A Confluent Hypergeometric System Associated with $\Phi_3$
and a Confluent Jordan-Pochhammer Equation

by

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Abstract. We treat a confluent hypergeometric system associated with $\Phi_3$. Near one of the singular loci of irregular type, asymptotic expansions and Stokes multipliers are obtained. Applying our results, we also clarify the asymptotic behaviour of linearly independent solutions of a confluent Jordan-Pochhammer equation.

1. Introduction

The series

\[
\Phi_3(\beta, \gamma, x, y) = \sum_{m,n \geq 0} \frac{(\beta)_m}{(\gamma)_{m+n} m! n!} x^m y^n \quad (|x| < \infty, |y| < \infty)
\]

with $(\beta)_m = \Gamma(\beta + m)/\Gamma(\beta)$ $(m \in \mathbb{Z})$ is one of the confluent hypergeometric functions derived from Appell’s hypergeometric function $F_1(\alpha, \beta, \beta', \gamma, x, y)$ ([5], [6], [9]). It satisfies a system of partial differential equations

\[
\begin{align*}
\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y} + (\gamma - x) \frac{\partial z}{\partial x} - \beta z &= 0, \\
\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + \gamma \frac{\partial z}{\partial y} - z &= 0
\end{align*}
\]

([2; §§5.7, 5.9]) for $(x, y) \in P^1(\mathbb{C}) \times P^1(\mathbb{C})$. Since $xz_{xy} - z_x + \beta z_y = 0$, this system is equivalent to a completely integrable Pfaffian system with respect to the unknown vector function $(z, xz_x, yz_y)$, which possesses the singular loci $x = 0, x = \infty, y = \infty$ of irregular type, and $y = 0$ of regular type. The solutions of (1.2), which are analytic in $\mathcal{R}^2$, constitute a three-dimensional vector space over $\mathbb{C}$, where $\mathcal{R}$ denotes the universal covering of $\mathbb{C} - \{0\}$. In [11], we studied the asymptotic behaviour of linearly independent solutions of (1.2) near the singular loci $x = \infty$ and $x = 0$. Eliminating the derivatives with respect to $x$ from (1.2), and putting $x = \kappa \in \mathbb{C} - \{0\}$, we obtain an ordinary differential equation of the form

\[
y \frac{d^3 z}{dy^3} - \left( \frac{y}{\kappa} + (\beta - \gamma - 1) \right) \frac{d^2 z}{dy^2} - \left( 1 + \frac{\gamma}{\kappa} \right) \frac{dz}{dy} + \frac{z}{\kappa} = 0,
\]

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which has the singular points $y = 0$ of regular type and $y = \infty$ of irregular type. It is easy to see that, for each fixed $x = \kappa (\in \mathbb{C} - \{0\})$, every solution of (1.2) satisfies equation (1.3). This is also derived from the Jordan-Pochhammer equation

\[
y(1 - y)(y - \kappa) \frac{d^3 z}{dy^3} + \left[((\beta' + 1)y + \beta \kappa)(1 - y) + ((\gamma + 1) - (\alpha + \beta' + 3)y)(y - \kappa)\right] \frac{d^2 z}{dy^2} + (\beta' + 1)(\gamma - (\alpha + \beta' + 1)y - \beta \kappa - (\alpha + 1)(y - \kappa)) \frac{dz}{dy} - \alpha \beta'(\beta' + 1)z = 0
\]

(see [4; §3.4]) by a process of making a confluence of singular points. In fact, replacing $(\alpha, \beta', \kappa, y)$ by $(1/\varepsilon, 1/\varepsilon, \varepsilon \kappa, \varepsilon^2 y)$ and letting $\varepsilon \to 0$, we arrive at equation (1.3), which is one of the confluent Jordan-Pochhammer equations.

The present paper gives asymptotic expansions and Stokes multipliers of linearly independent solutions of (1.2) near the irregular singular locus $y = \infty$, and clarify the global behaviour of the solutions of equation (1.3). As in [11] we assume that none of the complex numbers $\beta, \gamma, \beta - \gamma$ is an integer, and use the notation

\[
e^{(\lambda)} = \exp(2\pi i \lambda) \quad (\lambda \in \mathbb{C}).
\]

Recall solutions of (1.2) expressible in the form

\[
z_{-1} = z_{-1}(x, y) = (1 - e^{(-\beta)})^{-1} \int_{C_{-1}} f(\beta, \gamma; x, y, t) dt,
\]

\[
z_{a} = z_{a}(x, y) = \int_{C_{a}} f(\beta, \gamma; x, y, t) dt \quad (a = \pm 2),
\]

where

\[
f(\beta, \gamma; x, y, t) = t^{\beta - \gamma}(t - x)^{-\beta} \exp \left( t + \frac{y}{t} \right)
\]

([11; §2]). This integrand is obtained from one corresponding to $\Phi_2(\beta, 1/\varepsilon, \gamma, x, \varepsilon y)$ ([1]) by the limiting procedure $\varepsilon \to 0$. (For confluenes of the cycles of integral representations, see [3].) In each integral, the path and the branch of the integrand are taken under the condition

\[
0 \leq \arg y < \pi/2 < \arg x \leq \pi
\]
so that they have the following properties:

(1) The path $C_{-1}$ starts from $t = 0$, encircles $t = x$ in the positive sense, and returns to $t = 0$. Then, along $C_{-1}$, $(\arg t, \arg(t-x))$ varies from $(\pi + \arg y, -\pi + \arg x)$ to $(\pi + \arg y, \pi + \arg x)$.

(2) The path $C_2$ (or $C_{-2}$) starts from $t = 0$ and terminates in $t = \infty$. Then, along $C_2$ (or $C_{-2}$), $(\arg t, \arg(t-x))$ varies from $(\pi + \arg y, -\pi + \arg x)$ to $(\pi, -\pi)$ (or $(-\pi, -\pi)$).

