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Bell's inequality is violated in classical systems as well as quantum systems

by

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Abstract

Bell's inequality is usually considered to belong to mathematics and not quantum theory. We think that this complicates understanding of Bell's theorem. Thus in this paper, contrary to Bell's spirit (which inherits Einstein's spirit), we try to discuss Bell's inequality in the framework of quantum theory with the linguistic Copenhagen interpretation. And we clarify that whether or not Bell's inequality holds does not depend on whether classical systems or quantum systems, but depend on whether a kind of combined measurements exist or not. And further we assert that Bell's argument (based on Einstein's spirit) should be regarded as a philosophical representation of our scientific argument (based on the linguistic Copenhagen interpretation). Thus, we conclude that our inequality proposed in this paper is the true Bell's inequality.

Key phrases: EPR-paradox, Bell's inequality, Classical systems, Linguistic Copenhagen interpretation, Quantum Language

1 Review: Quantum language (= Measurement theory);

Quantum language has two formulations, i.e., the C^*-algebraic formulation and W^*-algebraic formulation, where the former is elementary, handy and somewhat rough. In refs. [20, 21], we discussed Bell's inequality in the W^*-algebraic formulation, which is exact, powerful and so mathematically difficult. In this paper, we discussed Bell's inequality in the C^*-algebraic formulation, which may be expected to be easy to understand.

Following refs. [12, 13, 21], we shall review quantum language (i.e., the linguistic Copenhagen interpretation of quantum mechanics, or measurement theory), which has the following form:

$$\text{(A)} \quad \underset{\text{(= measurement theory)}}{\boxed{\boxed{\text{Quantum langage}}}} = \underset{\text{(Axiom 1)}}{\boxed{\text{measurement}}} + \underset{\text{(Axiom 2)}}{\boxed{\text{causality}}} + \underset{\text{(how to use Axioms 1 and 2)}}{\boxed{\text{linguistic (Copenhagen) interpretation}}}$$

We assert that the location of quantum language in the history of world-description is as follows.

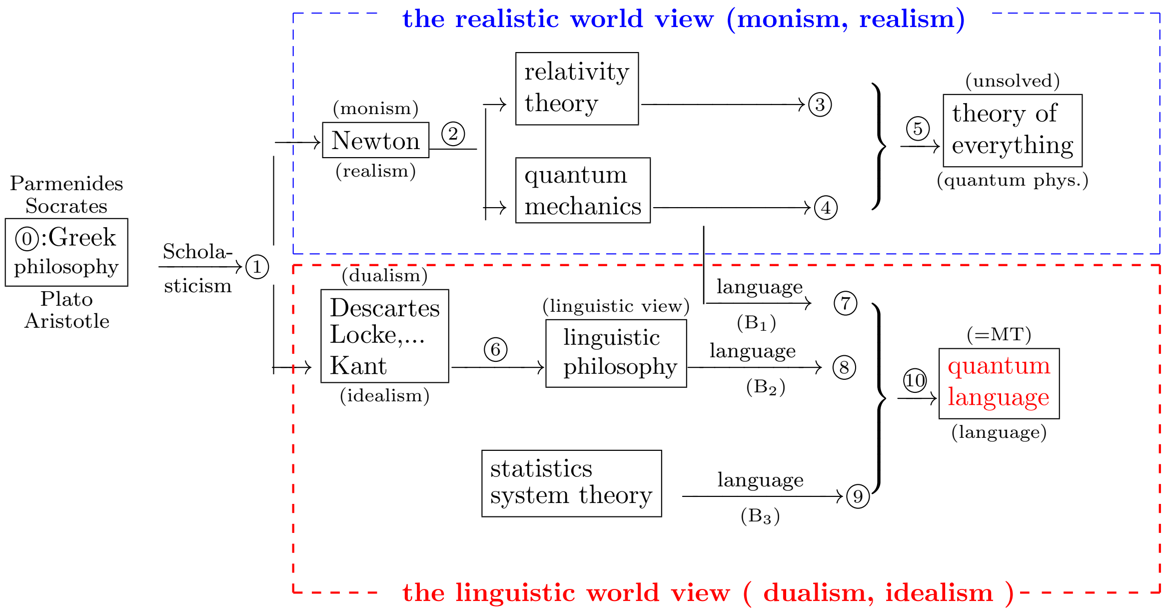

Figure 1: The history of the world-description

And we think that the following four are equivalent:

(B_0) to propose quantum language (*cf.* ⑩ in Figure 1, ref.[21])

(B_1) to clarify the Copenhagen interpretation of quantum mechanics (*cf.* ⑦ in Figure 1, refs.[12, 17, 8]), that is, the linguistic Copenhagen interpretation is the true figure of so-called Copenhagen interpretation

(B_2) to clarify the final goal of the dualistic idealism (Descartes=Kant epistemology) (*cf.* ⑧ in Figure 1, refs.[13, 18, 19])

(B_3) to reconstruct statistics (= dynamical system theory) in the dualistic idealism (*cf.* ⑨ in Figure 1, refs.[10, 14, 15, 16])

Bell's theorem (*cf.* ref. [1]) is clearly one of consequences of Bohr-Einstein debates (refs. [6, 2]). Also, our Figure 1 (= quantum language) is also another consequence of Bohr-Einstein debates as follows.

Einstein's standing-point (that is, "*the moon is there whether one looks at it or not*" (i.e., physics holds without observers)) is on the side of the realistic world view (in Figure 1). On the other hand, Bohr's standing-point (that is, "*To be is to be perceived*" due to Bishop Berkeley (i.e., there is no science without measurements)) is on the side of the linguistic world view (in Figure 1)[1].

Note that the great disputes in the history of the world view are always formed as follows:

$$\underset{\text{(monistic realism)}}{\overset{\text{Einstein,...}}{\boxed{\text{realistic world view}}}} \quad \underset{\text{v.s.}}{\longleftrightarrow} \quad \underset{\text{(dualistic idealism)}}{\overset{\text{Bohr,...}}{\boxed{\text{linguistic world view}}}}$$

For example,

Table 1 : The realistic world view vs. the linguistic world view

Dispute ╲ R vs. L	the realistic world view	the linguistic world view
Greek philosophy	Aristotle	Plato
Problem of universals	Nominalisme(William of Ockham)	Realismus(Anselmus)
Space·times	Clarke(Newton)	Leibniz
Quantum mechanics	Einstein (*cf.* [6])	Bohr (*cf.* [2])

(*cf.* Note 10.7 in Chapter 10 or ref. [21], or more precisely, ref. [19]).

Although Bohr-Einstein debates (and so, Bell's theorem) might intend unify the realistic world view and the linguistic world view, quantum language (= Figure 1) declares the **consistence** of the realistic world view and the linguistic world view.

[1]Bohr might believe that his theory (i.e., "Copenhagen interpretation") belongs to physics (i.e., the realistic world view), we think that his Copenhagen interpretation should belong to the linguistic world view as the linguistic Copenhagen interpretation.

