Arithmetic distributions of convergents arising from Jacobi-Perron algorithm

by

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Abstract
We study the distribution modulo m of the convergents associated with the d-dimensional Jacobi-Perron algorithm for a.e. real numbers in (0, 1)d by proving the ergodicity of a skew product of the Jacobi-Perron transformation; this skew product was initially introduced in [6] for regular continued fractions.

1 Introduction
For an irrational number x, 0 < x < 1, we denote by \( \frac{p_n}{q_n} \) the n-th convergent of x, which is defined by the regular continued fraction expansion coefficients of x. In 1988, H. Jager and P. Liardet [6] studied the distribution properties of the pairs \((p_n, q_n)\) modulo m. These properties were originally considered by P. Szüsz in [16], and then by R. Moeckel [7] who used the ergodicity of geodesic flows over the modular surfaces: more precisely, they proved that given any positive integer \( m \geq 2 \), for a.e. \( x \), the sequence \( \{(p_n, q_n) : n \geq 1\} \) is equidistributed modulo m over the set \( \{(p, q) \in \mathbb{Z}_m^2 : \langle p, q \rangle = \mathbb{Z}_m\} \), where \( \mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z} \) and where the notation \( \langle p_1, \ldots, p_k \rangle \) stands for the subgroup of \( \mathbb{Z}_m \) generated by the elements \( p_1, \ldots, p_k \). To prove this property, H. Jager and P. Liardet considered in [6] the group of \( 2 \times 2 \) matrices with entries from \( \mathbb{Z}_m \) and determinant \( \pm 1 \), that is,

\[
G(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_m, \ ad - bc = \pm 1 \right\}.
\]

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It is possible to show that \( \{ (0, 1) : a \in \mathbb{Z}_m \} \) generates \( G(m) \). This fact implies the ergodicity of a \( G(m) \)-extension (a skew product indeed) of the continued fraction transformation. The equidistribution property of \( \{(p_n, q_n) : n \geq 1\} \) modulo \( m \) is then an easy consequence of the individual ergodic theorem.

A natural extension of this skew product was then introduced in [3] to deduce the distribution of the approximation coefficients associated with the continued fraction algorithm; these results were also extended to the so-called \( S \)-expansions, in the sense of [4]; see also for connected results [1] and [10].

The aim of the present paper is to generalize these equidistribution results to the \( d \)-dimensional Jacobi-Perron algorithm. Note that the 1-dimensional Jacobi-Perron algorithm reduces to the regular continued fraction algorithm.

Let us start with the definition of the Jacobi-Perron algorithm. We fix a positive integer \( d \geq 2 \). Let \( X = [0, 1)^d \) be endowed with the Borel \( \sigma \)-algebra \( \mathcal{B} \). We first define the map \( T : X \to X \) by

\[
T(\mathbf{x}) = T((x_1, x_2, \ldots, x_d)) = \left( \frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}, \frac{1}{x_1} \right)
\]

for \( \mathbf{x} = (x_1, x_2, \ldots, x_d) \in X \) if \( x_1 \neq 0 \), and \( T(\mathbf{x}) = 0 \), otherwise; \((X, T)\) is called the \( d \)-dimensional Jacobi-Perron algorithm. Notice that there exists a unique absolutely continuous invariant probability measure \( \mu \) for \( T \) which is equivalent to the Lebesgue measure (see for instance [13]).

We put for \( \mathbf{x} \) in \( X \) with \( x_1 \neq 0 \)

\[
k(\mathbf{x}) = k(0)(\mathbf{x}) = (k_1, k_2, \ldots, k_d) = \left( \frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_d}{x_1}, \frac{1}{x_1} \right),
\]

if \( x_1 = 0 \), we set \( k(\mathbf{x}) = 0 \); we similarly define

\[
k(s)(\mathbf{x}) = (k_1(s), k_2(s), \ldots, k_d(s)) = k(T^{s-1}(\mathbf{x})) \quad \text{for} \quad s \geq 1.
\]

We then associate \((x_1, x_2, \ldots, x_d)\) with the column vector \( \begin{pmatrix} x_1 \\ \vdots \\ x_d \\ 1 \end{pmatrix} \) and consider the following matrix

\[
P = \begin{pmatrix}
-k_1 & 1 & 0 & \ldots & 0 \\
-k_2 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-k_d & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix} \tag{1}
\]

Then \( T((x_1, x_2, \ldots, x_d)) \) corresponds to \( P \begin{pmatrix} x_1 \\ \vdots \\ x_d \\ 1 \end{pmatrix} \). To construct the sequence

\[
\left\{ \left( \begin{pmatrix} p_1(k) \\ q(k) \\ \vdots \\ p_d(k) \\ q(k) \end{pmatrix}, \frac{p_1(k)}{q(k)}, \ldots, \frac{p_d(k)}{q(k)} : k \geq 1 - d \right) \right\}
\]
of simultaneous approximation convergents of $\mathbf{x}$ from the $d$-dimensional Jacobi-Perron algorithm, we first define $Q^{(0)}$ as the $(d + 1) \times (d + 1)$ identity matrix $I_{d+1}$; we then define recursively $Q^{(n)}$ for $n \geq 1$ as

$$Q^{(n)} := Q^{(n-1)} \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & k_1^{(n)} \\ 0 & 1 & \cdots & 0 & k_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & k_d^{(n)} \end{pmatrix}. $$

We thus set for $n \geq 1$

$$Q^{(n)} = \begin{pmatrix} p_1^{(n-d)} & p_1^{(n-d+1)} & \cdots & p_1^{(n-1)} & p_1^{(n)} \\ p_2^{(n-d)} & p_2^{(n-d+1)} & \cdots & p_2^{(n-1)} & p_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_d^{(n-d)} & p_d^{(n-d+1)} & \cdots & p_d^{(n-1)} & p_d^{(n)} \\ q_1^{(n-d)} & q_1^{(n-d+1)} & \cdots & q_1^{(n-1)} & q_1^{(n)} \end{pmatrix}.$$

Let us observe that

$$Q^{(1)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & k_1 \\ 0 & 1 & \cdots & 0 & k_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & k_d \end{pmatrix} = P^{-1}. \quad (2)$$

It is well-known that for any $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in X$,

$$\lim_{n \to \infty} \frac{p_i^{(n)}}{q^{(n)}} = x_i \quad \text{for } 1 \leq i \leq d$$

holds.

