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A SYSTEM THEORETICAL CHARACTERIZATION OF FACTOR ANALYSIS

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ABSTRACT. Our purpose is to characterize "factor analysis" and "principal components analysis" from the system theoretical point of view. We believe that our characterization is clearer than the conventional one since our approach is mechanical, and therefore, ambiguous concepts are automatically clarified in terms of mechanics.

1. INTRODUCTION

We have an opinion that the fundamental spirit of "system theory" is *the mechanical world view,* that is, to understand "all" phenomena by an analogy of mechanics. Thus our present purpose is to describe "factor analysis" and "principal components analysis" in terms of mechanics.

Usually dynamical system theory is formulated in the following form:

$$\begin{cases} \frac{d\vec{x}(t)}{dt} = f(\vec{x}(t), \vec{u}_1(t), t), \quad \vec{x}(0) = \vec{x}_0 & \cdots \quad (\text{ state equation }), \\ \vec{y}(t) = g(\vec{x}(t), \vec{u}_2(t), t) & \cdots \quad (\text{ measurement equation }) \end{cases} \tag{1.1}$$

where $\vec{u}_1$ and $\vec{u}_2$ are external forces. It is natural to consider that system theory is modeled on mechanics. Thus, we assume that the state equation is motivated by Newtonian equation. On the other hand, there seems to be no firm opinion for the source of the measurement equation. *What fundamental theory is in hiding behind the measurement equation ?* Recently, in [2,3] we proposed a foundation of measurements, which was called "fuzzy measurement theory", or in short "measurement theory". Also, motivated by quantum mechanics, i.e., "quantum mechanics" = "Heisenberg's kinetic equation" + "Born's measurement axiom", we proposed the following new frame of "system theory":

$$\text{"system theory"} = \text{"the rule of time evolution" (or more generally, "the rule of the relation among systems")} + \text{"measurement theory"}, \tag{1.2}$$

which, of course, includes the conventional system theory (1.1).

We are convinced that the system theory (1.2) is quite rich. It is not too much to say that "all" phenomena can be analized in the frame of (1.2). For example, we proposed the fine (system theoretical) formulations of several statistical inferences, e.g., maximal likelihood method, Bayes's method, regression analysis, Kalman filter and so on (cf. [3,5,6]). Also, in [4], we asserted that the system theory (1.2) has great power of expression. In other words, there is a good hope that we can obtain a proper translation from "natural

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language" into "system theoretical language". The translation is regarded as a kind of system theoretical modeling problem in a broad sense. We believe that the translation is the essence of "fuzzy logic" (= "fuzzy set theory"). That is, we believe that Zadeh's original aim [8] is precisely the translation. Also, we consider that *"fuzzy system theory"* is characterized as the study concerning grade quantities (i.e., membership functions) in the system theory (1.2).

2. PRELIMINARIES: FUZZY MEASUREMENT THEORY

The theory of operator algebras is a convenient mathematical tool to describe both classical and quantum mechanics. Thus, the system theory (1.2) is described in terms of C^*-algebras since our theory is essentially mechanics.