1.1 Mathematical Preparations

Consider an operator algebra $B(H)$ (i.e., an operator algebra composed of all bounded linear operators on a Hilbert space H with the norm $\|F\|_{B(H)} = \sup_{\|u\|_H=1} \|Fu\|_H$). Let $\mathcal{A}(\subseteq B(H))$ be a C^*-algebra, i.e, a $*$-subalgebra of $B(H)$ (*cf.* refs. [12, 21, 23]). For simplicity, in this paper we always assume that the C^*-algebra has the identity I. Let $\mathcal{A}^*$ be the dual Banach space of $\mathcal{A}$. That is, $\mathcal{A}^* = \{\rho \mid \ \rho$ is a continuous linear functional on $\mathcal{A}$ $\}$, and the norm $\|\rho\|_{\mathcal{A}^*}$ is defined by $\sup\{|\rho(F)| \mid \ F \in \mathcal{A}$ such that $\|F\|_{\mathcal{A}}(= \|F\|_{B(H)}) \le 1\}$. Define the *mixed state* ρ $(\in \mathcal{A}^*)$ such that $\|\rho\|_{\mathcal{A}^*} = 1$ and $\rho(F) \ge 0$ for all $F \in \mathcal{A}$ such that $F \ge 0$. And define the mixed state space $\mathfrak{S}^m(\mathcal{A}^*)$ such that

$$\mathfrak{S}^m(\mathcal{A}^*)=\{\rho \in \mathcal{A}^* \mid \rho \text{ is a mixed state}\}.$$

A mixed state $\rho(\in \mathfrak{S}^m(\mathcal{A}^*))$ is called a *pure state* if it satisfies that "$\rho = \theta\rho_1 + (1-\theta)\rho_2$ for some $\rho_1, \rho_2 \in \mathfrak{S}^m(\mathcal{A}^*)$ and $0 < \theta < 1$" implies "$\rho = \rho_1 = \rho_2$". Put

$$\mathfrak{S}^p(\mathcal{A}^*)=\{\rho \in \mathfrak{S}^m(\mathcal{A}^*) \mid \rho \text{ is a pure state}\},$$

which is called a *state space.*

Note that a commutative C^*-algebara can be characterizes as $C(\Omega)$, i.e., the algebra composed of all complex valued continuous functions on a compact set Ω. Also, in this paper, as a -commutative C^*-algebra, we devote ourselves to $B(\mathbb{C}^n)$, i.e., $\mathbb{C}^n$ is finite dimensional Hilbert space. Thus, in this paper, the measurement theory (=quantum language) is classified as follows.

$$\text{(C) measurement theory (A)} = \begin{cases} (\mathrm{C}_1)\text{: quantum systems (when } \mathcal{A} = B(\mathbb{C}^n),\ B(\mathbb{C}^2) \otimes B(\mathbb{C}^2)\text{, etc.)} \\ \qquad \text{pure state } \rho = |u\rangle\langle u| \text{ (where } u \in \mathbb{C}^n,\ \|u\|_{\mathbb{C}^n} = 1) \\ \\ (\mathrm{C}_2)\text{: classical systems (when } \mathcal{A} = C(\Omega),\ C(\Omega \times \Omega)\text{, etc.)} \\ \qquad \text{pure state } \rho = \delta_\omega \text{ (where } \delta_\omega \text{ is the point measure at } \omega,\ (\omega \in \Omega) \end{cases}$$

An *observable* O :=$(X, \mathcal{F}, F)$ in $\mathcal{A}$ (or, a *measuring instrument* O :=$(X, \mathcal{F}, F)$ in $\mathcal{A}$) is defined as follows:

(i) [field] X is a set, $\mathcal{F}(\subseteq \mathcal{P}(X)$, the power set of X) is a field of X, that is, "$\Xi_1, \Xi_2 \in \mathcal{F} \Rightarrow \Xi_1 \cup \Xi_2 \in \mathcal{F}$", "$\Xi \in \mathcal{F} \Rightarrow X \setminus \Xi \in \mathcal{F}$", "$\emptyset \in \mathcal{F}$".

(ii) [finite additivity] F is a mapping from $\mathcal{F}$ to $\mathcal{A}$ satisfying: (a): for every $\Xi \in \mathcal{F}$, $F(\Xi)$ is a non-negative element in $\mathcal{A}$ such that $0 \le F(\Xi) \le I$, (b): $F(\emptyset) = 0$ and $F(X) = I$, where 0 and I is the 0-element and the identity in $\mathcal{A}$ respectively. (c): for any $\Xi_1, \Xi_2 (\in \mathcal{F})$, it holds that $F(\Xi_1 \cup \Xi_2) = F(\Xi_1) + F(\Xi_2) - F(\Xi_1 \cap \Xi_2)$ in $\mathcal{A}$.

Remark 1. We have two formulations of quantum language, i.e., the C^*-algebraic formulation and the W^*-algebraic formulation. In the former, the above additivity (ii) is finite, on the other hand, in the later the countably additivity is required. Thus, from the mathematical point of view, the W^*-algebraic formulation is not easy than the C^*-algebraic formulation. However, since the C^*-algebraic formulation is handy, we devote ourselves to the C^*-algebraic formulation in this paper. If the readers want to the mathematically exact discussion, they are recommended to progress to the W^*-algebraic formulation in refs.[20, 21].

1.2 Axiom 1 [Measurement] and Axiom 2 [Causality] in Quantum Language

Quantum language (A) is composed of two axioms (i.e., Axioms 1 and 2) as follows. With any *system* S, a C^*-algebra $\mathcal{A}(\subseteq B(H))$ can be associated in which the measurement theory (A) of that system can be formulated. A *state* of the system S is represented by an element $\rho(\in \mathfrak{S}^p(\mathcal{A}^*))$ and an *observable* is represented by an observable $\mathsf{O} :=(X, \mathcal{F}, F)$ in $\mathcal{A}$. Also, the *measurement of the observable* O *for the system* S *with the state* ρ (or *the measurement for the system* S *with the state* ρ *by the measuring instrument* O) is denoted by $\mathsf{M}_{\mathcal{A}}(\mathsf{O}, S_{[\rho]})$ $\big($ or more precisely, $\mathsf{M}_{\mathcal{A}}(\mathsf{O} :=(X, \mathcal{F}, F), S_{[\rho]})$ $\big)$. An observer can obtain a measured value x $(\in X)$ by the measurement $\mathsf{M}_{\mathcal{A}}(\mathsf{O}, S_{[\rho]})$.

The Axiom 1 presented below is a kind of mathematical generalization of Born's probabilistic interpretation of quantum mechanics (*cf.* ref.[3]). And thus, it is a statement without reality.

Now we can present Axiom 1 in the C^*-algebraic formulation as follows.

Axiom 1 [Measurement]. *The probability that a measured value* x $(\in X)$ *obtained by the measurement* $\mathsf{M}_{\mathcal{N}}(\mathsf{O} :=(X, \mathcal{F}, F), S_{[\rho]})$ *(i.e., measurement of the observable* O *for the system* S *with the state* ρ, or *measurement for the system* S *with the state* ρ *by the measuring instrument* O *) belongs to a set* $\Xi(\in \mathcal{F})$ *is given by* $\rho(F(\Xi))$.

Next, we explain Axiom 2. Let $\mathcal{A}_1(\subseteq B(H_1))$ and $\mathcal{A}_2(\subseteq B(H_2))$ be C^*-algebras. A continuous linear operator $\Phi_{1,2} : \mathcal{A}_2 \to \mathcal{A}_1$ is called a *Markov operator,* if it satisfies that (i): $\Phi_{1,2}(F_2) \geq 0$ for any non-negative element F_2 in $\mathcal{A}_2$, (ii): $\Phi_{1,2}(I_2) = I_1$, where I_k is the identity in $\mathcal{A}_k$, $(k = 1, 2)$.