Each integral is continued analytically to the whole domain $\mathcal{R}^2$, if we modify the path continuously preserving conditions imposed on $(\arg t, \arg(t-x))$ at both ends of it. We consider the triplet of linearly independent solutions $z_{-1}, z_2, z_{-2}$ of (1.2) near the singular locus $y = \infty$. (The linearly independence follows from [11; Proposition 2.1 and Theorem 3.1].) The main results concerning asymptotic expansions and Stokes multipliers of these solutions are stated in Section 2. The proofs of them are given in Section 3 and Section 4. In the calculation of asymptotic expansions, the saddle point method is employed, and in the derivation of Stokes multipliers, the monodromy matrices obtained in [11] are used. It may be interesting to treat these Stokes multipliers from a group-theoretic point of view ([8]). In the final section, we apply our results to equation (1.3), and clarify the global behaviour of its solutions, namely asymptotic expansions, Stokes multipliers (near $y = \infty$), and convergent series expansions in $0 < |y| < \infty$. They are described explicitly by well-known special functions. These solutions of (1.3) are expected to be applicable to a global study of a third or higher order linear differential equation with one or more irregular singularities (cf. [7], [10]).

2. Main results

In what follows, $\delta$ denotes an arbitrary small positive constant, $R$ an arbitrary one satisfying $R \geq 2.44$, and $\delta_R$ an arbitrary one satisfying

\begin{equation}
\sin^{-1}(2R(R^2 - 1)^{-1}) + \sin^{-1}(R^{-2}) < \delta_R < \pi/2.
\end{equation}

For example, we can take $\delta_R = \pi/100$ (if $R \geq 65$), $\delta_R = \pi/5$ (if $R \geq 4$), and $\delta_R = \pi/2 - \pi/821$ (if $R \geq 2.44$).

2.1. Asymptotic expansions
Let $P^{(a,b)}_m(s)$ be the Jacobi polynomial

$$P^{(a,b)}_m(s) = \sum_{j=0}^{m} \binom{a+m}{j} \binom{b+m}{m-j} \left( \frac{s+1}{2} \right)^j \left( \frac{s-1}{2} \right)^{m-j}$$

(see [2; §10.8,(12),(16)]).

**Theorem 2.1.** The solution $z_{-1}$ admits an asymptotic expansion of the form

$$z_{-1} \sim U_{-1}(x,y) = -e^{(\beta)\Gamma(1 - \beta)x^{-\beta-\gamma+2}y^{\beta-1}}e^{y/x+x}$$

$$\times \sum_{m \geq 0} (1 - \beta)_m (1 - x^2y^{-1})^{\beta-1-2m} P_m^{(1-\gamma,\beta-1-2m)}(1 - 2x^2y^{-1})y^{-m}$$

uniformly for $|xy^{-1/2}| < 1/R$ as $y$ tends to $\infty$ through the sector $|\arg(y/x) + \pi| < 3\pi/2 - \delta_R$.

**Theorem 2.2.** (i) The solution $z_2$ admits an asymptotic expansion of the form

$$z_2 \sim U_2(x,y) = -\sqrt{\pi}e^{(2\beta-\gamma)\pi y^{-\gamma+2+1/4}} \exp(-2y^{1/2})$$

$$\times \sum_{m \geq 0} (m + 1)_m 4^{-m} (1 + xy^{-1/2})^{-\beta-2m} P_{2m}^{(-3/2+\gamma-m,-\beta-2m)}(1 + 2xy^{-1/2})y^{-m/2}$$

uniformly for $|xy^{-1/2}| < 1/R$ as $y$ tends to $\infty$ through the sector $|\arg(y) < 3\pi - \delta, |\arg(y/x)| < 3\pi/2 - \delta_R$.

(ii) The solution $z_{-2}$ admits an asymptotic expansion of the form

$$z_{-2} \sim U_{-2}(x,y) = e^{(\gamma-\beta)}U_2(x,e^{2\pi i}y)$$

uniformly for $|xy^{-1/2}| < 1/R$ as $y$ tends to $\infty$ through the sector $|\arg(y + 2\pi| < 3\pi - \delta, |\arg(y/x) + 2\pi| < 3\pi/2 - \delta_R$.

### 2.2. Stokes multipliers

Let $S = S(\theta_1, \theta_2)$ denote a sector defined by

$$S(\theta_1, \theta_2) = \{ (x,y) \in R^2 \mid |\arg y - \theta_1| < 2\pi - \delta, |\arg(y/x) - \theta_2| < \pi - \delta_R \}.$$  

We call a matrix $T(S)$ ($\in GL(3, \mathbb{C})$) a Stokes multiplier corresponding to the sector $S$ with respect to $(z_{-1}, z_2, z_{-2})$, if linearly independent solutions $z_{S_{-1}}, z_{S_2}, z_{S_{-2}}$ such that

$$T(z_{-1}, z_2, z_{-2}) = T(S)T(z_{S_{-1}}, z_{S_2}, z_{S_{-2}})$$

satisfy

$$z_{S_{-1}} \sim U_{-1}(x,y), \quad z_{S_2} \sim U_2(x,y), \quad z_{S_{-2}} \sim U_{-2}(x,y)$$

uniformly for $|xy^{-1/2}| < 1/R$ as $y$ tends to $\infty$ through the sector $S$. 