Let $\mathcal{A}$ be a C^*-algebra, i.e., a Banach *-algebra satisfying the C^*-condition, An element F in $\mathcal{A}$ is called *self-adjoint* if it holds that $F = F^*$, where F^* is the adjoint element of F. A self-adjoint element F in $\mathcal{A}$ is called *positive* (and denoted by $F \geq 0$) if there exists an element F_0 in $\mathcal{A}$ such that $F = F_0^* F_0$. Let $\mathcal{A}^*$ be the dual Banach space of $\mathcal{A}$. That is, $\mathcal{A}^* = \{\rho : \rho$ is a continuous linear functional on $\mathcal{A}\}$ with the norm $|| \cdot ||_{\mathcal{A}^*}$ ($\equiv \sup\{|\rho(F)| \colon ||F||_{\mathcal{A}} \leq 1\}$). $\Big($ The linear functional $\rho(F)$ is sometimes denoted by ${}_{\mathcal{A}^*}\Big\langle \rho, F \Big\rangle_{\mathcal{A}}$. $\Big)$ Define the *mixed state class* $\mathfrak{S}^m(\mathcal{A}^*)$ such that $\mathfrak{S}^m(\mathcal{A}^*) = \{\rho \in \mathcal{A}^* \colon ||\rho||_{\mathcal{A}^*} = 1$ and $\rho(F) \geq 0$ for all $F \geq 0$ $\}$. A mixed state ρ ($\in \mathfrak{S}^m(\mathcal{A}^*)$) is called a *pure state* if it satisfies that "$\rho = \theta\rho_1 + (1-\theta)\rho_2$ for some $\rho_1, \rho_2 \in \mathfrak{S}^m(\mathcal{A}^*)$ and $0 < \theta < 1$" implies "$\rho = \rho_1 = \rho_2$". Define $\mathfrak{S}^p(\mathcal{A}^*) \equiv \{\rho^p \in \mathfrak{S}^m(\mathcal{A}^*) : \rho^p$ is a pure state $\}$, which is called a *state space*.

As a natural generalization of Davies' idea [1] in quantum mechanics, a C^**-observable* (or in short, *observable, fuzzy observable*) $\mathbf{O} \equiv (X, \mathcal{R}, F)$ in a C^*-algebra $\mathcal{A}$ is defined such that it satisfies that

(i). X is a set, and $\mathcal{R}$ is the subring of the power set $\mathcal{P}(X)$ ($\equiv \{\Xi : \Xi \subseteq X\}$),

(ii). for every $\Xi \in \mathcal{R}$, $F(\Xi)$ is a positive element in $\mathcal{A}$ such that $F(\emptyset) = 0$, and there exists a sequence $\{\Xi_i\}_{i=1}^\infty$ in $\mathcal{R}$ such that $\Xi_{i_1} \subseteq \Xi_{i_2}$ (if $i_1 \leq i_2$), $\cup_{i=1}^\infty \Xi_i = X$ and $\lim_{i\to\infty} \rho^m(F(\Xi_i)) = 1$ ($\forall \rho^m \in \mathfrak{S}^m(\mathcal{A}^*)$),

(iii). for any countable decomposition $\{\Xi_1, \Xi_2, ..., \Xi_n, ...\}$ of Ξ, ($\Xi, \Xi_n \in \mathcal{R}$), it holds that $\rho^m\Big(F(\Xi)\Big) = \lim_{N\to\infty} \rho^m\Big(\sum_{n=1}^N F(\Xi_n)\Big)$ $\quad (\forall \rho^m \in \mathfrak{S}^m(\mathcal{A}^*))$.

Our starting point is as follows.

AXIOM 0. [Fundamental concepts]. *With any system S, a C^*-algebra $\mathcal{A}$ can be associated in which the fuzzy measurement theory (or more generally, the system theory (1.2)) of that system can be formulated. A state of the system S is represented by a pure state ρ^p ($\in \mathfrak{S}^p(\mathcal{A}^*)$), an observable is represented by a C^*-observable $\mathbf{O} \equiv (X, \mathcal{R}, F)$ in the C^*-algebra $\mathcal{A}$. Also, a quantity (or, mechanical quantity) is represented by a self-adjoint element in the $\mathcal{A}$. (It will be generalized by several ways. For example see Remarks 2.1 and 3.1 later) The measurement of the observable $\mathbf{O}$ for the system S with the state ρ^p is denoted by $\mathbf{M}_{\mathcal{A}}(\mathbf{O}, S_{[\rho^p]})$ in the C^*-algebra $\mathcal{A}$. We can obtain a measured value x ($\in X$) by the measurement $\mathbf{M}_{\mathcal{A}}(\mathbf{O}, S_{[\rho^p]})$.*

The axiom presented below is analogous to (or, a kind of generalizations of) Born's probabilistic interpretation of quantum mechanics. We, of course, assert that the axiom is a principle for all measurements, i.e., classical and quantum measurements. Cf. [2,3].