It is clear that the dual operator $\Phi_{1,2}^* : \mathcal{A}_1^* \to \mathcal{A}_2^*$ satisfies that $\Phi_{1,2}^*(\mathfrak{S}^m(\mathcal{A}_1^*)) \subseteq \mathfrak{S}^m(\mathcal{A}_2^*)$. If it holds that $\Phi_{1,2}^*(\mathfrak{S}^p(\mathcal{A}_1^*)) \subseteq \mathfrak{S}^p(\mathcal{A}_2^*)$, the $\Phi_{1,2}$ is said to be deterministic. If it is not deterministic, it is said to be non-deterministic. Also note that, for any observable $\mathsf{O}_2 :=(X, \mathcal{F}, F_2)$ in $\mathcal{A}_2$, the $(X, \mathcal{F}, \Phi_{1,2}F_2)$ is an observable in $\mathcal{A}_1$.

Now Axiom 2 is presented as follows. (For details, see ref. [21].)

Axiom 2 [Causality]. *Let* $t_1 \leq t_2$. *The causality is represented by a Markov operator* $\Phi_{t_1,t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}$.

Remark 1 In dualistic idealism we have the two most important problems. One is the mind-body problem (*cf.* ref. [18, 19]). Another is the causality problem, i.e., "What is causality?". Note (*cf.* ref. [21])that the solution to the causality problem is given by Axiom 2.

1.3 The linguistic (Copenhagen) interpretation (= the manual to use Axioms 1 and 2)

In the above, Axioms 1 and 2 are kinds of spells, (i.e., incantation, magic words, metaphysical statements), and thus, it is nonsense to verify them experimentally. Therefore, what we should do is not "to understand" but "to use". After learning Axioms 1 and 2 by rote, we have to improve how to use them through trial and error.

We can do well even if we do not know the linguistic interpretation (= the manual to use Axioms 1 and 2). However, it is better to know the linguistic interpretation , if we would like to make progress quantum language early. I believe that the linguistic interpretation is the true Copenhagen interpretation (though there are several opinions for "Copenhagen interpretation" *cf.* ref. [7]).
The linguistic interpretation says that

(D) **Only one measurement is permitted.** And thus, the state after a measurement is meaningless since it can not be measured any longer. Thus, the collapse of the wavefunction is prohibited. We are not concerned with anything after measurement. Strictly speaking, the phrase "*after the measurement*" should not be used (*cf.* [17]). Also, the causality should be assumed only in the side of system, however, a state never moves. Thus, the Heisenberg picture should be adopted, and thus, the Schrödinger picture should be prohibited. Also, there is no probability (thus, no probability space) without a measurement.

and so on. For details, see ref. [21].

1.4 Simultaneous measurement (or, product observable), parallel measurement

Definition 2. (i): Consider observables $\mathsf{O}_k = (X_k, \mathcal{F}_k, F_k)$ $(k = 1, 2, ..., K)$ in a C^*-algebra $\mathcal{A}$. Let $(\times_{k=1}^n X_k,\ \boxtimes_{k=1}^n \mathcal{F}_k)$ be the product measurable space of $(X_k, \mathcal{F}_k)$, $k = 1, 2, ..., n$. An observable $\mathsf{O} = \times_{k=1}^n \mathsf{O}_k = (\times_{k=1}^n X_k,\ \boxtimes_{k=1}^n \mathcal{F}_k, F)$ in $\mathcal{A}$ is called the simultaneous observable (or, product observable) of O_k $(k = 1, 2, ..., K)$, if it holds that

$$\mathop{\times}_{k=1}^{n} F_k(\Xi_k) = F(\mathop{\times}_{k=1}^{n} \Xi_k) \qquad (\forall \Xi_k \in \mathcal{F}_k) \tag{1}$$

Also, the measurement $\mathsf{M}_{\mathcal{A}}(\mathsf{O}, S_{[\rho_0]})$ is called a simultaneous measurement of measurements $\mathsf{M}_{\mathcal{A}}(\mathsf{O}_k, S_{[\rho_0]})$ $(k = 1, 2, ..., n)$. Note that a simultaneous observable $\mathsf{O} = (\times_{k=1}^n X_k,\ \boxtimes_{k=1}^n, F)$ in $\mathcal{A}$ always exists if observables $\mathsf{O} = (\times_{k=1}^n X_k,\ \boxtimes_{k=1}^n, F)$ commute, i.e.,

$$F_k(\Xi_k) F_l(\Xi_l) = F_l(\Xi_l) F_k(\Xi_k) \qquad (\Xi_k \in \mathcal{F}_l, \Xi_l \in \mathcal{F}_k, k \neq l) \tag{2}$$

(ii): Consider measurements $\mathsf{M}_{\mathcal{A}_k}(\mathsf{O}_k = (X_k, \mathcal{F}_k, F_k), S_{[\rho_k]})$ $(k = 1, 2, ..., n)$ in a C^*-algebra $\mathcal{A}_k$. Let $\otimes_{k=1}^n \mathsf{O}_k = (\mathop{\times}_{k=1}^n X_k,\ \boxtimes_{k=1}^n \mathcal{F}_k,\ \otimes_{k=1}^n F_k)$ be a parallel observable in a tensor C^*-algebra $\otimes_{k=1}^n \mathcal{A}_k$. And let $\otimes_{k=1}^n \rho_k$ $(\in \mathfrak{S}^p((\otimes_{k=1}^n \mathcal{A}_k)^*))$. Then, the measurement $\mathsf{M}_{\otimes_{k=1}^n \mathcal{A}_k}(\otimes_{k=1}^n \mathsf{O}_k = (X_k, \mathcal{F}_k, F_k), S_{[\otimes_{k=1}^n \rho_k]})$ is called a prallel measurement of $\mathsf{M}_{\mathcal{A}_k}(\mathsf{O}_k = (X_k, \mathcal{F}_k, F_k), S_{[\rho_k]})$ $(k = 1, 2, ..., K)$. Cf. [12, 21].

1.5 Bell's inequality in mathematics

Bell's inequality is generally considered to be important in the relation of "the hidden variable". However, in this paper I assert that Bell's inequality should be reconsidered in quantum language and not in mathematics.

Firstly, let us mention Bell's inequality in mathematics.

Theorem 3. [The conventional Bell's inequality (*cf.* refs. [1, 4, 22, 24])] *The mathematical Bell's inequality is as follows: Let $(\Theta, \mathcal{B}, P)$ be a probability space. Let $(f_1, f_2, f_3, f_4) : \Theta \to X^4 (\equiv \{-1, 1\}^4)$ be a measurable functions. Define the correlation functions $\widetilde{R}_{ij} (i = 1, 2, j = 3, 4)$ by $\int_\Theta f_i(\theta) f_j(\theta) P(d\theta)$. Then, the following mathematical Bell's inequality (or precisely, CHSH inequality (cf. ref. [4])) holds:*

$$|\widetilde{R}_{13} - \widetilde{R}_{14}| + |\widetilde{R}_{23} + \widetilde{R}_{24}| \le 2 \tag{3}$$

Proof. It is easy as follows.

$$\begin{aligned}
&\text{“the left-hand side of the above eq.(3)”}\\
&\le \int_\Theta |f_3(\theta) - f_4(\theta)| P(d\theta) + \int_\Theta |f_3(\theta) + f_4(\theta)| P(d\theta) \le 2
\end{aligned}$$

This completes the proof.

□

This theorem is too easy, but we must remember the linguistic interpretation:

(F) There is no probability (or, no probability space) without measurements.

Thus, in this paper, we discuss "What is the probability space in Theorem 3?".

2 Bell's inequality should be reconsidered in quantum language

2.1 Bell's inequality holds in both classical and quantum systems

Now let us consider a kind of generalization of the simultaneous observable as follows.