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Theorem 2.3. We write \( S_1 = S(-\pi, -3\pi/2), S_2 = S(-\pi, -\pi/2), S_3 = S(\pi, -3\pi/2), S_4 = S(\pi, -\pi/2) \). Then the Stokes multipliers \( T_j = T(S_j) \) \((j = 1, 2, 3, 4)\) corresponding to these sectors with respect to \((z_1, z_2, z_3)\) are given by

\[
T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 - e^{(\beta)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 - e^{(-\beta)} & 0 & 1 \end{pmatrix},
\]

\[
T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 - e^{(\beta)} & 1 & 0 \\ 0 & e^{(-\beta)} + e^{(\gamma - \beta)} & 1 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 - e^{(-\beta)} & e^{(-\beta)} + e^{(\gamma - \beta)} & 1 \end{pmatrix}.
\]

3. Proofs of Theorems 2.1 and 2.2

3.1. Preliminaries

Consider the functions

\[
\tau = g(t) = t + y/t,
\]

\[
\tau = h(t) = g(t) + (\beta - \gamma) \log t - \beta \log(t - x),
\]

where \( \text{Im} \log s = \text{arg} s \). Integrand (1.4) is written in the form \( f(\beta, \gamma; x, y, t) = \exp h(t) \). In the proof of Theorem 2.2, we use the saddle points of \( h(t) \) and \( g(t) \), namely the roots of \( h'(t) = 0 \) and \( g'(t) = 0 \). In the following three lemmas, we assume that \(|xy^{-1/2}| < 1/2\), and that \(|y|\) is sufficiently large.

Lemma 3.1. The saddle points of \( g(t) \) are \( t_\pm = \pm y^{1/2} \), and those of \( h(t) \) are \( t_{\pm 1} \) and \( t_2 \), where \( t_{\pm 1} = \pm y^{1/2} + O(1) \), \( t_2 = x - \beta x^2 y^{-1} (1 - x^2 y^{-1})^{-1} (1 + O(y^{-1/2})) \).

Lemma 3.2. Let \( \mu \) be an arbitrary positive constant.

(i) For \(|t - t_-| \leq |y|^{1/2-\mu}\),

\[
g(t) - g(t_-) = -y^{-1/2}(t - t_-)^2(1 + O(|y|^{-\mu})).
\]

(ii) For \(|t - t_{\pm 1}| \leq |y|^{1/2-\mu}\),

\[
h(t) - h(t_{\pm 1}) = \pm y^{-1/2}(t - t_{\pm 1})^2(1 + O(|y|^{-\mu}));
\]

\[
h'(t) = \pm 2y^{-1/2}(t - t_{\pm 1})(1 + O(|y|^{-\mu})).
\]

(iii) For \(|t - t_2| \leq |y|^{-\mu}|x^2 y^{-1}| \ (0 < \mu \leq 1/2)\),

\[
h(t) - h(t_2) = (2\beta)^{-1} x^{-4} y^2 (1 - x^2 y^{-1})^2 (t - t_2)^2(1 + O(|y|^{-\mu}))
\]

\[
h'(t) = \beta^{-1} x^{-4} y^2 (1 - x^2 y^{-1})^2 (t - t_2)(1 + O(|y|^{-\mu})).
\]
**Lemma 3.3.** We have

\begin{align}
(3.6) & \quad h(t_{\pm}) = \pm 2y^{1/2} - (\gamma/2) \log y + O(1), \\
(3.7) & \quad h(t_2) = x^{-1}y(1 + x^2y^{-1} + O(y^{-1/3})).
\end{align}

Let \(\alpha\) be an arbitrary complex constant. For every non-negative integer \(k\) and for a fixed positive integer \(N\), we write

\[ R_{N+1}(\alpha - k, \sigma) = (1 - \sigma)^{\alpha - k} - \sum_{n \leq N} \frac{(k - \alpha)n}{n!} \sigma^n, \]

where the branch of \((1 - \sigma)^{\alpha - k}\) is taken such that \(\arg(1 - \sigma) = 0\) for \(\sigma < 1\). The following lemma is a special case of [11; Lemma 5.1].

**Lemma 3.4.** If \(N \geq \Re \alpha\), then \(|R_{N+1}(\alpha - k, \sigma)| \leq 2^k(k+1)^{N+1}K_N|\sigma|^{N+1}\) in the domain \(|\sigma| < 1/2\), where \(K_N\) is a positive constant independent of \(k\).

**Lemma 3.5** ([11; Lemma 5.2]). For any complex numbers \(a, b\) and for any non-negative integer \(m\),

\begin{align}
(3.8) & \quad \sum_{k \geq 0} \frac{(b)k(a + k)^m}{k!} \xi^k = m!(1 - \xi)^{-b-m} P_m^{(a-1,-b-m)}(1 - 2\xi)
\end{align}

in the domain \(|\xi| < 1\).

In order to calculate the asymptotic expansion of \(z_2\) as \(y\) tends to \(\infty\), we need to modify the path of integration \(C_2\), for each \((x, y)\) satisfying \(|xy^{-1/2}| < 1/R\), in such a way that \(C_2\) possesses the following properties.

(a) \(C_2\) consists of three curves \(\Gamma_{-}, \Gamma_{0}, \Gamma_{+}\) such that

(a.1) \(\Gamma_{0}\) is an arc passing through \(t = t_{-}\) and lying inside the circle \(K_{0}\) defined by \(|t - t_{-}| = |y|^{1/3}\);

(a.2) both ends \(a_{+}, a_{-}\) of \(\Gamma_{0}\) are located on \(K_{0}\);

(a.3) \(\Gamma_{-}\) (or \(\Gamma_{+}\)) is a curve starting from \(a_{-}\) (or \(a_{+}\)), tending to \(\infty\) (or 0), and lying outside the circle \(K_{0}\).

(b) \(C_2\) lies outside the circles \(|t - t_{1}| = |y|^{1/4}, |t - t_{2}| = |\beta||y|^{-1/4}x^2y^{-1}|.

(c) \(g(t) - g(t_{-}) \leq 0\) for \(t \in \Gamma_{0}\).

(d) \((d/d\rho) \Re h(t) \leq -c\), for \(t \in \Gamma_{-}\) (or \(t \in \Gamma_{+}\)), in which \(c\) is a positive constant and \(\rho = \rho(t)\) denote the length of a part of \(h(\Gamma_{-})\) (or \(h(\Gamma_{+})\)) from \(h(a_{-})\) (or \(h(a_{+})\)) to \(h(t)\).
Lemma 3.6. If \((x, y) \in \mathbb{R}^2\) satisfies \(|xy^{-1/2}| < 1/R \quad (R \geq 2.44), \ |y| > R_\infty\),
\begin{equation}
(3.9) \quad |\arg y| < 3\pi - \delta, \quad |\arg(y/x)| < 3\pi/2 - \delta_R,
\end{equation}
then we can modify the path \(C_2\) continuously with respect to \((x, y)\) preserving the properties above, where \(\delta\) and \(\delta_R\) are positive constants given in Section 2 and \(R_\infty\) is a sufficiently large positive constant.