In this paper, we prove that for almost every $\mathbf{x} \in X$ the sequences of vectors\{(q_1^{(n-d)}, q_1^{(n-d+1)}, \ldots, q_1^{(n)}): n \geq 1\} and \{(p_1^{(n)}, p_2^{(n)}, \ldots, p_d^{(n)}, q^{(n)}): n \geq 1\} are both equidistributed modulo $m$ for any integer $m \geq 2$.

More precisely we put

$$\mathbb{Z}_m^{d+1} = \{ (\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \in \mathbb{Z}_m^{d+1} : (\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) = \mathbb{Z}_m \}$$

and

$$c_m = \# \mathbb{Z}_m^{d+1} \quad (\text{the cardinality of } \mathbb{Z}_m^{d+1}).$$

One easily sees that

$$c_m = \varphi_{d+1}(m) \quad \times \{ (a_1, a_2, \ldots, a_{d+1}) \in \{1, \ldots, m\}^{d+1} : \gcd(a_1, \ldots, a_{d+1}, m) = 1 \}, \quad (3)$$

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where \( \varphi_{d+1} \) denotes the Jordan totient function of order \( d+1 \); we thus have (see for instance [15] or [11])

\[
c_m = m^{d+1} \prod_{p|m}(1 - p^{-(d+1)}),
\]

where the notation \( \prod_{p|m} \) stands in all that follows for the product over the prime numbers \( p \) that divide \( m \). We then have the following:

**Theorem 1.** Let \( m \geq 2 \) be a nonnegative integer. For almost every \( x \in X \) and for any \( (\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \in \mathbb{Z}_{m}^{d+1}, \) we have

\[
\lim_{N \to \infty} \frac{\sharp \{1 \leq n \leq N : (q^{(n-d)}, q^{(n-d+1)}, \ldots, q^{(n)}) \equiv (\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \pmod{m}\}}{N} = \frac{1}{c_m} \frac{1}{\varphi_{d+1}(m)} = \frac{1}{m^{d+1} \prod_{p|m}(1 - p^{-(d+1)})}.
\]

To prove this theorem, we consider for a given integer \( m \geq 2 \), the group \( G(m) \) defined in a similar way as in [6]:

\[
G(m) = \begin{cases} 
SL(d+1, \mathbb{Z}_m) & \text{if } d \text{ is even,} \\
SL_\pm(d+1, \mathbb{Z}_m) & \text{if } d \text{ is odd,}
\end{cases}
\]

where \( SL(d+1, \mathbb{Z}_m) \) stands for the matrices with entries in \( \mathbb{Z}_m \) with determinant 1, whereas \( SL_\pm(d+1, \mathbb{Z}_m) \) stands for the matrices with entries in \( \mathbb{Z}_m \) with determinant \( \pm 1 \). Let us recall that (see for instance [11] or [9]) that

\[
\sharp SL(d+1, \mathbb{Z}_m) = m^{(d+1)^2-1} \prod_{i=2}^{d+1} \prod_{p|n}(1 - p^{-i}) = m^{d(d+1)/2} \prod_{i=2}^{d+1} \varphi_i(m).
\]

Let \( C_m \) denote the cardinality of \( G(m) \). Since \( SL(d+1, \mathbb{Z}_m) \) is a subgroup of \( SL_\pm(d+1, \mathbb{Z}_m) \) of index 2 if \( d \) is odd and \( m \neq 2 \), one thus gets

\[
C_m = \begin{cases} 
m^{(d+1)^2-1} \prod_{i=2}^{d+1} \prod_{p|n}(1 - p^{-i}) & \text{if } d \text{ is even or } m = 2 \\
2m^{(d+1)^2-1} \prod_{i=2}^{d+1} \varphi_i(m) & \text{if } d \text{ is odd and } m \neq 2.
\end{cases}
\]

We identify \( Q^{(1)} \) with the matrix with coefficients in \( \mathbb{Z}_m \) obtained by reducing modulo \( m \) its entries. Here we note that \( \det Q^{(1)} = 1 \) or \( -1 \) if \( d \) is respectively even or odd, which implies that \( Q^{(1)} \) belongs to the group \( G(m) \), whatever may be the parity of \( d \).
We define the map $T_m$ on $X \times G(m)$ by
\[ T_m(x, A) = (T(x), AQ^{(1)}) \]
$T_m$ is said to be a $G(m)$-extension of the map $T$.