Definition 4. [Combinable, Combined observable(*cf.* ref. [10])] Let $\mathcal{A}(\subseteq B(H))$ be a C^*-algebra. Let $\{S_1, S_2, ..., S_j\}$ be a family (i.e., a set of sets) such that $S_l \subseteq \{1, 2, ..., n\}$ $(\forall l = 1, 2, ..., j\})$. For each $l \in \{1, 2, ..., j\}$, consider an observable $\mathsf{O}_l = (\times_{s \in S_l} X_s, \ \boxtimes_{s \in S_l} \mathcal{F}_s, \ F_l)$ in a C^*-algebra $\mathcal{A}$, and define a natural map $\pi_l : \times_{k=1,2,...,n} X_k \to \times_{s \in S_l} X_s$ such that

$$\times_{k=1,2,...,n} X_k \ni (x_k)_{k=1,2,...n} \mapsto (x_k)_{k \in S_l} \in \times_{k \in S_l} X_k$$

Here, the $\{\mathsf{O}_l : l = 1, 2, ..., j\}$ is said to be **combinable**, if there exists an observable $\mathsf{O} = (\times_{k=1,2,...,n} X_k, \boxtimes_{k=1,2,...,n} \mathcal{F}_k, F)$ in $\mathcal{A}$ such that

$$F(\pi_l^{-1}(\times_{s \in S_l} \Xi_s)) = F_l(\times_{s \in S_l} \Xi_s) \quad (\Xi_s \in \mathcal{F}_s, s \in S_l)$$

Also, the observable O is called a **combined observable** of $\{\mathsf{O}_l : l = 1, 2, ..., j\}$

Note that, for each l, a measurement $\mathsf{M}_{\mathcal{A}}(\mathsf{O}_l, S_{[\rho_0]})$ is included in $\mathsf{M}_{\mathcal{A}}(\mathsf{O}, S_{[\rho_0]})$. That is, it suffices to take only a measurement $\mathsf{M}_{\mathcal{A}}(\mathsf{O}, S_{[\rho_0]})$.

Quantum language (i.e., the linguistic Copenhagen interpretation) says that a combined observable is the most fundamental than a simultaneous observable , a quesi-product observable, etc.

Example 5. [Quasi-product observable] Let $\mathcal{A}(\subseteq B(H))$ be a C^*-algebra. Let $\{S_1, S_2, ..., S_n\}$ be a family (i.e., a set of sets) such that $S_l = \{l\}$ $(\forall l = 1, 2, ..., n\})$. For each $l \in \{1, 2, ..., n\}$, consider an observable O_l $= (X_l, \mathcal{F}_l, F_l)$ in a C^*-algebra $\mathcal{A}$, and define a natural map $\pi_l : \times_{k=1,2,...,n} X_k \to X_l$ such that

$$\times_{k=1,2,...,n} X_k \ni (x_k)_{k=1,2,...n} \mapsto x_l \in X_l$$

Here, the $\{\mathsf{O}_l : l = 1, 2, ..., l\}$ is said to be combinable, if there exists an observable $\mathsf{O} = (\times_{k=1,2,...,n} X_k, \boxtimes_{k=1,2,...,n} \mathcal{F}_k, F)$ in $\mathcal{A}$ such that

$$F(\pi_l^{-1}(\Xi_l)) = F(X_1 \times X_2 \times ... \times X_{l-1} \times \Xi_l \times X_{l+1} \times ... \times X_n)) = F_l(\Xi_l) \quad (\Xi_l \in \mathcal{F}_l, l = 1, 2, ..., n)$$

Also, the combined observable observable O is called a **quasi-product observable** of $\{\mathsf{O}_l : l = 1, 2, ..., n\}$. Note that a simultaneous observable is a kind of quasi-product observable.

Example 6. [Combined observable in syllogism] Let $\mathcal{A}(\subseteq B(H))$ be a C^*-algebra. Let $\{S_1, S_2\}$ be a family (i.e., a set of sets) such that $S_1 = \{l, 2\}$ and $S_2 = \{2, 3\}$. For each S_1, S_2, consider observables $\mathsf{O}_{S_1} = (X_1 \times X_2, \mathcal{F}_1 \boxtimes \mathcal{F}_2, F_l)$ and $\mathsf{O}_{S_2} = (X_2 \times X_3, \mathcal{F}_2 \boxtimes \mathcal{F}_3, F_{S_2})$ in a C^*-algebra $\mathcal{A}$, and define natural maps $\pi_1 : \times_{k=1,2,3} X_k \to \times_{k=1,2} X_1$ and $\pi_2 : \times_{k=1,2,3} X_k \to \times_{k=2,3} X_k$ such that

$$(x_1, x_2, x_3) \underset{\pi_1}{\mapsto} (x_1, x_2), \qquad (x_1, x_2, x_3) \underset{\pi_2}{\mapsto} (x_2, x_3),$$

Here, the $\{\mathsf{O}_{S_1}, \mathsf{O}_{S_2}\}$ is said to be combinable, if there exists an observable $\mathsf{O} = (\times_{k=1,2,3} X_k, \boxtimes_{k=1,2,3} \mathcal{F}_k,$

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F) in $\mathcal{A}$ such that

$$F(\Xi_1 \times \Xi_2 \times X_3) = F_{S_1}(\Xi_1 \times \Xi_2), \quad F(X_1 \times \Xi_2 \times \Xi_3) = F_{S_2}(\Xi_2 \times \Xi_3) \quad (\Xi_k \in \mathcal{F}_k, k = 1, 2, 3)$$

This combined observable observable O plays an important role in the proof of the classical syllogism (*cf.* ref. [10]). That is, syllogism (i.e., $[[A \Rightarrow B] \wedge [B \Rightarrow C]] \Rightarrow [A \Rightarrow C]$) does not hold in quantum systems but in classical systems (*cf.* ref. [21]).

In this paper we devote ourselves to the following combined observable.

Example 7. [Combined observable related to Bell test experiment] Let $\mathcal{A}(\subseteq B(H))$ be a C^*-algebra. **Put** $X = \{-1, 1\}$. Let $\mathsf{O}_1 = (X, \mathcal{P}(X), F_1)$, $\mathsf{O}_2 = (X, \mathcal{P}(X), F_2)$, $\mathsf{O}_3 = (X, \mathcal{P}(X), F_3)$, $\mathsf{O}_4 = (X, \mathcal{P}(X), F_3)$ be observables in $\mathcal{A}$. Consider four observables: $\mathsf{O}_{13} = (X^2, \mathcal{P}(X^2), F_{13})$, $\mathsf{O}_{14} = (X^2, \mathcal{P}(X^2), F_{14})$, $\mathsf{O}_{23} = (X^2, \mathcal{P}(X^2), F_{23})$, $\mathsf{O}_{24} = (X^2, \mathcal{P}(X^2), F_{24})$ in $\mathcal{A}$ such that

$$\begin{aligned}
&F_{13}(\{x\} \times X) = F_{14}(\{x\} \times X) = F_1(\{x\}) \\
&F_{23}(\{x\} \times X) = F_{24}(\{x\} \times X) = F_2(\{x\}) \\
&F_{13}(X \times \{x\}) = F_{23}(X \times \{x\}) = F_3(\{x\}) \\
&F_{14}(X \times \{x\}) = F_{24}(X \times \{x\}) = F_4(\{x\})
\end{aligned} \tag{4}$$

for any $x \in \{-1, 1\}$. The four observables O_{13}, O_{14}, O_{23} and O_{24} are said to be combinable if there exists an observable $\mathsf{O} = (X^4, \mathcal{P}(X^4), F)$ in $\mathcal{A}$ such that

$$\begin{aligned}
&F_{13}(\{(x_1, x_3)\}) = F(\{x_1\} \times X \times \{x_3\} \times X), \quad F_{14}(\{(x_1, x_4)\}) = F(\{x_1\} \times X \times X \times \{x_4\}) \\
&F_{23}(\{(x_2, x_3)\}) = F(X \times \{x_2\} \times \{x_3\} \times X), \quad F_{24}(\{(x_2, x_4)\}) = F(X \times \{x_2\} \times X \times \{x_4\})
\end{aligned} \tag{5}$$

for any $(x_1, x_2, x_3, x_4) \in X^4$. (Note that the formula (5) implies (4). The condition (4) is not needed.) The observable O is said to be a combined observable of O_{ij} $(i = 1, 2, j = 3, 4)$. Also, the measurement $\mathsf{M}_{\mathcal{A}}(\mathsf{O} = (X^4, \mathcal{P}(X^4), F), S_{[\rho_0]})$ is called the combined measurement of $\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{13}, S_{[\rho_0]})$, $\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{14}, S_{[\rho_0]})$, $\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{23}, S_{[\rho_0]})$ and $\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{24}, S_{[\rho_0]})$.