Proof. First consider the special case where \(\arg x = \arg y = 0, \ |xy^{-1/2}| < 1/R,\) and \(\beta, \beta - \gamma \in \mathbb{R} - \mathbb{Z}\). Take the path \(C_2\) to be the negative real axis passing through \(t = t_- = -y^{1/2}\). It is expressed as \(C_2 = \Gamma_+ \cup \Gamma \cup \Gamma_0\) with \(\Gamma_- : t \leq a_-^0, \ \Gamma_0 : a_-^0 < t < a_+^0, \ \Gamma_+ : a_+^0 < t < 0\), where \(a_-^0 = t_- - y^{1/3}, \ a_+^0 = t_+ + y^{1/3}\). Then the images \(S_0 = g(\Gamma_0), \ T_0 = h(\Gamma_-), \ T_+ = h(\Gamma_+), \) are included in the negative real axis and expressed as \(S_0 : g(a_-^0) \leq \tau \leq g(t_-) = -2y^{1/2}, \ T_0 : \tau \leq h(a_-^0), \ T_+ : \tau \leq h(a_+^0)\), respectively. Observing that \(t_1 - (-t_-) = O(1)\), we can verify that \(C_2\) has the properties above.

Next we consider the case where \(\arg x = \arg y = 0\) is not necessarily satisfied and \(\beta, \beta - \gamma \in \mathbb{C} - \mathbb{Z}\). Take the segment \(S : \tau = g(t_-) - \sigma = (-2|y|^{1/6} \leq \sigma \leq 0)\) in the \(\tau\)-plane. By (3.1) the inverse image \(g^{-1}(S)\) passes through \(t_-\) and intersects the circle \(|t - t_-| = |y|^{1/3}\) at \(a_-, \ a_+\), which are continuous in \(y\) and, in case \(\arg x = \arg y = 0\), coincide with \(a_-^0, \ a_+^0\), respectively. We wish to choose curves \(T_-\) and \(T_+\) in the \(\tau\)-plane with the following properties.

(i) \(T_-\) (or \(T_+\)) is a curve starting from \(h(a_-)\) (or \(h(a_+)\)) and tending to \(\infty\), and lies outside the circles \(|\tau - h(t_1)| = 2, \ |\tau - h(t_2)| = |\beta|(1 + R^2)^2\).

(ii) \(d/d\rho \Re \tau \leq -c,\) for \(\tau \in T_-\) (or \(\tau \in T_+\)), where \(\rho\) denotes the length of a part of \(T_-\) (or \(T_+\)) from \(a_-\) (or \(a_+\)) to \(\tau\).

(iii) \(T_-\) (or \(T_+\)) is a continuous modification of \(T_0^-\) (or \(T_0^+\)).

Let \(\delta_R^*\) be a sufficiently small positive constant such that
\begin{equation}
(3.10) \quad \delta_R > \sin^{-1}(2R(R^2 - 1)^{-1}) + \sin^{-1}(R^2) + \delta_R^*,
\end{equation}
(cf. (2.1)). Note that \(g(\pm t_-) = \mp 2y^{1/2},\) and that \((a_\pm - t_-)^2/(a_\pm - t_-)^2 = 1 + O(|y|^{-1/2})\). We have \(H_\pm = (h(a_\pm) - h(t_-))/g(a_\pm) - g(t_-)) = 1 + o(1)\) (cf. (3.1), (3.2)). Hence, by (3.1) and by the definition of \(S\) given above,
\begin{equation}
(3.11) \quad h(a_\pm) = h(t_-) + H_\pm(g(a_\pm) - g(t_-)) = h(t_-) - |y|^{1/6}(1 + o(1))
\end{equation}
Furthermore (3.6) implies that

\[(3.12) \quad h(t_{\pm 1}) = g(\pm t_-) - (\gamma/2) \log y + O(1).\]

Since \(|xy^{-1/2}| < 1/R, R \geq 2.44\), it follows from (3.7) that

\[(3.13) \quad |g(\pm t_-)|/|h(t_2)| < 2R(R^2 - 1)^{-1} + o(1) < 0.99 + o(1).\]

By these estimates, if \(|y|\) is sufficiently large, as long as

\[(3.14) \quad |\arg g(-t_-)| < 3\pi/2 - \delta/2,\]
\[(3.15) \quad |\arg h(t_2)| < 3\pi/2 - \theta(x, y) - \delta^*_R\]

with \(\theta(x, y) = \sin^{-1}(|g(\pm t_-)|/|h(t_2)|) < \pi/2\), we can draw the curves \(T_-\) and \(T_+\) with the properties above (cf. Figures 3.1 and 3.2).

**Figure 3.1.**

**Figure 3.2.**

Once these curves are constructed, we obtain the desired modification \(C_2 = \Gamma_- \cup \Gamma_0 \cup \Gamma_+\), where \(\Gamma_-\) (or \(\Gamma_+\)) is one of the connected components of the inverse image \(h^{-1}(\Gamma_-)\) (or \(h^{-1}(\Gamma_+)\)) tending to \(t = \infty\) (or \(t = 0\)), and \(\Gamma_0 = \{t \in g^{-1}(S) \mid |t-t_-| \leq |y|^{1/3}\}.\) Since (3.14) is written as \(|\arg y| < 3\pi - \delta\), it remains to verify that (3.15) is valid in sector (3.9). Note that \(\arg h(t_2) = \arg(y/x) + \arg(1 + x^2y^{-1} + o(1))\) (cf. (3.7)). For sufficiently large \(|y|\), using (3.10), (3.13) and the inequality \(|\arg(1 + x^2y^{-1} + o(1))| < \sin^{-1}(R^{-2}) + o(1),\) we derive (3.15) from (3.9). Thus the lemma is proved. □
Lemma 3.7. Under the same hypotheses as in Lemma 3.6, for the path $C_2 = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$ given above, we have
\[
\int_{\Gamma_- \cup \Gamma_+} \exp(h(t)) \, dt = y^{-\gamma/2+1/4} \exp(-2y^{1/2}) E(x, y)
\]
with $E(x, y) = O(\exp(-|y|^{1/6}/2))$.