We define the probability measure $\delta_m$ on $G(m)$ by $(1, \ldots, 1)$. Then it is easy to see that $\mu \times \delta_m$ is an invariant probability measure for $T_m$. Our question is whether $(T_m, \mu \times \delta_m)$ is ergodic or not. In Section 2, we show that the set of matrices of the form (2) (reduced modulo $m$) generates $G(m)$. Then in Section 3, we prove the ergodicity of $T_m$, from which we deduce the following proposition and then Theorem 1 (in the same way as in [6]):

**Proposition 1.** For a.e. $x \in X$ and any $A \in G(m)$,
\[
\lim_{N \to \infty} \frac{1}{N} \sharp \{ 1 \leq n \leq N : Q^{(n)} \equiv A \pmod{m} \} = \frac{1}{C_m}.
\]

Finally we have the following

**Corollary 1.** For a.e. $x \in X$ and any $a \in \mathbb{Z}_m$
\[
\lim_{N \to \infty} \frac{1}{N} \sharp \{ 1 \leq n \leq N : q^{(n)} \equiv a \pmod{m} \} = \frac{m^d \cdot \varphi_d(\gcd(a, m))}{\gcd(a, m) \cdot \varphi_{d+1}(m)}.
\]

In all that follows, we simply denote by $0, 1, \ldots, m-1$ the elements of $\mathbb{Z}_m$ if it is clear that the elements are in $\mathbb{Z}_m$ according to the context. In this case, one has obviously $m-1 = -1$.

## 2 Basic properties of $G(m)$

We first define
\[ \Gamma_m = \{ A \in SL(d+1, \mathbb{Z}) : A \equiv I_{d+1} \pmod{m} \}. \]
Then it is well-known that
\[ SL(d+1, \mathbb{Z}_m) \cong \Gamma_m \setminus SL(d+1, \mathbb{Z}) \]
e.g., see G. Shimura [15], p. 21. From this property, it easily follows that
\[ SL_\pm(d+1, \mathbb{Z}_m) \cong \Gamma_m \setminus GL(d+1, \mathbb{Z}). \]

We respectively say that a $(d+1) \times (d+1)$ matrix with $\mathbb{Z}$ (or $\mathbb{Z}_m$)-entries of the form (2) is a J-P matrix, and that a matrix of the form (1) is a J-P$^*$ matrix; a J-P matrix is the inverse of a J-P$^*$ matrix.

In the sequel of this section, we show that the monoid generated by the set of J-P matrices with $\mathbb{Z}_m$-entries is equal to $G(m)$:
Theorem 2. For any \( B \in G(m) \), there exist J-P matrices \( A_1, A_2, \ldots, A_s \) such that

\[
B = A_1 A_2 \cdots A_s.
\]

For this purpose, we first need some notation and some preliminary lemmas. We put

\[
\Delta(k_1, k_2, \ldots, k_d) := \begin{pmatrix}
k_1 & 1 & 0 & \ldots & 0 \\
k_2 & 0 & 1 & \ldots & 0 \\
    & \vdots & \ddots & \ddots & \vdots \\
k_d & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

in particular,

\[
\Delta = \Delta(0, 0, \ldots, 0) = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
    & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

We thus have

\[
(a_1, a_2, \ldots, a_{d+1}) \Delta = (a_{d+1}, a_1, a_2, \ldots, a_d),
\]

where \( a_1, a_2, \ldots, a_{d+1} \) are either elements in \( \mathbb{Z}_m \) or \((d+1)\)-dimensional vectors with \( \mathbb{Z}_m \)-entries. Let us notice that the matrices \( \Delta(k_1, k_2, \ldots, k_d) \) are J-P matrices.

Lemma 1. For any \((\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_{m}^{d+1} \), there exist J-P matrices \( A_1, A_2, \ldots, A_s \) with \( \mathbb{Z}_m \)-entries such that

\[
(\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) = (0, \ldots, 0, 1) A_1 A_2 \cdots A_s,
\]

where \( s \) depends on \((\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \).

Proof of Lemma 1. We define the following natural order \( \prec \) on \( \mathbb{Z}_m \) by

\[
0 \prec 1 \prec \cdots \prec m - 1.
\]

Let \((\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_{m}^{d+1} \). We denote by \( \alpha^* \) the element in \( \mathbb{Z}_m \) such that \((\alpha^*) = (\alpha_1, \alpha_2, \ldots, \alpha_d) \). Let us prove by induction on \( \alpha^* \) (considered then as an element in \( \{1, \ldots, m\} \)) that there exists a finite number of J-P matrices \( \Delta_1, \ldots, \Delta_t \) and \((\alpha'_1, \ldots, \alpha'_{d}) \in \mathbb{Z}_m^{d} \) such that

\[
(\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \Delta_1 \cdots \Delta_t = (1, \alpha'_1, \ldots, \alpha'_{d}).
\]
If \( \alpha^* = 1 \), then \( (\alpha_1, \alpha_2, \ldots, \alpha_d) = \mathbb{Z}_m \) and there exist \( k_1, \ldots, k_d \in \mathbb{Z}_m \) such that 
\[
\sum_{i=1}^d k_i \alpha_i + \alpha_{d+1} = 1.
\]
We thus have 
\[
(\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \Delta(k_1, k_2, \ldots, k_d) = (1, \alpha_1, \ldots, \alpha_d).
\]
Suppose now that \( \alpha^* \neq 1 \). Since \( (\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \neq (\alpha^*) \), there exist \( k_1, \ldots, k_d \in \mathbb{Z}_m \) such that \( 0 < \sum_{i=1}^d k_i \alpha_i + \alpha_{d+1} < \alpha^* \). We thus have 
\[
(\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \Delta(k_1, k_2, \ldots, k_d, 1) = \left( \sum_{i=1}^d k_i \alpha_i + \alpha_{d+1}, \alpha_1, \ldots, \alpha_d \right).
\]
We can now conclude inductively since \( (\sum_{i=1}^d k_i \alpha_i + \alpha_{d+1}, \alpha_1, \ldots, \alpha_d) = \mathbb{Z}_m \).
Now we have 
\[
(1, \alpha'_1, \ldots, \alpha'_d) \cdot \Delta(-\alpha'_d, 0, \ldots, 0) \cdot \Delta(0, -\alpha'_{d-1}, 0, \ldots, 0) \cdots \Delta(0, \ldots, 0, -\alpha'_1) = (0, \ldots, 0, 1)
\]
Since a J-P* matrix is the inverse of a J-P matrix, we get the assertion of this lemma.