The following theorem is all of our insistence concerning Bell's inequality. We assert that this is the true Bell's inequality.

Theorem 8. [Bell's inequality in quantum language] *Let $\mathcal{A}(\subseteq B(H))$ be a C^*-algebra. Put $X = \{-1, 1\}$. Fix the pure state $\rho_0(\in \mathfrak{S}^p(\mathcal{A}^*))$. And consider the four measurements*

$\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{13} = (X^2, \mathcal{P}(X^2), F_{13}), S_{[\rho_0]})$, $\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{14} = (X^2, \mathcal{P}(X^2), F_{14}), S_{[\rho_0]})$, $\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{23} = (X^2, \mathcal{P}(X^2), F_{23}), S_{[\rho_0]})$ *and* $\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{24} = (X^2, \mathcal{P}(X^2), F_{24}), S_{[\rho_0]})$. *Or equivalently, consider the parallel measurement* $\otimes_{i=1,2,j=3,4}\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{ij} = (X^2, \mathcal{P}(X^2), F_{ij}), S_{[\rho_0]})$. *Define four correlation functions* $(i = 1, 2, j = 3, 4)$ *such that*

$$R_{ij} = \sum_{(u,v)\in X\times X} u \cdot v \ \ \rho_0(F_{ij}(\{(u, v)\}))$$

Assume that four observables $\mathsf{O}_{13} = (X^2, \mathcal{P}(X^2), F_{13})$, $\mathsf{O}_{14} = (X^2, \mathcal{P}(X^2), F_{14})$, $\mathsf{O}_{23} = (X^2, \mathcal{P}(X^2), F_{23})$ *and* $\mathsf{O}_{24} = (X^2, \mathcal{P}(X^2), F_{24})$ *are combinable, that is, we have the combined observable* $\mathsf{O} = (X^4, \mathcal{P}(X^4), F)$ *in* $\mathcal{A}$ *such that it satisfies the formula (5). Then we have a combined measurement* $\mathsf{M}_{\mathcal{A}}(\mathsf{O} = (X^4, \mathcal{P}(X^4), F), S_{[\rho_0]})$ *of* $\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{13}, S_{[\rho_0]})$, $\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{14}, S_{[\rho_0]})$, $\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{23}, S_{[\rho_0]})$ *and* $\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{24}, S_{[\rho_0]})$. *And further, we have Bell's inequality in quantum language as follows.*

$$|R_{13} - R_{14}| + |R_{23} + R_{24}| \leqq 2 \tag{6}$$

In this paper, we assert that this (6) is the true Bell's inequality.

Proof. Clearly we see, $i = 1, 2, j = 3, 4,$

$$R_{ij} = \sum_{(x_1,x_2,x_3,x_4)\in X\times X\times X\times X} x_i \cdot x_j \ \ \rho_0(F(\{(x_1, x_2, x_3, x_4)\})) \tag{7}$$

$\big($ for example, $R_{13} = \sum_{(x_1,x_2,x_3,x_4)\in X\times X\times X\times X} x_1 \cdot x_3 \ \ \rho_0(F(\{(x_1, x_2, x_3, x_4)\}))$ $\big)$. Therefore, we see that

$$\begin{aligned}
&|R_{13} - R_{14}| + |R_{23} + R_{24}| \\
=& \sum_{(x_1,x_2,x_3,x_4)\in X\times X\times X\times X} \Big[|x_1 \cdot x_3 - x_1 \cdot x_4| + |x_2 \cdot x_3 + x_2 \cdot x_4|\Big]\rho_0(F(\{(x_1, x_2, x_3, x_4)\})) \\
=& \sum_{(x_1,x_2,x_3,x_4)\in X\times X\times X\times X} \Big[|x_3 - x_4| + |x_3 + x_4|\Big]\rho_0(F(\{(x_1, x_2, x_3, x_4)\})) \leq 2
\end{aligned}$$

This completes the proof. □

As the corollary of this theorem, we have the followings:

Corollary 9. *Consider the parallel measurement* $\otimes_{i=1,2,j=3,4}\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{ij} = (X^2, \mathcal{P}(X^2), F_{ij}), S_{[\rho_0]})$ *as in Theorem 8. Let*

$$x = \big((x^1_{13}, x^2_{13}), (x^1_{14}, x^2_{14}), (x^1_{23}, x^2_{23}), (x^1_{24}, x^2_{24})\big) \in X^8 (\equiv \{-1, 1\}^8)$$

be a measured value of the parallel measurement $\otimes_{i=1,2,j=3,4}\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{ij} = (X^2, \mathcal{P}(X^2), F_{ij}), S_{[\rho_0]})$. *Let* N *be sufficiently large natural number. Consider* N*-parallel measurement* $\bigotimes_{n=1}^N$ $[$ $\otimes_{i=1,2,j=2,3}$ $\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{ij} :=$

$(X^2, \mathcal{P}(X^2), F_{ij}), S_{[\rho_0]})$]. Let $\{x^n\}_{n=1}^N$ be the measured value. That is,

$$\{x^n\}_{n=1}^N = \begin{bmatrix} ((x_{13}^{1,1}, x_{13}^{2,1}), (x_{14}^{1,1}, x_{14}^{2,1}), (x_{23}^{1,1}, x_{23}^{2,1}), (x_{24}^{1,1}, x_{24}^{2,1})) \\ ((x_{13}^{1,2}, x_{13}^{2,2}), (x_{14}^{1,2}, x_{14}^{2,2}), (x_{23}^{1,2}, x_{23}^{2,2}), (x_{24}^{1,2}, x_{24}^{2,2})) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \Big((x_{13}^{1,N}, x_{13}^{2,N}), (x_{14}^{1,N}, x_{14}^{2,N}), (x_{23}^{1,N}, x_{23}^{2,N}), (x_{24}^{1,N}, x_{24}^{2,N})\Big) \end{bmatrix} \in (X^8)^N$$

Here, note that the law of large numbers says: for sufficiently large N,

$$R_{ij} \approx \frac{1}{N} \sum_{n=1}^N x_{ij}^{1,n} x_{ij}^{2,n} \qquad (i = 1, 2, j = 3, 4).$$

Then, it holds, by the formula (6), that

$$|\sum_{n=1}^N \frac{x_{13}^{1,n} x_{13}^{2,n}}{N} - \sum_{n=1}^N \frac{x_{14}^{1,n} x_{14}^{2,n}}{N}| + |\sum_{n=1}^N \frac{x_{23}^{1,n} x_{23}^{2,n}}{N} + \sum_{n=1}^N \frac{x_{24}^{1,n} x_{24}^{2,n}}{N}| \le 2, \tag{8}$$

which is also called Bell's inequality in quantum language.

