟶ ⑦ ⟶ ⑪ in Fig. 1). Also, it is clear that classical system theory (= dynamical system theory, *cf.* (B_2)) plays a central role in engineering.

(Y_6) statistical mechanics is to describe statistical mechanical phenomena by quantum language (*cf.* ref. [12])

(Y_7) quantum mechanics (i.e., quantum information theory) is to describe quantum mechanical phenomena by quantum language (*cf.* (B_1)).

(Y_8) As mentioned in the (iv) and (v) of Remark 7, a lot of paradoxes (e.g., Bertrand's paradox, McTaggart's paradox, Zeno's paradox, Monty Hall problem, etc. (*cf.* refs. [11, 17, 18])) are clarified in quantum language.

(Y_9) As shown in Sections 3∼6, fundamental problems in philosophy of science (i.e., Hempel's raven paradox, Hume's problem of induction, Peirce's abduction, the flagpole problem, etc.) are easily solved in the linguistic quantum mechanical worldview. Therefore, quantum language guarantees that these problems are scientific. On the other hand, Goodman's grue paradox (R_1) cannot be described by quantum language. Thus, it is not scientific. Also, it should be noted that these results are consequences of Axiom 1 [measurement].

etc. Quantum language has the advantage of having the concept of "measurement". And thus, as seen in this paper, "logic" can be paraphrased in detail in terms of measurement, and thus, precise expression is obtained.

Remark 36. (i): [Can logic and statistics be regarded as kinds of worldviews?] Logic and statistics has respectively various aspects. However, when we say roughly, logic is the language of mathematics, and statistics is a quite useful mathematical theory. If so, how can we regard logic and statistics as kinds of worldviews? In this paper, we see that logic and statistics respectively has aspects such as simplified forms of quantum language. That is,

$$\left.\begin{array}{c} \text{logic} \\ \\ \text{statistics} \end{array}\right\} \xleftarrow[\text{elimination of dualism} (\approx \text{measurement})]{\text{simplified form}} \text{quantum language}$$

(*e.g.* see (B_3) in Fig. 1 and the formula (11)). If so, logic and statistics are scientific as simplified forms of quantum language. Thus, we conclude that so-called logic (i.e., non-mathematical logic) is essential in usual conversation (e.g., argument in a trial, etc.) and not in science.

(ii):[The histry of world descriptionss] Thus, we want to consider the history of world descriptions as follows.

$$\begin{array}{ccc} \overset{\text{⓪ in Fig. 1}}{\boxed{\text{logical world description}}} & \longrightarrow & \overset{\text{②,⑨ in Fig. 1}}{\boxed{\text{classical mechanical world description}}} \\ \text{Aristotle,...} & & \text{Newton, Fisher,...} \end{array}$$

$$\begin{array}{c} \overset{\text{⑪ in Fig. 1}}{\longrightarrow \boxed{\text{quantum mechanical world description}}} \\ \text{Born,...} \end{array} \tag{12}$$
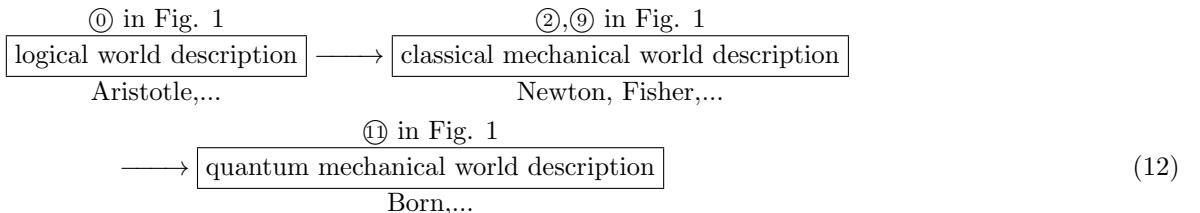
7.2 Summing up; Quantum language is the language of science

In this paper, we clarify the following unsolved problems in the linguistic quantum mechanical worldview:

- Hempel's raven paradox in Section 3, Hume's problem of induction in Section 4.1, Goodman's grue paradox in Section 4.2, Peirce's abduction in Section 5, the flagpole problem in Section 6

That is, we sum up as follows:

(Z_1) [Hempel's raven paradox in Section 3]: Hempel's raven paradox urges us to investigate the problem: "What is the meaning od 'All ravens are black'?". And we conclude that 'All ravens are black' is defined by $\Omega_R \subseteq \Omega_B$. And therefore, in most cases, it can not verified experimentally. Further, roughly speaking we say that

($\flat$) the word "all" cannot be used logically in a posteriori statement

which is being often talked about. Also we show that the ($\flat$) can be justified by quantum language. Therefore, in order to believe in 'All ravens are black', we should rely on hypothesis testing.

(Z_2) [Hume's problem of induction in Section 4.1]: In the linguistic quantum mechanical worldview, Hume's problem of induction (= Theorem 30 [Inductive reasoning]) is essentially the same as the law of large numbers (which is the most basic theorem in science). It may be reasonable. That is because Hume's problem of induction should be the most basic claim in science if there is the answer to this problem.