Proof. Since $1/h'(t)$ is analytic at $t \neq t_{\pm 1}$, $t_2$, from (b), (3.3), (3.5) combined with the maximum modulus principle, it follows that $|dt| = |1/h'(t)||dh/d\rho|d\rho = O(|y|^{1/4})d\rho$ for $t \in \Gamma_-$. The property (d) yields $\text{Re}(h(t) - h(a_-)) \leq -c\rho$ for $t \in \Gamma_-$. Using (3.6), (3.11) and this inequality, we obtain
\[
|\exp h(t)| \leq e^{-c\rho} |\exp h(a_-)| = e^{-c\rho} \left| \exp(h(t_{-1}) - |y|^{1/6}(1 + o(1))) \right| \\
\leq e^{-c\rho} |y^{-\gamma/2} \exp(-2y^{1/2})| \exp(-|y|^{1/6}/2)
\]
for $t \in \Gamma_-$. From this estimate and a similar one for $t \in \Gamma_+$, the lemma immediately follows. \(\square\)

3.2. Proof of Theorem 2.2

It is sufficient to show the asymptotic representation of $z_2$, from which we can derive that of $z_{-2}$ by using the relation
\[
z_{-2}(x, y) = e^{(\gamma-\beta)} z_2(x, e^{2\pi i} y)
\]
(see [11; Theorem 3.2]). Assume that $(x, y)$ satisfies the hypotheses of Lemma 3.6, and that the path $C_2$ has the properties (a),..., (d). Consider an integral of the form
\[
I = \int_{\Gamma_0} t^{\beta-\gamma}(t-x)^{-\beta} \exp(t + y/t) \, dt.
\]
We put $t = y^{1/2}(\sigma - 1)$, in which $\sigma$ moves along a curve $\Gamma_0^*$ inside the circle $|\sigma| = |y|^{-1/6}$. Taking $\arg t$ and $\arg(t-x)$ into consideration, we can write $t = e^{\pi i} y^{1/2}(1 - \sigma)$, $t-x = e^{-\pi i} y^{1/2}(1 - \sigma)(1 + xy^{-1/2}(1 - \sigma)^{-1})$ along $\Gamma_0$, where $\arg(1-\sigma) \to 0$, $\arg(1 + xy^{-1/2}(1 - \sigma)^{-1}) \to 0$ as $\sigma \to 0$, $xy^{-1/2} \to 0$. Observe that $g(t) = t + y/t = -2y^{1/2} - y^{1/2} \sigma^2 - y^{1/2} \sigma^3(1 - \sigma)^{-1}$. We wish to calculate an asymptotic expansion of the integral
\[
J = e^{(\gamma-2\beta)\pi i} y^{(\gamma-1)/2} \exp(2y^{1/2}) I = \int_{\Gamma_0^*} w(x, y, \sigma) \exp(-y^{1/2} \sigma^2) \, d\sigma,
\]
where

\[(3.19) \quad w(x,y,\sigma) = (1 - \sigma)^{-\gamma} \left(1 + \frac{xy^{-1/2}}{1-\sigma}\right)^{-\beta} \exp\left(-\frac{y^{1/2} \sigma^3}{1-\sigma}\right)\]

\[= \sum_{k \geq 0, p \geq 0} \frac{(\beta)_k}{k!p!} (-xy^{-1/2})^k (-y^{1/2})^p \sigma^{3p} (1 - \sigma)^{-\gamma - k - p}\]

for \(|xy^{-1/2}| < 1/R, |\sigma| \leq |y|^{-1/6}\). Let \(N\) be an arbitrary large fixed positive integer. By Lemma 3.4,

\[(1 - \sigma)^{-\gamma - k - p} = \sum_{n=0}^{N} \frac{(\gamma + k + p)_n}{n!} \sigma^n + O\left(2^{k+p}(k+p+1)^{N+1}\sigma^{N+1}\right)\]

Hence series (3.19) is written in the form

\[(3.20) \quad \sum_{k \geq 0} \sum_{p \geq 0} \sum_{n=0}^{N} \frac{(\beta)_k}{k!p!n!} (-xy^{-1/2})^k (-y^{1/2})^p \sigma^{3p+n} + E(x, y, \sigma)\]

Here, for \(|xy^{-1/2}| < 1/R, |\sigma| \leq |y|^{-1/6}\),

\[E(x, y, \sigma) = O\left(\sum_{k \geq 0} \frac{(|\beta|)_k}{k!} G_k(N, y, \sigma) R^{-k}\right)\]

with

\[G_k(N, y, \sigma) = \sum_{p \geq N+1} \sum_{n=0}^{N} \frac{|(\gamma + k + p)_n| y^{1/2} \sigma^3 |p|}{p!n!} + 2^k \sum_{p \geq 0} \frac{(k+p+1)^{N+1}}{p!} |\sigma|^{N+1} |2y^{1/2} \sigma^3|^p.\]

Observing that \(\sum_{n=0}^{N} (1/n!) (\gamma + k + p)_n = O(p^N (k + |\gamma| + N + 1)^N)\) uniformly for \(p \geq N + 1, k \geq 0\), and that \((k+p+1)^{N+1} \leq (k+1)^{N+1} (p+1)^{N+1}\) uniformly for \(p \geq 0, k \geq 0\), we have

\[(3.21) \quad E(x, y, \sigma) = O(|y^{1/2} \sigma^3|^{N+1} + |\sigma|^{N+1}).\]