Remark 10. [(i):The conventional Bell's inequality (*cf.* refs. [4, 22, 24])] From the mathematical point of view, the formulas (3) and (6) are the same. However, the probability space $(X^4, \mathcal{P}(X^4), \rho_0(F(\cdot)))$ in Theorem 8 is visible and concrete.

[(ii): "true value" (or, "hidden value")] In Theorem 8, we have the combined measurement $\mathsf{M}_{\mathcal{A}}(\mathsf{O} = (X^4, \mathcal{P}(X^4), F), S_{[\rho_0]})$. Thus, some may consider that

- the true value (x_1, x_2, x_3, x_4) (of observables O_k, $k = 1, 2, 3, 4$ in Example 7) can be obtained by the measurement $\mathsf{M}_{\mathcal{A}}(\mathsf{O} = (X^4, \mathcal{P}(X^4), F), S_{[\rho_0]})$.

No-Go theorem (*cf.* [22]) is usually mentioned in terms of Einstein's world view. However,

- If No-Go theorem is mentioned in terms of Bohr's world view, we think that No-Go theorem is the existence theorem of the combined observable.

2.2 "Bell's inequality" is violated in classical systems as well as quantum systems

In the previous section, we show that Theorem 8 (or Corollary 9) says

(F_1) Under the combinable condition (*cf.* Example 7), Bell's inequality (6) (or, (8)) holds in both classical systems and quantum systems.

Or, equivalently,

(F_2) If Bell's inequality (6) (or, (8)) is violated, then the combined observable does not exist, and thus, we cannot obtain the measured value (by the combined measurement).

Remark 11. [Heisenberg's uncertainty principle (*cf.* [8])] The above (F_2) is similar to the following well-known statement in quantum mechanics:

(F'_2) We have no simultaneous measurement (= combined measurement) of the position observable Q and the momentum observable P, and thus we cannot obtain the *measured value* (by the simultaneous measurement),

which may be, from Einstein's point of view, represented that "*true value* (or, *hidden variable*) of the position and momentum" does not exist. That is, we have the following correspondence:

$$\underset{\text{Bohr's point of view}}{\boxed{\text{non-existence of combined observable}}} \Longleftrightarrow \underset{\text{Einstein's point of view}}{\boxed{\text{non-existence of true value}}}$$

Since the error Δ is usually defined by $\Delta = |\text{rough measured value} - \text{true value}|$, it is not easy to define the errors Δ_Q and Δ_P in Heisenberg's uncertainty principle $\Delta_Q \cdot \Delta_P \geq \hbar/2$. Therefore, if we define the error Δ by $\Delta = |\text{rough measured value} - \text{true value}|$, Heisenberg's uncertainty principle includes paradox (*cf.* page 403 of ref. [9]). The definition of Δ_Q and Δ_P was completed and Heisenberg's uncertainty principle was proved (*cf.* Corollary 1 in ref. [8]). Also, according to the maxim of dualism: "To be is to be perceived" due to G. Berkeley, we think that it is not necessary to name that does not exist (or equivalently, that is not measured).

The above statement (F_2) makes us expect that

(G) Bell's inequality (6) (or, (8)) is violated in classical systems as well as quantum systems without the combinable condition.

This (G) was already shown in my previous paper [12]. However, I received a lot of questions concerning (G) from the readers. Thus, in this section, we again explain the (G) more precisely.

2.2.1 Bell test experiment

In order to show the (G), three steps ([Step:I] ∼[Step:III]) are prepared in what follows.

[Step: I]. Put $X = \{-1, 1\}$. Define complex numbers $a_k (= \alpha_k + \beta_k \sqrt{-1} \in \mathbb{C}$: the complex field) ($k = 1, 2, 3, 4$) such that $|a_k| = 1$. Define the probability space $(X^2, \mathcal{P}(X^2), \nu_{a_i a_j})$ such that ($i = 1, 2, j = 3, 4$)

$$\begin{aligned} \nu_{a_i a_j}(\{(1,1)\}) = \nu_{a_i a_j}(\{(-1,-1)\}) = (1 - \alpha_i \alpha_j - \beta_i \beta_j)/4 \\ \nu_{a_i a_j}(\{(-1,1)\}) = \nu_{a_i a_j}(\{(1,-1)\}) = (1 + \alpha_i \alpha_j + \beta_i \beta_j)/4 \end{aligned} \tag{9}$$

The correlation $R(a_i, a_j)$ ($i = 1, 2, j = 3, 4$) is defined as follows:

$$R(a_i, a_j) \equiv \sum_{(x_1, x_2) \in X \times X} x_1 \cdot x_2 \nu_{a_i a_j}(\{(x_1, x_2)\}) = -\alpha_i \alpha_j - \beta_i \beta_j \tag{10}$$

Now we have the following problem:

(H) Find a measurement $\mathsf{M}_{\mathcal{A}}(\mathsf{O}_{a_i a_j} := (X^2, \mathcal{P}(X^2), F_{a_i a_j}), S_{[\rho_0]})$ $(i = 1, 2, j = 3, 4)$ such that

$$\rho_0(F_{a_i a_j}(\Xi)) = \nu_{a_i a_j}(\Xi) \qquad (\forall \Xi \in \mathcal{P}(X^2)) \tag{11}$$

and

$$\begin{aligned} &F_{a_1 a_3}(\{x_1\} \times X) = F_{a_1 a_4}(\{x_1\} \times X) \quad F_{a_2 a_3}(\{x_2\} \times X) = F_{a_2 a_4}(\{x_2\} \times X) \\ &F_{a_1 a_3}(X \times \{x_3\}) = F_{a_2 a_3}(X \times \{x_3\}) \quad F_{a_1 a_4}(X \times \{x_4\}) = F_{a_2 a_4}(X \times \{x_4\}) \\ &\qquad (\forall x_k \in X(\equiv \{-1, 1\}), k = 1, 2, 3, 4) \end{aligned}$$

which is the same as the condition (4)

[Step: II].

Let us answer this problem (H) in the two cases (i.e., classical case and quantum case), that is,

- $\begin{cases} \text{(i):the case of quantum systems: } [\mathcal{A} = B(\mathbb{C}^2) \otimes B(\mathbb{C}^2)(\equiv B(\mathbb{C}^2 \otimes \mathbb{C}^2))] \\ \text{(ii):the case of classical systems: } [\mathcal{A} = C(\Omega) \otimes C(\Omega)(\equiv C(\Omega \times \Omega))\,] \end{cases}$

(i):the case of quantum system: $[\mathcal{A} = B(\mathbb{C}^2) \otimes B(\mathbb{C}^2)]$

Put

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (\in \mathbb{C}^2).$$

For each a_k $(k = 1, 2, 3, 4)$, define the observable $\mathsf{O}_{a_k} \equiv \big(X, \mathcal{P}(X), G_{a_k}\big)$ in $B(\mathbb{C}^2)$ such that

$$G_{a_k}(\{1\}) = \frac{1}{2} \begin{bmatrix} 1 & \bar{a}_k \\ a_k & 1 \end{bmatrix}, \quad G_{a_k}(\{-1\}) = \frac{1}{2} \begin{bmatrix} 1 & -\bar{a}_k \\ -a_k & 1 \end{bmatrix}.$$

where $\bar{a}_k = \alpha_k - \beta_k \sqrt{-1}$. Then, we have four observable:

$$\widehat{\mathsf{O}}_{a_i} = (X, \mathcal{P}(X), G_{a_i} \otimes I), \quad \widehat{\mathsf{O}}_{a_j} = (X, \mathcal{P}(X), I \otimes G_{a_j}) \quad (i = 1, 2, j = 3, 4) \tag{12}$$

and further,

$$\mathsf{O}_{a_i a_j} = (X^2, \mathcal{P}(X^2), F_{a_i a_j} := G_{a_i} \otimes G_{a_j}) \qquad (i = 1, 2, j = 3, 4) \tag{13}$$

in $B(\mathbb{C}^2) \otimes B(\mathbb{C}^2)$, where it should be noted that $F_{a_i a_j}$ is separated by G_{a_i} and G_{a_j}.