(Z_3) [Goodman's grue paradox in Section 4.2]: We believe that if Hume's problem of induction is solved, Goodman's grue paradox is immediately solved as its corollary. In fact, Goodman's grue paradox can be easily solved as the corollary of Theorem 30 [Inductive reasoning] in the linguistic quantum mechanical worldview. Thus, Goodman's grue paradox is due to the lack of the understanding of Hume's problem of induction.

(Z_4) [Peirce's abduction in Section 5]: As shown in the formula (11), Peirce's abduction is characterized as the simplified form of Fisher's maximum likelihood method in quantum language.

(Z_5) [The flagpole problem in Section 6]: The confusion concerning the flagpole problem is due to relying only on "logic", and not using quantum language.

Hence we conclude that the reason that these problems are not yet clarified depends on lack of the worldview with the concept of measurement in philosophy of science. Again we emphasize the importance of worldview in science. That is because, if we do not have the worldview, we do not know what to rely on to proceed with the discussion. Therefore, it is no exaggeration to say that there is no science without a scientific worldview.

As mentioned in Remark 1, quantum language does not cover all sciences. However, we consider that familiar sciences are described by quantum language. And further, we believe that quantum language (i.e., the probabilistic interpretation of science) plays a central role in almost familiar sciences. That is, we believe that quantum language gives the mathematical foundations to science.

However, the answer to "What is science?" may not necessarily be determined uniquely. Although there are several aspects of philosophy of science, we believe that it is the central theme of the philosophy of science to find a mathematical structure that is common to almost all sciences. If so, we feel like knowing other interpretations (i.e., other scientific worldview) besides ours (i.e., the probabilistic interpretation of science). Thus, we hope that various mathematical foundations (e.g., category theoretical approach, modal logic approach, etc.) of scientific philosophy will be proposed. And we expect that fundamental problems such as raven problem, problem of induction, grue paradox, etc. will be investigated in these interpretations. And we hope that philosophy of science will progress with such competition.

We hope that our proposal will be examined from various points of view[*3]

[*3] For the further information, see my homepage (http://www.math.keio.ac.jp/~ishikawa/indexe.html)

References

[1] Hempel, C.G. (1945) Studies in the Logic of Confirmation (I), Mind, 54 1-26

[2] Hempel, C.G. (1965) Aspects of Scientific Explanations and other Essays in the Philosophy of Science, Free Press

[3] Ishikawa, S. (1991) Uncertainty relation in simultaneous measurements for arbitrary observables, Rep. Math. Phys., **9**, 257-273, doi: 10.1016/0034-4877(91)90046-P
(`http://dx.doi.org/10.1016/0034-4877(91)90046-P`)

[4] Ishikawa, S. (1997) Fuzzy inferences by algebraic method, Fuzzy Sets and Systems 87, 181–200
doi:10.1016/S0165-0114(96)00035-8
(`http://www.sciencedirect.com/science/article/pii/S0165011496000358`)

[5] Ishikawa, S. (2000) *Statistics in measurements*, Fuzzy sets and systems, Vol. 116, No. 2, 141-154,
doi:10.1016/S0165-0114(98)00280-2

[6] Ishikawa, S. (2006) Mathematical Foundations of Measurement Theory, Keio University Press Inc.
335pages, (`http://www.keio-up.co.jp/kup/mfomt/`)

[7] Ishikawa, S. (2011) A New Interpretation of Quantum Mechanics, Journal of quantum information science, Vol. 1, No. 2, 35-42, doi: 10.4236/jqis.2011.12005
(`http://www.scirp.org/journal/PaperInformation.aspx?paperID=7610`)

[8] Ishikawa, S. (2012) Quantum Mechanics and the Philosophy of Language: Reconsideration of traditional philosophies, Journal of quantum information science, Vol. 2, No. 1, 2-9,
doi: 10.4236/jqis.2012.21002
(`http://www.scirp.org/journal/PaperInformation.aspx?paperID=18194`)

[9] Ishikawa, S. (2012) The linguistic interpretation of quantum mechanics,
arXiv:1204.3892v1[physics.hist-ph] (`http://arxiv.org/abs/1204.3892`)

[10] Ishikawa, S. (2012) A Measurement Theoretical Foundation of Statistics, Applied Mathematics, Vol. 3, No. 3, 283-292, doi: 10.4236/am.2012.33044
(`http://www.scirp.org/journal/PaperInformation.aspx?paperID=18109&`)

[11] Ishikawa, S. (2012) Monty Hall Problem and the Principle of Equal Probability in Measurement Theory, Applied Mathematics, Vol. 3 No. 7, pp. 788-794, doi: 10.4236/am.2012.37117.
(`http://www.scirp.org/journal/PaperInformation.aspx?paperID=19884`)

[12] Ishikawa, S. (2012) Ergodic Hypothesis and Equilibrium Statistical Mechanics in the Quantum Mechanical World View, World Journal of Mechanics, Vol. 2, No. 2, pp. 125-130. doi: 10.4236/wjm.2012.22014.