The following lemmas are essential and easily proved.

**Lemma 2.** For any \((d+1)\)-dimensional vectors with \(\mathbb{Z}_m\)-entries \((a_1, a_2, \ldots, a_{d+1})\), we have 
\[
(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_d, a_{d+1}) \cdot \Delta(0, \ldots, 0, -1, 0, \ldots, 0) \cdot \Delta^{d-i} \\
\cdot \Delta(0, \ldots, 0, 1, 0, \ldots, 0) \cdot \Delta^{i-1} \cdot \Delta(0, \ldots, 0, -1, 0, \ldots, 0) \cdot \Delta^d \\
= (a_1, \ldots, a_{i-1}, a_{i+1}, a_{i+1}, \ldots, a_d, -a_i).
\]

**Lemma 3.** We have 
\[
(a_1, \ldots, a_{d+1}) \cdot \Delta(0, \ldots, 0, 1) \cdot \Delta(-1, 0, \ldots, 0) \cdot \Delta^{d-1} \cdot \Delta(0, \ldots, 0, 1) \\
= (a_d, a_1, \ldots, a_{d-1}, -a_{d+1}).
\]
In particular, when \( d \) is odd 
\[
(a_1, \ldots, a_{d+1}) \cdot [\Delta(0, \ldots, 0, 1) \cdot \Delta(-1, 0, \ldots, 0) \cdot \Delta^{d-1} \cdot \Delta(0, \ldots, 0, 1)]^d \\
= (a_1, a_2, \ldots, a_d, -a_{d+1}).
\]

**Proof of Theorem 2.** Let us fix 
\[
B = \begin{pmatrix}
    b_{11} & b_{12} & \cdots & b_{1(d+1)} \\
    b_{21} & b_{22} & \cdots & b_{2(d+1)} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{(d+1)1} & b_{(d+1)2} & \cdots & b_{(d+1)(d+1)}
\end{pmatrix}
\]
in \( G(m) \). We want to prove that there exist J-P* matrices \( \Delta_1, \ldots, \Delta_s \) such that
\[
B \Delta_1 \cdots \Delta_s = I_{d+1},
\]
which implies immediately the desired result. For that purpose, let us prove by induction on \( 1 \leq j \leq d \) that there exist J-P* matrices \( \Delta_1, \ldots, \Delta_{s_j} \) such that
\[
B_j := B \Delta_1 \cdots \Delta_{s_j} = \begin{pmatrix} I_j & 0 \\ 0 & B^{J_P} \end{pmatrix},
\]
where \( I_j \) is the \( j \times j \) identity matrix. Indeed, if this property holds for \( j = d \), then we obtain that there exist J-P* matrices \( \Delta_1, \ldots, \Delta_{s_d} \) such that
\[
B_{d+1} := B \Delta_1 \cdots \Delta_{s_d} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \ldots & 0 & \pm 1 \end{pmatrix},
\]
If \( d \) is even, then all J-P* matrices are of determinant 1. Thus the \((d+1, d+1)\)-entry of the right hand side is equal to 1. If \( d \) is odd and the \((d+1, d+1)\)-entry of the right hand side is equal to \(-1\), then by Lemma 3 we can reduce it to 1 by application of J-P* matrices. In either case, we get the desired result.

It thus remains to prove the induction property. Let us first prove that it holds for \( j = 1 \). Since \( \det B = \pm 1 \), then \( (b_{11}, b_{12}, \ldots, b_{1(d+1)}) = \mathbb{Z}_m \), and there thus exist J-P* matrices \( \Delta_1, \ldots, \Delta_{s_1} \) such that
\[
(b_{11}, b_{12}, \ldots, b_{1(d+1)}) \Delta_1 \cdots \Delta_{s_1} = (0, \ldots, 0, 1)
\]
by Lemma 1. Thus
\[
B_1 := B \Delta_1 \cdots \Delta_{s_1} = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ b_{11}^{(1)} & \ldots & b_{1d}^{(1)} & b_{1(d+1)}^{(1)} \\ \vdots & \ddots & \vdots & \vdots \\ b_{d1}^{(1)} & \ldots & b_{dd}^{(1)} & b_{d(d+1)}^{(1)} \end{pmatrix}.
\]
We set
\[
B^{(1)} = \begin{pmatrix} b_{11}^{(1)} & \ldots & b_{1d}^{(1)} \\ \vdots & \ddots & \vdots \\ b_{d1}^{(1)} & \ldots & b_{dd}^{(1)} \end{pmatrix}.
\]
Since \( \det B^{(1)} = \pm 1 \), then there exist \( k_1, \ldots, k_d \in \mathbb{Z}_m \) such that
\[
B \Delta_1 \cdots \Delta_{s_1} \Delta(k_1, \ldots, k_d) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & b_{11}^{(1)} & \ldots & b_{1d}^{(1)} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & b_{d1}^{(1)} & \ldots & b_{dd}^{(1)} \end{pmatrix}.
\]
It remains to set \( \Delta_{s_1} = \Delta(k_1, \ldots, k_d) \) to conclude the proof of the induction property for \( j = 1 \).