Further define the singlet state $\rho_0 = |\psi_s\rangle\langle\psi_s| \ \big(\in \mathfrak{S}^p(B(\mathbb{C}^2 \otimes \mathbb{C}^2)^*)\big)$, where

$$\psi_s = (e_1 \otimes e_2 - e_2 \otimes e_1)/\sqrt{2}$$

Thus we have the measurement $\mathsf{M}_{B(\mathbb{C}^2\otimes\mathbb{C}^2)}(\mathsf{O}_{a_ia_j}, S_{[\rho_0]})$ in $B(\mathbb{C}^2)\otimes B(\mathbb{C}^2)$ $(i=1,2, j=3,4)$. The followings are clear: for each $(x_1,x_2)\in X^2(\equiv\{-1,1\}^2)$,

$$\rho_0(F_{a_ia_j}(\{(x_1,x_2)\})) = \langle\psi_s, (G_{a_i}(\{x_1\})\otimes G_{a_j}(\{x_2\}))\psi_s\rangle = \nu_{a_ia_j}(\{(x_1,x_2)\}) \quad (i=1,2, j=3,4) \tag{14}$$

For example, we easily see:

$$\begin{aligned}
&\rho_0(F_{a_ib_j}(\{(1,1)\})) = \langle\psi_s, (G_{a_i}(\{1\})\otimes G_{a_j}(\{1\}))\psi_s\rangle \\
=&\frac{1}{8}\langle(e_1\otimes e_2 - e_2\otimes e_1), (\begin{bmatrix} 1 & \bar{a}_i \\ a_i & 1\end{bmatrix}\otimes\begin{bmatrix} 1 & \bar{a}_j \\ a_j & 1\end{bmatrix})(e_1\otimes e_2 - e_2\otimes e_1)\rangle \\
=&\tfrac{1}{8}\langle(\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}), (\begin{bmatrix} 1 & \bar{a}_i \\ a_i & 1\end{bmatrix}\otimes\begin{bmatrix} 1 & \bar{a}_j \\ a_j & 1\end{bmatrix})(\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix})\rangle \\
=&\frac{1}{8}\langle(\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}), (\begin{bmatrix}1\\a_i\end{bmatrix}\otimes\begin{bmatrix}\bar{a}_j\\1\end{bmatrix} - \begin{bmatrix}\bar{a}_i\\1\end{bmatrix}\otimes\begin{bmatrix}1\\a_j\end{bmatrix})\rangle \\
=&\frac{1}{8}(2 - a\bar{a}_j - \bar{a}_ia_j) = (1-\alpha_i\alpha_j - \beta_i\beta_j)/4 = \nu_{a_ia_j}(\{(1,1)\}).
\end{aligned}$$

Therefore, the measurement $\mathsf{M}_{B(\mathbb{C}^2\otimes\mathbb{C}^2)}(\mathsf{O}_{a_ia_j}, S_{[\rho_0]})$ satisfies the condition (H).

(ii):the case of classical systems: $[\mathcal{A} = C(\Omega)\otimes C(\Omega) = C(\Omega\times\Omega)]$

Put $\omega_0(=(\omega_0', \omega_0''))\in\Omega\times\Omega$、$\rho_0=\delta_{\omega_0}$ $(\in\mathfrak{S}^p(C(\Omega\times\Omega)^*)$, i.e., the point measure at ω_0)). Define the observable $\mathsf{O}_{a_ia_j} := (X^2, \mathcal{P}(X^2), F_{a_ia_j})$ in $L^\infty(\Omega\times\Omega)$ such that

$$[F_{a_ia_j}(\{(x_1,x_2)\})](\omega) = \nu_{a_ia_j}(\{(x_1,x_2)\}) \quad (\forall(x_1,x_2)\in X^2, i=1,2, j=3,4, \forall\omega\in\Omega\times\Omega)$$

Thus, we have four observables

$$\mathsf{O}_{a_ia_j} = (X^2, \mathcal{P}(X^2), F_{a_ia_j}) \qquad (i=1,2, j=3,4) \tag{15}$$

in $L^\infty(\Omega\times\Omega)$ (though the variables are not separable (*cf.* the formula (13)). Then, it is clear that the measurement $\mathsf{M}_{C_0(\Omega\times\Omega)}(\mathsf{O}_{a_ia_j}, S_{[\delta_{\omega_0}]})$ satisfies the condition (H).

(ii)$'$:the case of classical systems: $[\mathcal{A} = C(\Omega)\otimes C(\Omega) = C(\Omega\times\Omega)]$

It is easy to show a lot of different answers from the above (ii). For example, as a slight generalization of (9), define the probability measure $\nu^t_{a_ia_j}$ $(0\le t\le 1)$ such that

$$\begin{aligned}
&\nu^t_{a_ia_j}(\{(1,1)\}) = \nu^t_{a_ia_j}(\{(-1,-1)\}) = (1 - t(\alpha_i\alpha_j + \beta_i\beta_j))/4 \\
&\nu^t_{a_ia_j}(\{(-1,1)\}) = \nu^t_{a_ia_j}(\{(1,-1)\}) = (1 + t(\alpha_i\alpha_j + \beta_i\beta_j))/4
\end{aligned} \tag{16}$$

And consider the real-valued continuous function $t(\in C(\Omega\times\Omega))$ such that $0\le t(\omega',\omega'')\le 1$ $(\forall\omega=(\omega',\omega'')\in\Omega\times\Omega)$. And assume that $t(\omega_0)=1$ for some $\omega_0(=(\omega_0',\omega_0''))\in\Omega\times\Omega$、$\rho_0=\delta_{\omega_0}$ $(\in\mathfrak{S}^p(C(\Omega\times\Omega)^*)$, i.e., the point measure at ω_0)). Define the observable $\mathsf{O}_{a_ia_j} := (X^2, \mathcal{P}(X^2), F_{a_ia_j})$ in $L^\infty(\Omega\times\Omega)$ such that

$$[F_{a_ia_j}(\{(x_1,x_2)\})](\omega) = \nu^{t(\omega)}_{a_ia_j}(\{(x_1,x_2)\}) \quad (\forall(x_1,x_2)\in X^2, i=1,2, j=3,4, \forall\omega\in\Omega\times\Omega) \tag{17}$$

Thus, we have four observables

$$\mathsf{O}_{a_i a_j} = (X^2, \mathcal{P}(X^2), F_{a_i a_j}) \qquad (i = 1, 2, j = 3, 4)$$

in $L^\infty(\Omega \times \Omega)$ (though the variables are not separable (*cf.* the formula (13)). Then, it is clear that the measurement $\mathsf{M}_{L^\infty(\Omega\times\Omega)}(\mathsf{O}_{a_i a_j}, S_{[\delta_{\omega_0}]})$ satisfies the condition (H).

[Step: III].