AXIOM 1. [Measurement axiom]. *Consider a measurement* $\mathbf{M}_{\mathcal{A}}(\mathbf{O} \equiv (X, \mathcal{R}, F), S_{[\rho^p]})$ *formulated in a* C^**-algebra* $\mathcal{A}$. *Assume that the measured value* x $(\in X)$ *is obtained by the measurement* $\mathbf{M}_{\mathcal{A}}(\mathbf{O}, S_{[\rho^p]})$. *Then, the probability that the* x $(\in X)$ *belongs to a set* Ξ $(\in \mathcal{R})$ *is given by* $\rho^p(F(\Xi))$ $(\equiv {}_{\mathcal{A}^*}\left\langle \rho^p, F(\Xi) \right\rangle_{\mathcal{A}})$.

Of course it corresponds to "measurement theory" in (1.2). Also in this paper we omit to state Axiom 2, which says "the rule of the relation among systems" in (1.2). For the precise description, see [6,7].

Since our present concern is classical systems, we add the following remark.

Remark 2.1. (Commutative C^*-algebras). When $\mathcal{A}$ is a commutative C^*-algebra, that is, $F_1 \cdot F_2 = F_2 \cdot F_1$ holds for all F_1, $F_2 \in \mathcal{A}$, by Gelfand theorem, we can put $\mathcal{A}$ $= C_0(\Omega)$, the algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space Ω. It is well known that $C_0(\Omega)^* = \mathcal{M}(\Omega)$, i.e., the Banach space composed of all regular complex-valued measures on Ω (with the Borel σ-field $\mathcal{B}_\Omega$). And therefore, $\mathfrak{S}^m(\mathcal{M}(\Omega)) = \{\rho \in \mathcal{M}(\Omega) : \rho \geq 0, ||\rho||_{\mathcal{M}(\Omega)} = 1\}$, which is denoted by $\mathcal{M}_{+1}(\Omega)$. Also, it is well known that $\mathfrak{S}^p(\mathcal{M}(\Omega)) = \Big\{\delta_\omega \in \mathcal{M}(\Omega) :$ δ_ω is a point measure at $\omega \in \Omega$, i.e., ${}_{\mathcal{M}(\Omega)}\left\langle \delta_\omega, f \right\rangle_{C_0(\Omega)} = f(\omega)$ ($\forall f \in C_0(\Omega)$, $\forall \omega \in \Omega$) $\Big\}$, which is denoted by $\mathcal{M}^p_{+1}(\Omega)$. And therefore, we have the identification: $\Omega \ni \omega \longleftrightarrow \delta_\omega \in \mathcal{M}^p_{+1}(\Omega)$. Thus the locally compact Hausdorff space Ω may be also called a *state space*. A *quantity* may be generalized as a real-valued continuous function on Ω. Let $\mathbf{O} \equiv (X, \mathcal{R}, F)$ be an observable in a commutative C^*-algebra $\mathcal{A}$ $(\equiv C_0(\Omega)$). Note that, for any fixed Ξ $(\in \mathcal{R})$, the $F(\Xi)$ is a membership function on Ω, i.e., a continuous function on Ω such that $0 \leq [F(\Xi)](\omega) \leq 1$ $(\forall \omega \in \Omega)$. Thus, the $F(\cdot)$ will be sometimes denoted by $F_{(\cdot)}$, that is, $[F(\Xi)](\omega) = F_\Xi(\omega)$ $(\forall \Xi, \forall \omega)$. And therefore, $(X, \mathcal{R}, F)$ is often denoted by $(X, \mathcal{R}, F_{(\cdot)})$ in a commutative C^*-algebra $C_0(\Omega)$.