From (c) and the fact that, in case \(\arg y = 0\), the path \(\Gamma^*_0\) coincides with the segment from \(t = t_- + y^{-1/6}\) to \(t = t_- - y^{-1/6}\), it follows that

\[(3.22) \quad \int_{\Gamma^*_0} \sigma^{q} \exp(-y^{1/2} \sigma^2) \, d\sigma =\begin{cases} -\Gamma((q+1)/2)y^{-(q+1)/4} + O(\exp(-|y|^{1/6})) & (q : \text{even}), \\
O(\exp(-|y|^{1/6})) & (q : \text{odd}).\end{cases}\]
Substitute (3.20) and (3.21) into (3.18), and put \( N = 2M, \ n + p = 2m \). Then, by (3.22), the integral \( J \) becomes

\[
-\sqrt{\pi} y^{-1/4} \sum_{m=0}^{M} (1/2)_m y^{-m/2} \sum_{k \geq 0} (\beta)_k \frac{k!}{k!} K_{k,m}(-xy^{-1/2})^k + O(y^{-(M+1)/2}),
\]

where

\[
K_{k,m} = \sum_{p=0}^{2m} \frac{(m+1/2)_p (-\gamma - k - 2m + 1)_{2m-p}}{p!(2m-p)!} = \frac{(\gamma - 1/2 + k - m)_{2m}}{(2m)!}.
\]

Using (3.8), we have an asymptotic expansion of \( I \):

\[
I = e^{(2\beta - \gamma)\pi i} y^{-(\gamma - 1)/2} \exp(-2y^{1/2}) J \sim U_2(x, y)
\]

uniformly for \( |xy^{-1/2}| < 1/R \) as \( y \) tends to \( \infty \) through (3.9). Combining this formula with Lemma 3.7, we arrive at the asymptotic representation of \( z_2 \).

### 3.3. Proof of Theorem 2.1

By [11; Proposition 2.2,(2.8)], we have

\[
z_{-1}(\beta, \gamma; x, y) = e^{\beta \pi i} e^{(\beta - \gamma)x - \beta y^{3/2}} z_1(\beta, \beta - \gamma + 2; e^{2\pi i} y/x, y),
\]

in which \( z_1(\beta, \gamma; x, y) \) is a solution of (1.2) admitting an asymptotic expansion of the form

\[
z_1(\beta, \gamma; x, y) \sim -e^{-\beta \pi i} \Gamma(1 - \beta) x^{\beta - \gamma} e^{x+y/x}
\]

\[
\times \sum_{m \geq 0} (1 - \beta)_m (1 - x^{-2} y)^{3/2 - 2m} P_m^{(\gamma - 1/2)}(1 - 2x^{-2} y)x^{-m}
\]

uniformly for \( |x^{-2} y| < 1/R_1 \) as \( x \) tends to \( \infty \) through the sector \( |\arg x - \pi| < 3\pi/2 - \delta_{R_1} \) (cf. [11; Theorem 4.1]). Here \( R_1 \) and \( \delta_{R_1} \) are arbitrary constants satisfying \( R_1 > 2 \) and \( 2 \sin^{-1}(R_1^{-1/2}) < \delta_{R_1} < \pi/2 \). Putting \( R_1 = R^2, \delta_{R_1} = \delta_R \), from the fact above, we obtain the desired asymptotic representation as \( y/x \) tends to \( \infty \).

### 4. Proof of Theorem 2.3

#### 4.1. Preliminaries

Consider the column vector functions \( \mathbf{u}(x, y) = \text{trans}(z_{-1}, z_2, z_{-2}), \mathbf{U}(x, y) = \text{trans}(U_{-1}(x, y), U_2(x, y), U_{-2}(x, y)) \) (cf. Section 2). From [11; Proposition 2.1,(2.3) and Theorem 3.2], we derive the following lemma.
Lemma 4.1. We have $u(xe^{2\pi i}, y) = M_1' u(x, y)$, $u(x, ye^{2\pi i}) = M_2' u(x, y)$, where

$$M_1' = \begin{pmatrix} e^{(-\gamma)} & e^{(-\beta)} & -e^{(-\gamma)} \\ e^{(-\gamma)} - e^{(-\beta)} & e^{(-\beta)} & e^{(-\beta - \gamma)} - e^{(-\gamma)} \\ e^{(-\beta)} - 1 & 0 & 1 \end{pmatrix},$$

$$M_2' = \begin{pmatrix} 1 & -1 & e^{(\beta - \gamma)} \\ 0 & 0 & e^{(\beta - \gamma)} \\ 0 & -1 & e^{(\beta - \gamma)} + 1 \end{pmatrix}.$$

The formal monodromy matrices are given by the following lemma.

Lemma 4.2. We have $U(xe^{2\pi i}, y) = P_1 U(x, y)$, $U(x, ye^{2\pi i}) = P_2 U(x, y)$, where

$$P_1 = \begin{pmatrix} e^{(-\beta - \gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} e^{(\beta)} & 0 & 0 \\ 0 & 0 & e^{(\beta - \gamma)} \\ 0 & -e^{(-\beta)} & 0 \end{pmatrix}.$$