As defined by (9), consider four complex numbers $a_k(= \alpha_k + \beta_k\sqrt{-1}; k = 1, 2, 3, 4)$ such that $|a_k| = 1$. Thus we have four observables

$$\mathsf{O}_{a_1 a_3} := (X^2, \mathcal{P}(X^2), F_{a_1 a_3}), \quad \mathsf{O}_{a_1 a_4} := (X^2, \mathcal{P}(X^2), F_{a_1 a_4}),$$
$$\mathsf{O}_{a_2 a_3} := (X^2, \mathcal{P}(X^2), F_{a_2 a_3}), \quad \mathsf{O}_{a_2 a_4} := (X^2, \mathcal{P}(X^2), F_{a_2 a_4}),$$

in $\mathcal{A}$. Thus, we have the parallel measurement $\otimes_{i=1,2,j=3,4} \mathsf{M}_{\mathcal{A}}(\mathsf{O}_{a_i a_j} := (X^2, \mathcal{P}(X^2), F_{a_i a_j}), S_{[\rho_0]})$ in $\otimes_{i=1,2,j=3,4}\mathcal{A}$.

Thus, putting

$$a_1 = \sqrt{-1},\ a_2 = 1,\ a_3 = \frac{1+\sqrt{-1}}{\sqrt{2}},\ a_4 = \frac{1-\sqrt{-1}}{\sqrt{2}},$$

we see, by (10), that

$$|R(a_1, a_3) - R(a_1, a_4)| \ + \ |R(a_2, a_3) + R(a_2, a_4)| = 2\sqrt{2} \tag{18}$$

Further, assume that the measured value is $x(\in X^8)$. That is,

$$x = \big((x^1_{13}, x^2_{13}), (x^1_{14}, x^2_{14}), (x^1_{23}, x^2_{23}), (x^1_{24}, x^2_{24})\big) \in \mathop{\times}_{i,j=1,2} X^2 (\equiv \{-1, 1\}^8)$$

Let N be sufficiently large natural number. Consider N-parallel measurement $\bigotimes_{n=1}^N [\otimes_{i=1,2,j=3,4} \mathsf{M}_{\mathcal{A}}(\mathsf{O}_{a_i a_j} := (X^2, \mathcal{P}(X^2), F_{a_i a_j}), S_{[\rho_0]})\ ]$. Assume that its measured value is $\{x^n\}_{n=1}^N$. That is,

$$\{x^n\}_{n=1}^N = \begin{bmatrix} ((x^{1,1}_{13}, x^{2,1}_{13}), (x^{1,1}_{14}, x^{2,1}_{14}), (x^{1,1}_{23}, x^{2,1}_{23}), (x^{1,1}_{24}, x^{2,1}_{24})) \\ ((x^{1,2}_{13}, x^{2,2}_{13}), (x^{1,2}_{14}, x^{2,2}_{14}), (x^{1,2}_{23}, x^{2,2}_{23}), (x^{1,2}_{24}, x^{2,2}_{24})) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \Big((x^{1,N}_{13}, x^{2,N}_{13}), (x^{1,N}_{14}, x^{2,N}_{14}), (x^{1,N}_{23}, x^{2,N}_{23}), (x^{1,N}_{24}, x^{2,N}_{24})\Big) \end{bmatrix} \in \big(\mathop{\times}_{i=1,2,j=3,4} X^2\big)^N (\equiv \{-1, 1\}^{8N})$$

Then, the law of large numbers says that

$$R(a_i, a_j) \approx \frac{1}{N}\sum_{n=1}^N x^{1,n}_{ij} x^{2,n}_{ij} \quad (i = 1, 2, j = 3, 4)$$

This and the formula (18) say that

$$|\sum_{n=1}^N \frac{x^{1,n}_{13} x^{2,n}_{13}}{N} - \sum_{n=1}^N \frac{x^{1,n}_{14} x^{2,n}_{14}}{N}| + |\sum_{n=1}^N \frac{x^{1,n}_{23} x^{2,n}_{23}}{N} + \sum_{n=1}^N \frac{x^{1,n}_{24} x^{2,n}_{24}}{N}| \approx 2\sqrt{2} \tag{19}$$

Therefore, Bell's inequality (6) (or, (8)) is violated in classical systems as well as quantum systems.

Remark 12. For completeness, note that the observables $\mathsf{O}_{a_i a_j}$ $(i = 1, 2, j = 3, 4)$ in the classical $L^\infty(\Omega\times\Omega)$ are not combinable in spite that these commute. Also, note that the formulas (16) and (17) imply that

$$
\begin{aligned}
&[F_{a_1a_3}(\{x\}\times X)](\omega) = [F_{a_1a_4}(\{x\}\times X)](\omega) = 1/2, \quad [F_{a_2a_3}(\{x\}\times X)](\omega) = [F_{a_2a_4}(\{x\}\times X)](\omega) = 1/2,\\
&[F_{a_1a_3}(X\times\{x\})](\omega) = [F_{a_2a_3}(X\times\{x\})](\omega) = 1/2, \quad [F_{a_1a_4}(X\times\{x\})](\omega) = [F_{a_2a_4}(X\times\{x\})](\omega) = 1/2\\
&\qquad\qquad (\forall x \in X, \forall \omega \in \Omega\times\Omega),
\end{aligned}
$$

which is similar as (4).

3 Conclusion

In Bohr-Einstein debates (refs. [6, 2]), Einstein's standing-point (that is, "*the moon is there whether one looks at it or not*" (i.e., physics holds without observers)) is on the side of the realistic world view in Figure 1. On the other hand, we think that Bohr's standing point (that is, "*to be is to be perceived*" (i.e., there is no science without measurements)) is on the side of the linguistic world view in Figure 1.1.

In this paper, contrary to Bell's spirit (which inherits Einstein's spirit), we try to discuss Bell's inequality in Bohr's spirit (i.e., in the framework of quantum language). And we show Theorem 8 (Bell's inequality in quantum language), which says the statement (F_2), that is,

(I_1) ($\equiv$ (F_2)): [from Bohr's standing-point]:

If Bell's inequality (6) (or, (8)) is violated, then the combined observable does not exist, and thus, we cannot obtain the *measured value* (by the measurement of the combined observable).

Also, recall that Bell's original argument (which is under the influence of Bohr-Einstein debates) says, roughly speaking, that

(I_2) [from Einstein's standing-point]:

If the mathematical Bell's inequality (3) is violated in Bell test experiment (the quantum case of Section 4.5.3), then *hidden variables* do not exist.

It should be note that the concept of "hidden variable" is independent of measurements, thus, the (I_2) is a philosophical statement in Einstein's spirit, or precisely, the (I_2) may says that quantum mechanical phenomenon (i.e., Bell test experiment) cannot be described in Einstein's spirit. On the other hand, our (I_1) is not related Einstein's spirit, that is, it is a statement in Bohr's spirit (i.e., there is no science without measurements). It is sure that Bell's answer (I_2) is philosophically attractive, however, we believe in the scientific superiority of our answer (I_1). For example, consider the following problem:

(J) [Problem]: Why is Bell's inequality violated in the Bell test experiment (mentioned in Section 2.2)?

It is sure that everybody agrees to the answer (I_1) and not (I_2). Thus, the scientific superiority of our answer (I_1) is clear. That is, we think that Bell's (I_2) is a philosophical view of the scientific (I_1). If so, we can, for the first time, understand Bell's inequality from the practical point of view.

That is,

Theorem 8 is the true Bell's inequality.

And we conclude that whether or not Bell's inequality holds does not depend on whether classical systems or quantum systems (in Sections 2,2), but depend on whether the combined measurement exists or not (in Section 2.1).

Readers are recommended to progress to the W^*-algebraic approach to Bell's inequality [20, 21]. For the recent information concerning quantum language, see my home page [26].

I hope that quantum language (= Figure 1) will be accepted as one of the main results of Bohr-Einstein debates.

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