Let us assume now that the induction property holds for \( 1 \leq j \leq d - 1 \) (if \( d = 1 \), the proof is finished); one thus deduces that the determinant of \( B^{(j)} \) (defined in (6)) is equal to \( \pm 1 \). We set

\[
B^{(j)} = \begin{pmatrix}
    b^{(j)}_{11} & \cdots & b^{(j)}_{1(d+1-j)} \\
    \vdots & \ddots & \vdots \\
    b^{(1)}_{(d+1-j)1} & \cdots & b^{(1)}_{(d+1-j)(d+1-j)}
\end{pmatrix}.
\]

Let us divide the induction proof into two steps for clarity issues.

**Step 1.** Let us first prove that we can find \( \text{J-P}^* \) matrices \( \Delta_{s_j+1}, \ldots, \Delta_{s_{j+t}} \) such that \( B_j \Delta_{s_{j+1}} \cdots \Delta_{s_{j+t}} \) is equal to

\[
\begin{pmatrix}
    0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\
    g_{(j+1)}^{(j+1)} & g_{(j+1)}^{(j+1)} & g_{(d+1-j)}^{(j+1)} & g_{(j+1)}^{(j+1)} & g_{(d+1-j)}^{(j+1)} & \cdots & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
    g_{(d-j)(d-j)}^{(j+1)} & \cdots & g_{(d-j)(d-j)}^{(j+1)} & g_{(j+1)}^{(j+1)} & g_{(d-j)(d-j)}^{(j+1)} & \cdots & 0 & \cdots & 0
\end{pmatrix},
\]

for some \( l, 0 \leq l < d - j + 1 \).

According to the proof of Lemma 1, we can find \( (d - j + 1) \times (d - j + 1) \) \( \text{J-P}^* \) matrices \( \Delta(k^{(1)}_1, \ldots, k^{(1)}_{d-j}), \ldots, \Delta(k^{(t)}_1, \ldots, k^{(t)}_{d-j}) \) such that

\[
(b^{(j)}_{11}, \ldots, b^{(j)}_{1(d+1-j)}) \Delta(k^{(1)}_1, \ldots, k^{(1)}_{d-j}) \cdots \Delta(k^{(u)}_1, \ldots, k^{(u)}_{d-j}) = (0, \ldots, 0, 1).
\]

Now

\[
\begin{pmatrix}
    I_j & 0 \\
    0 & B^{(j)}
\end{pmatrix} \Delta(0, \ldots, 0, *, \ldots, *) = \begin{pmatrix}
    0 & I_j & 0 \\
    * & 0 & *
\end{pmatrix},
\]

and one checks more generally that for \( 0 \leq v \leq d - j \)

\[
\begin{pmatrix}
    I_j & 0 \\
    0 & B^{(j)}
\end{pmatrix} \Delta(0, \ldots, 0, *, \ldots, *) \cdots \Delta(*, \ldots, *, 0, \ldots, *, 0, \ldots, *, *) = \begin{pmatrix}
    v + 1 & I_j & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & 0 \\
    v + 1 & \cdots & *
\end{pmatrix}.
\]

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One thus gets that if \( u \leq d - j + 1 \), then
\[
\begin{pmatrix}
I_j \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
B^{(j)}
\end{pmatrix}
\begin{pmatrix}
\Delta(0, \ldots, 0, k_{1}^{(1)}, \ldots, k_{d-j}^{(1)}) \\
\vdots \\
\Delta(0, \ldots, 0, k_{1}^{(2)}, \ldots, k_{d-j}^{(2)}) \\
\vdots \\
\Delta(0, \ldots, 0, k_{1}^{(u)}, \ldots, k_{d-j}^{(u)})
\end{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\]

has the desired form (7).

Now if \( u \geq d - j + 2 \), then using (5), one gets
\[
\begin{pmatrix}
0 \\
I_j
\end{pmatrix}
\begin{pmatrix}
0 \\
B^{(j)}
\end{pmatrix}
\begin{pmatrix}
\Delta(0, \ldots, 0, k_{1}^{(1)}, \ldots, k_{d-j}^{(1)}) \\
\vdots \\
\Delta(0, \ldots, 0, k_{1}^{(2)}, \ldots, k_{d-j}^{(2)}) \\
\vdots \\
\Delta(0, \ldots, 0, k_{1}^{(u)}, \ldots, k_{d-j}^{(u)})
\end{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\]

and suitable insertions of \( \Delta^j \) such as
\[
\begin{pmatrix}
0 \\
I_j
\end{pmatrix}
\begin{pmatrix}
0 \\
B^{(j)}
\end{pmatrix}
\begin{pmatrix}
\Delta(0, \ldots, 0, k_{1}^{(1)}, \ldots, k_{d-j}^{(1)}) \\
\vdots \\
\Delta(0, \ldots, 0, k_{1}^{(2)}, \ldots, k_{d-j}^{(2)}) \\
\vdots \\
\Delta(0, \ldots, 0, k_{1}^{(u)}, \ldots, k_{d-j}^{(u)})
\end{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\]

provide the desired form (7), which ends the proof of Step 1.

**Step 2.** By (5),

\[
B_j \Delta_{s_j+1} \cdots \Delta_{s_j+t} \Delta^{l+j}
\]

\[
\begin{pmatrix}
0 & \cdots & 0 & 1 \\
g_{1(t+1)} & \cdots & g_{1(d-j)} & g_{1(d+1-j)} \\
\vdots & \ddots & \vdots & \vdots \\
g_{(d-j)(t+1)} & \cdots & g_{(d-j)(d-j)} & g_{(d-j)(d+1-j)}
\end{pmatrix}
\begin{pmatrix}
I_j \\
0
\end{pmatrix}
\begin{pmatrix}
0 & \cdots & 0 \\
g_{1(t+1)} & \cdots & g_{1(d-j)} & g_{1(d+1-j)} \\
\vdots & \ddots & \vdots & \vdots \\
g_{(d-j)(t+1)} & \cdots & g_{(d-j)(d-j)} & g_{(d-j)(d+1-j)}
\end{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
I_j \\
0
\end{pmatrix}
\begin{pmatrix}
0 & \cdots & 0 & 1 \\
g_{1(t+1)} & \cdots & g_{1(d-j)} & g_{1(d+1-j)} \\
\vdots & \ddots & \vdots & \vdots \\
g_{(d-j)(t+1)} & \cdots & g_{(d-j)(d-j)} & g_{(d-j)(d+1-j)}
\end{pmatrix}
\]