4.2. Proof of Theorem 2.3

By Lemma 4.1,

$$z_2(xe^{-2\pi i}, y) = (e^{(\beta)} - 1)z_{-1}(x, y) + z_2(x, y),$$

$$z_{-2}(xe^{2\pi i}, y) = (e^{(-\beta)} - 1)z_{-1}(x, y) + z_{-2}(x, y).$$

Assume that $(x, y) \in S_1$. By Theorems 2.1 and 2.2, we have

$$z_{-1} \sim U_{-1}(x, y), \quad z_{-2} \sim U_{-2}(x, y).$$

Since $|\arg(y/(xe^{2\pi i}))| < \pi/2 < \pi - \delta_R$, it follows from Theorem 2.2 and Lemma 4.2 that $z_2(xe^{-2\pi i}, y) \sim U_2(xe^{-2\pi i}, y) = U_2(x, y)$. This relation and (4.3) combined with (4.1) yield the matrix $T_1$. In the sector $S_2$, observing that $|\arg(y/(xe^{2\pi i}))| + 5\pi/2 < \pi - \delta_R$ and using (4.2), we can derive $T_2$ in a similar way. If $(x, y) \in S_3$ then $(xe^{-2\pi i}, ye^{-2\pi i}) \in S_1$. Hence, by Lemmas 4.1 and 4.2, we have $u(x, y) = M_1' M_2' u(xe^{-2\pi i}, ye^{-2\pi i}) = M_1' M_2' T_1 U(xe^{-2\pi i}, ye^{-2\pi i}) = M_1' M_2' T_1 P_1^{-1} P_2^{-1} U(x, y)$, from which $T_3 = M_1' M_2' T_1 P_1^{-1} P_2^{-1}$ follows. Using the fact that $(x, y) \in S_4$ implies $(xe^{-2\pi i}, ye^{-2\pi i}) \in S_2$, we also derive $T_4 = M_1' M_2' T_2 P_1^{-1} P_2^{-1}$. Thus the proof is completed.
5. Confluent Jordan-Pochhammer equation (1.3)

In equation (1.3), assume that the constant $\kappa \in \mathbb{C} - \{0\}$ satisfies $-\varepsilon < \arg \kappa < 2\pi + \varepsilon$, where $\varepsilon$ is a small positive constant. Consider the triplet of linearly independent solutions $(z_{-1}^\kappa, z_2^\kappa, z_{-2}^\kappa)$ of (1.3) near $y = \infty$, in which $z_{-1}^\kappa = z_{-1}(\kappa, y)$, $z_2^\kappa = z_2(\kappa, y)$, and $z_{-2}^\kappa = z_{-2}(\kappa, y)$. By $L_{\nu}^{(\alpha)}(\xi)$ we denote the Laguerre polynomial

$$L_{\nu}^{(\alpha)}(\xi) = \frac{1}{\nu!} e^{\xi - \xi^\alpha} \left( \frac{d}{d\xi} \right)^\nu (e^{-\xi} \xi^\nu) = \sum_{j=0}^{\nu} \left( \begin{array}{c} \nu + \alpha \\ \nu - j \end{array} \right) \frac{(-\xi)^j}{j!}.$$

Let $\delta$ be an arbitrary small positive constant.

**Theorem 5.1.** The solution $z_{-1}^\kappa$ admits an asymptotic expansion of the form

$$z_{-1}^\kappa \sim U_{-1}^\kappa(y) = -e^{(1 - \beta)} \Gamma(1 - \beta) e^{\kappa - \beta - \gamma + 2 \gamma - 1} e^{y/\kappa} \sum_{n \geq 0} (1 - \beta)n^{\kappa} L_n^{(1 - \gamma)}(-\kappa)y^{-n}$$

as $y$ tends to $\infty$ through the sector $|\arg y - \arg \kappa + \pi| < 3\pi/2 - \delta$.

**Theorem 5.2.** (i) The solution $z_2^\kappa$ admits an asymptotic expansion of the form

$$z_2^\kappa \sim U_2^\kappa(y) = -\sqrt{\pi} e^{(2\beta - \gamma)\pi i} y^{-\gamma/2 + 1/4} \exp(-2y^{1/2}) \sum_{n \geq 0} Q_n(\beta, \gamma; \kappa) y^{-n/2}$$

with

$$Q_n(\beta, \gamma; \kappa) = \sum_{m=0}^{n} \frac{(\beta)_{n-m}(3/2 - \gamma - n)_{2m}}{4^m (n-m)! m!} (-\kappa)^{n-m}$$

as $y$ tends to $\infty$ through the sector $-3\pi/2 + \arg \kappa + \delta < \arg y < \min\{3\pi/2 + \arg \kappa, 3\pi\} - \delta$.

(ii) The solution $z_{-2}^\kappa$ admits an asymptotic expansion of the form

$$z_{-2}^\kappa \sim U_{-2}^\kappa(y) = e^{(\gamma - \beta)} U_2^\kappa(e^{2\pi i} y)$$

as $y$ tends to $\infty$ through the sector $-7\pi/2 + \arg \kappa + \delta < \arg y < \min\{-\pi/2 + \arg \kappa, \pi\} - \delta$.

**Proofs of Theorems 5.1 and 5.2.** It is sufficient to show that, after rearranging the terms of the formal series $U_{-1}(\kappa, y)$ (or $U_2(\kappa, y)$) in Theorem 2.1 (or Theorem 2.2), we obtain the asymptotic expression $U_{-1}^\kappa(y)$ (or $U_2^\kappa(y)$). By (3.8),

$$\left[-e^{(1 - \beta)} \Gamma(1 - \beta) e^{\kappa - \beta - \gamma + 2 \gamma - 1} e^{y/\kappa + \kappa} \right]^{-1} U_{-1}(\kappa, y) = \sum_{m \geq 0} \sum_{k \geq 0} (1 - \beta)_m e^{(2 - \gamma + k)m} \frac{m!}{m! k!} \kappa^{m+2k} y^{-m-k} = \sum_{n \geq 0} (1 - \beta)n^{\kappa} L_n^{(1 - \gamma)}(-\kappa)y^{-n}$$
\((n = m + k)\), which implies Theorem 5.1. Observing that
\[
[-\sqrt{\pi e^{(2\beta - \gamma)\pi i} y^{1/4 - \gamma/2} \exp(-2y^{1/2})}]^{-1} U_2(\kappa, y)
= \sum_{m \geq 0} \sum_{k \geq 0} \frac{(\beta)_{k} (\gamma - 1/2 + k - m)_{2m}}{4^m m! k!} (-\kappa)^k y^{-(m+k)/2},
\]
and putting \(m + k = n\), we obtain the asymptotic series \(U_2^\kappa(y)\). Thus the theorems are verified. \(\square\)