Example 2.2. (Gaussian observable). Put $\Omega = \mathbf{R}$, the real line. And let σ be any positive real. Define the *Gaussian observable* $\mathbf{O}_{G^\sigma} \equiv (\mathbf{R}, \mathcal{B}^{\mathrm{bd}}_{\mathbf{R}}, G^\sigma_{(\cdot)})$ in $C_0(\Omega)$ such that

$$G^\sigma_\Xi(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_\Xi e^{-\frac{(v-u)^2}{2\sigma^2}} dv \qquad (\forall \Xi \in \mathcal{B}_{\mathbf{R}}, \quad \forall u \in \Omega \equiv \mathbf{R}),$$

where $\mathcal{B}^{\mathrm{bd}}_{\mathbf{R}} = \{B : B$ is a bounded Borel set in $\mathbf{R}$ $\}$. Also, define the *finest "observable"* $\mathbf{O}_{\mathrm{FIN}} = (\mathbf{R}, \mathcal{B}^{\mathrm{bd}}_{\mathbf{R}}, \chi_{(\cdot)})$ in $C_0(\mathbf{R})$, which $\chi_\Xi : \Omega \to \{0, 1\}$ is the characteristic function of Ξ $(\in \mathcal{B}^{\mathrm{bd}}_{\mathbf{R}})$. The finest observable $\mathbf{O}_{\mathrm{FIN}}$ should be understood as the $\mathbf{O}_{G^\sigma}$ for sufficiently small σ.

3. A SYSTEM THEORETICAL APPROACH TO FACTOR ANALYSIS

Let Ω be a compact Hausdorff space. Consider quantities f_1, f_2,...,f_N in $C(\Omega)$ $(\equiv C_0(\Omega))$. For example, we may consider that the Ω represents the set of persons (or precisely, the state space representing the states of persons) with the discrete topology, and the height [resp. weight, etc.] of a person ω is represented by $f_1(\omega)$ [resp. $f_2(\omega)$, etc.].

Define the (real) linear space Span$[\{f_1, ..., f_N\}]$ of $C(\Omega)$ such that Span$[\{f_1, ..., f_N\}]$ $= \{\sum_{n=1}^N \alpha_n f_n : \alpha_n \in \mathbf{R}\}$. The Span$[\{f_1, ..., f_N\}]$ may be called the *"theoretical factor space"* (concerning $\{f_1, ..., f_N\}$). And the linear basis $\{e_1, e_2, ..., e_{N_0}\}$ $(1 \leq \exists N_0 \leq N)$ of Span$[\{f_1, ..., f_N\}]$ may be called the *"theoretical factors"*. However, it is, of course, too theoretical. Our concern is the "practical factor analysis" presented below.

Let $\nu \in \mathcal{M}_{+1}(\Omega)$. This normalized measure ν is assumed to be induced by some sequence $\{\omega_k\}_{k=1}^K$ in Ω (with sufficiently large K) such as $\nu \approx (1/K)\sum_{k=1}^K \delta_{\omega_k}$. In this sense, the ν may be called a *weight* (or, *mixed state*). Of course, the uniqueness of the $\{\omega_k\}_{k=1}^K$ is not guaranteed in general. For each n $(1 \le n \le N)$, consider an observable $\mathbf{O}_{\epsilon^n} \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}^{\rm bd}, \epsilon_{(\cdot)}^n)$ in $C_0(\mathbf{R})$ such that $\int_{\mathbf{R}} x\epsilon_{dx}^n(u) = u$ $(\forall u \in \mathbf{R})$, and $\Delta[\mathbf{O}_{\epsilon^n}]$ $\equiv [\int_{\mathbf{R}} |x-u|^2 \epsilon_{dx}^n(u)]^{1/2}$ is independent of u. In most cases, it suffices to consider that $\mathbf{O}_{\epsilon^n}$ $= \mathbf{O}_{G^{\sigma(n)}}$ (cf. Example 2.2).