...
We put
\[ G = \begin{pmatrix}
g_{11} & \cdots & g_{1(d-j)} \\
\vdots & \ddots & \vdots \\
g_{(d-j)1} & \cdots & g_{(d-j)(d-j)}
\end{pmatrix}.\]

Since the determinant of \( G \) is equal \( \pm 1 \), there exist \( k'_1, \ldots, k'_d \in \mathbb{Z}_m \) such that
\[
\begin{pmatrix}
I_j \\
0 \\
\vdots \\
0
\end{pmatrix}
\begin{pmatrix}
0 & \cdots & 0 & 1 \\
g_{11} & \cdots & g_{1(d-j)} & g_{1(d-j+1)} \\
\vdots & \ddots & \vdots \\
g_{(d-j)1} & \cdots & g_{(d-j)(d-j)} & g_{(d-j)(d-j+1)}
\end{pmatrix}
\Delta(0, \ldots, 0, k'_1, \ldots, k'_d) = \begin{pmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0
\end{pmatrix}.
\]

By applying (5), we get
\[
\begin{pmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 \\
I_j \\
\vdots \\
0
\end{pmatrix}
\begin{pmatrix}
G
\end{pmatrix}.
\]

We thus have proved that there exist J-P* matrices \( \Delta_{s_j+t+1}, \ldots, \Delta_{s_j+t'} \) such that
\[
B_j \Delta_{s_j+1} \cdots \Delta_{s_j+t'} = \begin{pmatrix}
I_j & 0 & 0 \\
0 & \cdots & 0 \\
0 & \cdots & 0 \\
0 & G & \cdots
\end{pmatrix}.
\]

It remains to apply Lemma 2; there thus exist J-P* matrices \( \Delta_{s_j+t'+1}, \ldots, \Delta_{s_j+1} \) such that
\[
B_j \Delta_{s_j+1} \cdots \Delta_{s_j+1} = \begin{pmatrix}
I_{j+1} & 0 \\
0 & B_{j+1}
\end{pmatrix},
\]

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where we put  
\[
B^{(j+1)} = \begin{pmatrix}
g_{12} & g_{13} & \cdots & g_{1(d-j)} & -g_{11} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
g_{(d-j)2} & g_{(d-j)3} & \cdots & g_{(d-j)(d-j)} & -g_{(d-j)1}
\end{pmatrix},
\]
which concludes the induction proof. \[\square\]

3 Ergodicity of \(T_m\) and proof of Theorem 1

3.1 Ergodicity

Let us recall some fundamental facts about Jacobi-Perron algorithm. For an integer vector \(a = (a_1, a_2, \ldots, a_d)\) with \(a_i \geq 0\) for \(1 \leq i \leq d\), we put  
\[
X_a = \{x \in X : k(x) = a\}.
\]

Then we see that \(X_a \neq \emptyset\) if and only if \(0 \leq a_i \leq a_d\) for any \(1 \leq i \leq d - 1\) and \(a_d > 0\). For a finite sequence of integer vectors \(\{a^{(l)}\} = (a^{(l)}_1, a^{(l)}_2, \ldots, a^{(l)}_d), 1 \leq l \leq n\) such that \(a^{(l)}_1 \leq a^{(l)}_d, 1 \leq l \leq d - 1, \) and \(a^{(l)}_d > 0\) for \(1 \leq l \leq n\), we define the cylinder set of rank \(n\) by  
\[
X_{a^{(1)}a^{(2)}\ldots a^{(n)}} = \{x \in X : k^{(l)}(x) = a^{(l)} \text{ for } 1 \leq l \leq n\}.
\]

A cylinder set \(X_{a^{(1)}a^{(2)}\ldots a^{(n)}}\) is said to be proper (or full) if \(T^n(X_{a^{(1)}a^{(2)}\ldots a^{(n)}}) = X\). It is easy to see that \(X_{a^{(1)}a^{(2)}\ldots a^{(n)}}\) is proper if \(a^{(l)}_1 \leq a^{(l)}_d\) for all \(1 \leq l \leq n\) and \(1 \leq i \leq d - 1\). The following is essential.

**Lemma 4.** For almost every \(x \in X\) there exists a sequence of positive integers \(n_1 < n_2 < \ldots\) such that \(X_{k^{(1)}(x)k^{(n_1)}(x)}\) is proper for any \(i \geq 1\).

This shows the exactness of the dynamical system \((X, T, \mu)\); the exactness means here that \(\bigcap_{n=1}^{\infty} T^{-n}X = \emptyset\) \((\mu\text{-mod } 0)\). In particular, \((X, T, \mu)\) is ergodic and strong mixing. We refer to F. Schweiger [12] or [14] about the theory of Jacobi-Perron algorithm. Now we will show the ergodicity of \(T_m\).