For a sector \(\Sigma (\subset \mathbb{R})\), we call a matrix \(T^\kappa(\Sigma) (\in \text{GL}(3, \mathbb{C}))\) a Stokes multiplier corresponding to \(\Sigma\) with respect to \((z_{-1}^\kappa, z_2^\kappa, z_{-2}^\kappa)\), if linearly independent solutions \(z_{-1}^\kappa, \Sigma^{-1} z_{-1}^\kappa, z_2^\kappa, \Sigma^{-2} z_{-2}^\kappa\) such that
\[
t((z_{-1}^\kappa, z_2^\kappa, z_{-2}^\kappa) = T^\kappa(\Sigma) t((z_{-1}^\kappa, \Sigma^{-1} z_{-1}^\kappa, z_{-2}^\kappa, z_{-2}^\kappa))\)

satisfy
\[
z_{-1}^\kappa, \Sigma \sim U_{-1}^\kappa(y), \quad z_2^\kappa, \Sigma \sim U_2^\kappa(y), \quad z_{-2}^\kappa, \Sigma \sim U_{-2}^\kappa(y)
\]
as \(y\) tends to \(\infty\) through \(\Sigma\). Let \(\Sigma_-, \Sigma_+\) be sectors defined by
\[
\Sigma_- = \{ y \in \mathbb{R} \mid -5\pi/2 + \arg \kappa + \delta < \arg y < -\pi/2 + \arg \kappa - \delta \},
\Sigma_+ = \{ y \in \mathbb{R} \mid -3\pi/2 + \arg \kappa + \delta < \arg y < \pi/2 + \arg \kappa - \delta \}.
\]
Then the Stokes multipliers corresponding to these sectors with respect to \((z_{-1}^\kappa, z_2^\kappa, z_{-2}^\kappa)\) are given by the following theorem, which immediately follows from Theorem 2.3.

**Theorem 5.3.** We have
\[
T^\kappa(\Sigma_-) = T_1, \quad T^\kappa(\Sigma_+) = T_2, \quad \text{if } -\epsilon < \arg \kappa \leq \pi/2,
T^\kappa(\Sigma_-) = T_1, \quad T^\kappa(\Sigma_+) = T_4, \quad \text{if } \pi/2 < \arg \kappa \leq 3\pi/2,
T^\kappa(\Sigma_-) = T_3, \quad T^\kappa(\Sigma_+) = T_4, \quad \text{if } 3\pi/2 < \arg \kappa < 2\pi + \epsilon,
\]
where \(T_j (j = 1, \ldots, 4)\) are matrices given in Theorem 2.3.

**Remark.** When \(\arg \kappa = \pi/2\) (or \(\arg \kappa = 3\pi/2\)), we may also take \(T^\kappa(\Sigma_+) = T_4\) (or \(T^\kappa(\Sigma_-) = T_3\)).
Around the regular singular point \( y = 0 \), we consider linearly independent solutions \( z_{-3}^\kappa, z_0^\kappa, z_3^\kappa \) expressible by the connection formulas

\[
\begin{align*}
  z_{-3}^\kappa &= (1 - e^{(-\beta)}) z_{-1}^\kappa + e^{(-\beta)} z_2^\kappa - z_{-2}^\kappa, \\
  z_0^\kappa &= (e^{(\gamma-\beta)} - 1) z_{-1}^\kappa - e^{(\gamma-\beta)} z_2^\kappa + z_{-2}^\kappa, \\
  z_3^\kappa &= -e^{(\gamma-\beta)} z_2^\kappa + z_{-2}^\kappa
\end{align*}
\]

(cf. [11; Proposition 2.1 and (3.2)]). From [11; Theorem 3.1], we obtain the convergent series expansions of these solutions.

**Theorem 5.4.** For \( y \in \mathbb{R} \), we have

\[
\begin{align*}
  z_{-3}^\kappa &= 2\pi i \sum_{n \geq 0} \frac{1}{\Gamma(\gamma)n!} F(\beta, \gamma + n, \kappa)y^n, \\
  z_0^\kappa &= \frac{2\pi i e^{\gamma \pi i} \Gamma(1 - \beta)}{\Gamma(\gamma - \beta) \Gamma(2 - \gamma)} \kappa^{1-\gamma} \\
  &\times \sum_{n \geq 0} \frac{(\gamma - 1)n(-\kappa)^{-n}}{\Gamma(\gamma - \beta)n!} F(\beta - \gamma + 1 - n, 2 - \gamma - n, -\kappa)y^n, \\
  z_3^\kappa &= -\frac{2\pi i e^{\beta \pi i}}{\Gamma(\beta - \gamma + 2)} \kappa^{-\beta - \gamma + 1} \\
  &\times \sum_{n \geq 0} \frac{1}{\Gamma(\beta - \gamma + 2)n!} \left( \sum_{m=0}^{n} \frac{(\beta)_m (-n)_m (-\kappa)^{-m}}{m!} \right) y^n,
\end{align*}
\]

where \( F(a, c, x) = \, _1F_1(a, c, x) \) is Kummer’s confluent hypergeometric function.

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\[ g(-t_-) \quad g(-t_-) \quad g(t_-) \quad g(t_-) \]
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\[ h(t_1) \quad h(t_1) \quad h(t_2) \quad h(t_2) \quad h(a_-) \quad h(a_-) \]
\[ \theta(x, y) \quad \theta(x, y) \quad T_- \quad T_- \quad 0 \quad 0 \quad \theta(x, y) \quad \theta(x, y) \quad T_- \quad T_- \quad 0 \quad 0 \]
\[ \pi/2 < \arg g(-t_-) < 3\pi/2 - \delta/2, \]
\[ -3\pi/2 + \theta(x, y) + \delta_R' < \arg h(t_2) < -\pi/2 \]
\[ -\pi/2 \leq \arg g(-t_-) \leq \pi/2, \]
\[ -3\pi/2 + \theta(x, y) + \delta_R' < \arg h(t_2) < -\pi/2 \]