(`http://www.scirp.org/journal/PaperInformation.aspx?PaperID=18861#.VKevmiusWap`)

[13] Ishikawa, S. (2015) Linguistic interpretation of quantum mechanics; Projection Postulate, Journal of quantum information science, Vol. 5, No.4 , 150-155, DOI: 10.4236/jqis.2015.54017
(`http://www.scirp.org/Journal/PaperInformation.aspx?PaperID=62464`)

[14] Ishikawa, S. (2017) History of Western Philosophy from the quantum theoretical point of view, Research Report (Department of mathematics, Keio university, Yokohama,
[Ver.2] (KSTS-RR-17/004, 2017, 141 pages)
(`http://www.math.keio.ac.jp/academic/research_pdf/report/2017/17004.pdf`)

[15] Ishikawa, S., (2017) A Final solution to mind-body problem by quantum language, Journal of quantum information science, Vol. 7, No.2 , 48-56, DOI: 10.4236/jqis.2017.72005 (`http://www.scirp.org/Journal/PaperInformation.aspx?PaperID=76391`)

[16] Ishikawa, S., (2017) Bell's inequality should be reconsidered in quantum language , Journal of quantum information science, Vol. 7, No.4 , 140-154, DOI: 10.4236/jqis.2017.74011
(`http://www.scirp.org/Journal/PaperInformation.aspx?PaperID=80813`)

[17] Ishikawa, S., (2018) Leibniz-Clarke correspondence, Brain in a vat, Five-minute hypothesis, McTaggart's

paradox, etc. are clarified in quantum language, Open Journal of Philosophy, Vol. 8, No.5 , 466-480, 2018, DOI: 10.4236/ojpp.2018.85032

(`https://www.scirp.org/Journal/PaperInformation.aspx?PaperID=87862`)

[18] Ishikawa, S. (2018) Linguistic Copenhagen interpretation of quantum mechanics; Quantum Language, Research Report, Dept. Math. Keio University, [Ver.4]; KSTS/RR-18/002 (2018); 449 p

(`http://www.math.keio.ac.jp/academic/research_pdf/report/2018/18002.pdf`)

[Ver. 5] is in preparation now. For the information, see my homepage (http://www.math.keio.ac.jp/~ishikawa/indexe.html

[19] Ishikawa, S., (2019) Philosophy of science for scientists; The probabilistic interpretation of science, Journal of quantum information science, Vol. 3, No.3 , 140-154, DOI: 10.4236/jqis.2019.93007

(`https://www.scirp.org/Journal/paperinformation.aspx?paperid=95447`)

[20] Putnam, H. (1981) Reason, Truth and History, Cambridge University Press.

[21] von Neumann, J. (1932) Mathematical foundations of quantum mechanics Springer Verlag, Berlin

[22] Prugovečki, E. (1982) Quantum mechanics in Hilbert space, Academic Press, New York.

[23] Sakai, S. (1971) C^*-algebras and W^*-algebras, Ergebnisse der Mathematik und ihrer Grenzgebiete (Band 60), Springer-Verlag, Berlin, Heidelberg, New York

[24] Yosida, K. (1980) Functional analysis, Springer-Verlag (Sixth Edition)

[25] Davies, E.B. (1976) Quantum theory of open systems, Academic Press

[26] Howard, D. (2004) Who invented the "Copenhagen Interpretation"? A study in mythology, Philosophy of Science, 71 669-682

[27] Kolmogorov, A. (1960) Foundations of the Theory of Probability (Translation), Chelsea Pub Co. Second Edition, New York,

[28] Popper, K.R. (1959) The logic of Scientific Discover, London Hutchinson

[29] Goodman, Nelson (1983) Fact, fiction, and forecast, Harvard University Press

[30] Peirce. C.S. (1958) Collected Papers of Charles Sanders Pierce, Harvard Univ. Press.

Department of Mathematics
Faculty of Science and Technology
Keio University

Research Report

2018

[18/001] Shiro Ishikawa,
Leibniz-Clarke correspondence, Brain in a vat, Five-minute hypothesis, McTaggart's
paradox, etc. are clarified in quantum language,
KSTS/RR-18/001, September 6, 2018 (Revised October 29, 2018)

[18/002] Shiro Ishikawa,
Linguistic Copenhagen interpretation of quantum mechanics: Quantum Language
[Ver. 4],
KSTS/RR-18/002, November 22, 2018

2019

[19/001] Sumiyuki Koizumi,
Contribution to the N. Wiener generalized harmonic analysis and its application to
the theory of generalized Hilbert transforms ,
KSTS/RR-19/001, August 2, 2019 (Revised September 20, 2019)

[19/002] Shiro Ishikawa,
Hempel's raven paradox, Hume's problem of induction, Goodman's grue paradox,
Peirce's abduction, Flagpole problem are clarified in quantum language,
KSTS/RR-19/002, September 12, 2019 (Revised November 1, 2019)