**Theorem 3.** The skew product \((X \times G(m), T_m, \mu \times \delta_m)\) is ergodic.

**Proof.** For any non-empty cylinder set \(X_{a^{(1)}a^{(2)}\ldots a^{(n)}}\), we see from Proposition 2 in [14] that  
\[
\sup_{x \in X_{a^{(1)}a^{(2)}\ldots a^{(n)}}} |DT^n(x)| < (d + 1)^{d+1} \inf_{x \in X_{a^{(1)}a^{(2)}\ldots a^{(n)}}} |DT^n(x)|, \tag{8}
\]
where \(|DT^n|\) is the Jacobian of \(T^n\). Suppose that \(\mathcal{M}\) is a \(T_m\)-invariant set of \((\mu \times \delta_m)\)-positive measure. Since \(T_m\) acts as \(T\) on the first coordinate, the ergodicity of \(T\) shows
\[
\{x \in X : (x, A) \in \mathcal{M} \text{ for some } A \in G(m)\} = X \ (\mu\text{-mod } 0).
\]
Thus there exists $A \in G(m)$ such that $(X \times \{A\}) \cap \mathcal{M}$ has positive $(\mu \times \delta_m)$-measure. We fix such a set $A$. By the density theorem and Lemma 4, for a given sequence $\varepsilon_i \searrow 0$ there exists a sequence of proper cylinder sets $W_i$ of rank $n_i$ and $B \in G(m)$ such that for all $i$
\[
\frac{(\mu \times \delta_m)((W_i \times \{A\}) \cap \mathcal{M})}{(\mu \times \delta_m)(W_i \times \{A\})} > 1 - \varepsilon_i
\] (9)
and
\[
A \cdot \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & a^{(1)}_1 \\
0 & 1 & \cdots & 0 & a^{(1)}_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & a^{(1)}_{d_i}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & a^{(2)}_1 \\
0 & 1 & \cdots & 0 & a^{(2)}_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & a^{(2)}_{d_i}
\end{pmatrix}
= B \pmod{m},
\]
where $(a^{(1)}_1, a^{(1)}_2, \ldots, a^{(1)}_{d_i}), \ldots, (a^{(n_i)}_1, a^{(n_i)}_2, \ldots, a^{(n_i)}_{d_i})$ are sequences of integers which define $W_i$. From (8) we see that (9) implies
\[
\frac{(\mu \times \delta_m)(T_m^n(W_i \times \{A\}) \cap \mathcal{M})}{(\mu \times \delta_m)(T_m^n(W_i \times \{A\}))} > 1 - (d + 1)^{d+1} \varepsilon_i.
\]
Since $W_i$ is proper and $\mathcal{M}$ is $T_m$-invariant, we conclude that
\[(X \times \{B\}) \cap \mathcal{M} = X \times \{B\} \pmod{m} \cdot (\mu \times \delta_m)\]
From Theorem 2, for any $C \in G(m)$ there exist J-P matrices $A_1, \ldots, A_s$ such that
\[C = BA_1 \cdots A_s \pmod{m}\]
with
\[
A_i = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & a^{(i)}_1 \\
0 & 1 & \cdots & 0 & a^{(i)}_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & a^{(i)}_{d_i}
\end{pmatrix}
\text{for } 1 \leq i \leq s.
\]
Moreover we can choose $A_1, \ldots, A_s$ so that the corresponding cylinder set $X_{a^{(i)}_1 \ldots a^{(i)}_{d_i}}$ with $a^{(i)}_1 = \ldots = a^{(i)}_{d_i}$, $1 \leq i \leq s$, is proper. This means
\[T_m^n(X \times \{B\}) \supset X \times \{C\}\]
and so $\mathcal{M} = X \times G(m) \cdot (\mu \times \delta_m)\pmod{0}$. Thus we get the assertion of the theorem. 

□

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3.2 Proofs

We are now able to give proofs of Proposition 1, Theorem 1, and Corollary 1.

**Proof of Proposition 1.** Let us recall that $C_m$ denotes the cardinality of $G(m)$. From Theorem 3 and the individual ergodic theorem, we have

$$
\lim_{N \to \infty} \frac{1}{N} \sharp \{1 \leq n \leq N : T_m^m(x, B) \in X \times \{A\}\} = \frac{1}{C_m}
$$

for $(\mu \times \delta_m)$-a.e. $(x, B)$. In particular, it holds for $(x, I_{d+1})$ for $\mu$-a.e. $x$. Since $T_m^m(x, I_{d+1}) = (T^n x, Q^m)$, we get the assertion.

**Proof of Theorem 1.**

For any $(\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \in \mathbb{Z}_m^{d+1}$, we denote by $N(\alpha_1, \alpha_2, \ldots, \alpha_{d+1})$ the number of elements in $G(m)$ such that the $(d+1)$th row is $(\alpha_1, \alpha_2, \ldots, \alpha_{d+1})$. We will show that

$$
N(\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) = C_m \cdot m^d, \quad (10)
$$

where $C_m$ denotes the cardinality of $SL(d, \mathbb{Z}_m)$ or $SL_\pm(d, \mathbb{Z}_m)$ if $d$ is even or odd, respectively. It is easy to see that

$$
N(0, \ldots, 0, 1) = C_m \cdot m^d. \quad (11)
$$

From Lemma 1, we note that there always exists $D \in G(m)$ such that the $(d+1)$th row is $(\alpha_1, \alpha_2, \ldots, \alpha_{d+1})$ for any $(\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \in \mathbb{Z}_m^{d+1}$. For any matrix $E$ of the form

$$
\begin{pmatrix}
* \\
0 & \ldots & 0 & 1
\end{pmatrix},
$$

$ED$ is of the form

$$
\begin{pmatrix}
* \\
\alpha_1 & \ldots & \alpha_d & \alpha_{d+1}
\end{pmatrix}.
$$

This implies

$$
N(\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \geq N(0, \ldots, 0, 1).
$$

On the other hand, for any matrix $D'$ of the form

$$
\begin{pmatrix}
* \\
\alpha_1 & \ldots & \alpha_d & \alpha_{d+1}
\end{pmatrix},
$$

we have

$$
\lim_{N \to \infty} \frac{1}{N} \sharp \{1 \leq n \leq N : T_m^m(x, B) \in X \times \{A\}\} = \frac{1}{C_m}
$$

for $(\mu \times \delta_m)$-a.e. $(x, B)$. In particular, it holds for $(x, I_{d+1})$ for $\mu$-a.e. $x$. Since $T_m^m(x, I_{d+1}) = (T^n x, Q^m)$, we get the assertion.

\[ \square \]
$D' \cdot D^{-1}$ is of the form
\[
\begin{pmatrix}
* \\
0 \ldots 0 1
\end{pmatrix},
\]
which implies
\[
N(\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \leq N(0, \ldots, 0, 1).
\]
Thus we have (10).