Now we define the observable $\mathbf{O}_n \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}^{\rm bd}, \epsilon^n \circ f_n)$ in $C(\Omega)$ such that $(\epsilon^n \circ f_n)_\Xi(\omega) =$ $\epsilon_\Xi^n(f_n(\omega))$ $(\forall \Xi \in \mathcal{B}_{\mathbf{R}}^{\rm bd}, \forall \omega \in \Omega)$. Note that the $\Delta[\mathbf{O}_{\epsilon^n}]$ is interpretated by the measurement error of the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}_n, S_{[\delta_\omega]})$ for the quantity f_n (cf. Definition 4.12 in [3]). And furthermore, define the product observable $\mathbf{O} \equiv \times_{n=1}^N \mathbf{O}_n \equiv (\mathbf{R}^N, \mathcal{B}_{\mathbf{R}^N}^{\rm bd}, \times_{n=1}^N \epsilon^n \circ f_n)$ in $C(\Omega)$, that is, $(\times_{n=1}^N \epsilon^n \circ f_n)_{\Xi_1 \times \cdots \times \Xi_N}(\omega) = \Pi_{n=1}^N \; \epsilon_{\Xi_n}^n(f_n(\omega))$. Thus, we have the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[\delta_{\omega_k}]})$. And assume that, for each k $(1 \le k \le K)$, a measured value $\widehat{x}^k \equiv (x_1^k, x_2^k, ..., x_N^k)$ $(\in \mathbf{R}^N)$ is obtained by the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[\delta_{\omega_k}]})$. Axiom 1 says that $P_{\omega_k}(\Xi_1 \times \cdots \times \Xi_N)$, the probability that the $\widehat{x}^k$ belongs to $\Xi_1 \times \cdots \times \Xi_N$, is given by $(\times_{n=1}^N \epsilon^n \circ f_n)_{\Xi_1 \times \cdots \times \Xi_N}(\omega_k)$. Put $P_\nu(\Xi_1 \times \cdots \times \Xi_N) = \int_\Omega P_\omega(\Xi_1 \times \cdots \times \Xi_N)\nu(d\omega)$, which is considered to be the "averaging probability" of $P_\omega(\Xi_1 \times \cdots \times \Xi_N)$ concerning the weight ν. (Also, see Method 1, or its objective view, in [3].) A simple calculation shows the following equalities:

$$E_n^K \equiv \frac{1}{K}\sum_{k=1}^K x_n^k \approx \int_{\mathbf{R}^N} x_n P_\nu(dx_1 \cdots dx_N) \equiv E_n$$

for all n such that $1 \le n \le N$, and

$$V_{m,n}^K \equiv \frac{1}{K}\sum_{k=1}^K x_m^k \cdot x_n^k \approx \int_{\mathbf{R}^N} x_m \cdot x_n P_\nu(dx_1 \cdots dx_N) \equiv V_{m,n}$$

for all m and n such that $1 \le m, n \le N$, if K is sufficiently large.

We may denote that $E_n = E_n(f_n, \mathbf{O}_{\epsilon^n})$ and $V_{m,n} = V_{m,n}(\{f_n\}_{n=1}^N, \{\mathbf{O}_{\epsilon^n}\}_{n=1}^N)$. And we introduce the following equivalence relation $\approx_\nu$:

$$[\{f_n\}_{n=1}^N, \{\mathbf{O}_{\epsilon^n}\}_{n=1}^N] \approx_\nu [\{\check{f}_n\}_{n=1}^N, \{\mathbf{O}_{\check{\epsilon}^n}\}_{n=1}^N]$$

if and only if

$$E_n(f_n, \mathbf{O}_{\epsilon^n}) = E_n(\check{f}_n, \mathbf{O}_{\check{\epsilon}^n}) \text{ and } V_{m,n}(\{f_n\}_{n=1}^N, \{\mathbf{O}_{\epsilon^n}\}_{n=1}^N) = V_{m,n}(\{\check{f}_n\}_{n=1}^N, \{\mathbf{O}_{\check{\epsilon}^n}\}_{n=1}^N)$$

for all m and n such that $1 \le m, n \le N$. Under the above preparation, we can characterize "factor analysis" in what follows:

[♯] For given data $[x_n^k : 1 \le n \le N, 1 \le k \le K]$, an integer N_0 $(1 \le N_0 \le N)$ and a weight ν, find the pair $[\{f_n\}_{n=1}^N, \{\mathbf{O}_{\epsilon^n}\}_{n=1}^N]$ such that $E_n^K = E_n(f_n, \mathbf{O}_{\epsilon^n})$ $(1 \le \forall n \le N)$, $V_{m,n}^K = V_{m,n}(\{f_n\}_{n=1}^N, \{\mathbf{O}_{\epsilon^n}\}_{n=1}^N)$ $(1 \le \forall m, \forall n \le N)$, and the dimension of Span $[\{f_n\}_{n=1}^N]$ is equal to N_0. Or, under the same situation, find the smallest N_0.

The Span$[\{f_n\}_{n=1}^N]$ $(\subseteq L^2(\Omega, \nu))$ is called the "factor space" (concerning the data $[x_n^k]$). And the (orthonormal) basis of Span$[\{f_n\}_{n=1}^N]$ is called "factors".

Also we can consider another characterization of "factor analysis", which is mathematically equivalent to the above.

Remark 3.1. (Another characterization of "factor analysis"). For each n, define $\widehat{f}_n : \Omega \to \mathcal{M}_{+1}(\mathbf{R})$ such that $[\widehat{f}_n(\omega)](\Xi) = (\epsilon^n \circ f_n)_\Xi(\omega)$ $(\forall \Xi \in \mathcal{B}_{\mathbf{R}}^{\mathrm{bd}}, \forall \omega \in \Omega)$, which may be called *"Markov quantity"* or *"quantity with a noise"*. Also, the value $\Delta[\mathbf{O}_{\epsilon^n}]$ is called the *"noise part in $\widehat{f}_n$"*. Consider a statistical (or, average) measurement $\mathbf{M}_{C_0(\mathbf{R})}(\mathbf{O}_{\mathrm{FIN}} \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}^{\mathrm{bd}}, \chi_{(\cdot)}), S(\widehat{f}_n(\omega))$), which is equaivalent to the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}_n, S_{[\delta_\omega]})$ (cf. Method 1 in[3], or [7]). Thus, the above [♯] is also valid, and therefore, we can also understand "factor analysis" in terms of "quantity with a noise" (and not "measurement with an error"). This may be rather acceptable for statisticians. Of course we may also consider the mixed idea of both "quantity with a noise" and "measurement with an error".

Remark 3.2. (Principal components analysis). The "principal components analysis" may not be misleading even in the conventional formulation of statistics. However, it should be noted that the concept of "measurement error" (or, "noise in Markov quantity") is not clear in the conventional formulation but in the system theory (1.2). Thus, from the system theoretical point of view, we can add that "principal components analysis" must be used under the situation that the measurement error $\Delta[\mathbf{O}_{\epsilon^n}]$ can be ignored, i.e., $\Delta[\mathbf{O}_{\epsilon^n}] \approx 0$. Under the hypothesis, the system theoretical characterization is easy. Thus, details are left to the reader.

4. CONCLUSIONS

In this paper we proposed a mechanical (i.e., system theoretical) formulation of "factor analysis" and "principal components analysis". It should be noted that we started from the mechanical terms, *e.g., state, observable, quantity, measurement, measurement error, etc.,* and not the non-mechanical concepts, *e.g., Kolmogorov's mathematical probability space, random variable, common factor, unique factor, etc.* Note that "unique factor" is regarded as "measurement error" or "noise in Markov quantity" in our formulation. Thus, our assertion can be immediately understood in mechanics. Therefore, it is a matter of course that our proposal is clearer than the conventional one. And furthermore, we added the remark on "principal components analysis", that is, it must be used under the situation that the $\Delta[\mathbf{O}_{\epsilon^n}]$ can be ignored.

We hope that *our mechanical world view* will be examined and investigated from various view points.

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