From Proposition 1 together with (10), we have
\[
\lim_{N \to \infty} \frac{\# \{ 1 \leq n \leq N : (q^{(n-d)}, q^{(n-d+1)}, \ldots, q^{(n)}) \equiv (\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \pmod{m} \}}{N} = \frac{C_m \cdot m^d}{C_m} = \frac{1}{c_m} \text{ for } \mu\text{-a.e. } x.
\]

Indeed one easily checks according to (3) and (4) that $C_m \cdot m^d = \frac{1}{c_m}$ holds. Since $\mu$ is equivalent to the Lebesgue measure, this holds for a.e. $x$ with respect to the Lebesgue measure. If we consider the $(d+1)$th column, then the same argument shows the other equality. This completes the proof of Theorem 1.

Proof of Corollary 1. For a given $a \in \mathbb{Z}_m$, let $\Gamma_a(m)$ denote the cardinality of the subset of $G(m)$ of matrices whose $(d+1, d+1)$-entry is equal to $a$.

Let us first assume that $a$ and $m$ are coprime. We then deduce from (11) that
\[
\Gamma_a(m) = \sum_{(\alpha_1, \ldots, \alpha_d, a) \in \mathbb{Z}_m} N(\alpha_1, \ldots, \alpha_d, a) = m^d \cdot C_m \cdot \varphi_d(m).
\]

Let us assume now that $m$ is a power of a prime divisor $p$ of $a$. One has $\gcd(\alpha_1, \ldots, \alpha_d, a, m) = 1$ if and only if $\gcd(\alpha_1, \ldots, \alpha_d, m) = 1$. Hence
\[
\Gamma_a(m) = \sum_{(\alpha_1, \ldots, \alpha_d, a) \in \mathbb{Z}_m} N(\alpha_1, \ldots, \alpha_d, a) = \sum_{(\alpha_1, \ldots, \alpha_d) : \gcd(\alpha_1, \ldots, \alpha_d, m) = 1} N(\alpha_1, \ldots, \alpha_d, a) = m^d \cdot C_m \cdot \varphi_d(m).
\]

It easily deduced from the Chinese remainder lemma that the functions $m \mapsto \Gamma_a(m)$, $m \mapsto \varphi_d(m)$, and $m \mapsto C_m$ are arithmetic multiplicative function. Hence one checks that
\[
\Gamma_a(m) = \frac{C_m \cdot \varphi_d(\gcd(m, a)) \cdot m^{2d}}{\gcd(m, a)^d}.
\]

It remains now to apply Theorem 1 to obtain the result, that is, for a.e. $x \in X$
\[
\lim_{N \to \infty} \frac{1}{N} \# \{ 1 \leq n \leq N : q^{(n)} \equiv a \pmod{m} \} = \frac{\Gamma_a(m)}{C_m \cdot m^{2d} \cdot \varphi_d(\gcd(a, m))} = \frac{m^d \cdot \varphi_d(\gcd(a, m))}{\gcd(a, m) \cdot \varphi_{d+1}(m)}
\]
Remark. Let $\mathbb{F}_q$ denote the finite field of cardinality $q$ and let $\mathbb{F}_q[X]$ be the set of polynomials with $\mathbb{F}_q$-coefficients. We denote by $\mathbb{L}$ the set of formal Laurent power series with negative degree. Since $\mathbb{L}$ is a compact Abelian group, there exists a unique normalized Haar measure $m$. We can define the Jacobi-Perron algorithm on $\mathbb{L}^d$ for any $d \geq 1$. In this case, $m^d$ is invariant under this algorithm.

Suppose that $\left(\frac{P_1^{(n)}}{Q_1^{(n)}}, \ldots, \frac{P_d^{(n)}}{Q_d^{(n)}}\right)$ is the $n$-th convergent of $(f_1, \ldots, f_d) \in \mathbb{L}^d$. For any $R \in \mathbb{F}_q[X]$, it is possible to prove the following: for any $A_1, \ldots, A_d, A_{d+1} \in \mathbb{F}_q[X]$ such that $A_1, \ldots, A_d, A_{d+1}, R$ are relatively prime,

$$\lim_{N \to \infty} \frac{\sharp\{1 \leq n \leq N : (P_1^{(n)}, \ldots, P_d^{(n)}, Q^{(n)}) \equiv (A_1, \ldots, A_d, A_{d+1}) \pmod{R}\}}{N} = c_R \text{ for } m^d\text{-a.e. } (f_1, \ldots, f_d) \in \mathbb{L}^d,$$

where $c_R$ is a constant depending only on $d$ and $R$. The proof is essentially the same as that of Theorem 1 of this paper. We refer to K. Inoue and H. Nakada [5] for the study of the rates of convergence for Jacobi-Perron algorithm over $\mathbb{L}^d$ and to R. Natsui [8] for the $\mathbb{L}$-version of Jager-Liardet’s result in the case of continued fractions.

References


