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FUZZY LOGIC IN MEASUREMENTS*

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ABSTRACT. Our main interest in this paper is to translate from "natural language" into "system theoretical language". This is of course important since a statement in system theory can be analyzed mathematically or computationally. We assume that, in order to obtain a good translation, "system theoretical language" should have great power of expression. Thus we first propose a new frame of system theory, which includes the concepts of "measurement" as well as "state equation". And we show that a certain statement in usual conversation, i.e., fuzzy modus ponens with the word "very", can be translated into a statement in the new frame of system theory. Though our result is merely one example of the translation from "natural language" into "system theoretical language", we believe that our method is fairly general.

key words: Possibility theory, Membership Functions, Fuzzy Numbers, Approximate Reasoning, Linguistic Modeling, Measurements.

1. INTRODUCTION AND SYSTEM THEORY

Our main interest in this paper is to translate from "natural language" into "system theoretical language". This is, of course, important for some purpose since a statement in system theory can be analyzed mathematically or computationally. We assume that, in order to obtain a good translation, "system theoretical language" should have great power of expression. Also we believe that only the introduction of a new concept essentially enriches "system theoretical language". Recently, in [3] and [4] we proposed a foundation of measurements, which was also called "fuzzy measurement theory", or in short "measurement theory". This theory is a general measurement theory for both classical and quantum

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systems. That is, as a particular case, it includes Born's quantum measurement theory. Also, motivated by quantum mechanics, i.e., "quantum mechanics" = "Born's measurement axiom" + "Heisenberg's keinetic equation", in [4] we proposed the viewpoint of "mechanics" such as "mechanics" = "measurement axiom" + "keinetic equation". Thus we now propose the following frame of system theory:

$$\text{“(dynamical) system theory”} = \text{“measurement axiom”} + \text{“state equation”}, \qquad (1.1)$$

which is modeled on mechanics. We believe that this proposal (1.1) is natural. In fact the word "measurement" has been frequently used in system theory, but the theoretical foundation has never been proposed. Therefore, we consider that the above proposal (1.1) is merely the firm description of "usual system theory" and not a mathematical generalization of "usual system theory". Also, it should be noted that the above (1.1) determines the meaning of "system theoretical language".

The purpose of this paper is to show that the system theory (1.1) has great power of expression. If the system theory (1.1) is linguistically rich, there is a good hope that we can obtain a proper translation from "natural language" into "system theoretical language". This is precisely our motivation in this paper. Also note that the linguistic richness was already suggested in [3], or particularly, in Remark 3.3 of [3]. As an example of statements in a natural language, we consider the following "fuzzy modus ponens" :

If a tomato is red then the tomato is ripe.

This tomato is **very** red.

This tomato is **very** ripe. (1.2)

which often appears in our usual conversation. In this paper we try to translate this fuzzy modus ponens (1.2) in our usual conversation into a statement in the system theory (1.1), or particularly, in fuzzy measurement theory. Here note that we do not need the dynamical part of (1.1) for the present purpose though we can easily expect that it is essential for statements concerning time, for example, "fuzzy control".

In order to obtain a better translation, in Section 2 we prepare "fuzzy logic" in the system theory (1.1). And, in Section 3 we study "operation for grade quantities", which appears in a rough (or, coarse) measurement. Furthermore, we clarify the relation between "probability" and "grade". Note that these are consequences of "measurement axiom". Under these preparations, the translation of "fuzzy modus ponens (1.2)" is presented as Statement III in Section 3. Here we must note that Statement III is a mathematical theorem in the system theory (1.1). Thus, under the identification: (1.2) $\leftrightarrow$Statement III, for the first time we can say that "fuzzy modus ponens (1.2)" is true. Though our result is merely one example of the translation from "natural language" into "system theoretical language", we believe that our method is fairly general.

Zadeh's excellent ideas, or similar ideas, will be found here and there in this paper. Thus we consider that his and our proposals are closely connected, or they aim at the same target. However in this paper we are not concerned with the relation between the two. If the reader wants to compare them, we recommend him to try to represent "fuzzy modus ponens (1.2)" as a statement in Zadeh's theory. We expect that he uses some methods such as we prepare in Sections 2 and 3.

Now let us review the elementary mathematical results of C^*-algebras. Note that the theory of operator algebras is a convenient mathematical tool to describe both classical and quantum mechanics (cf. [1]). Thus our theory is described in terms of C^*-algebras as the proposal (1.1) is modeled on mechanics. Since our concern in this paper is classical systems and not quantum systems, it may suffice to consider only commutative C^*-algebras (cf. Remark 1.3 later). However, as our proposal (1.1) is originally motivated by a hint of quantum mechanics, we begin with general C^*-algebras. Note that the purpose of this section is to introduce "measurement axiom" in classical system theory, which is well known for quantum systems.

Let $\mathcal{A}$ be a C^*-algebra (cf. [1], [3], [4], [6]). For simplicity, in this paper we assume that $\mathcal{A}$ has the identity I. An element T $(\in \mathcal{A})$ is called *positive* (and denoted by $T \geq 0$

) if there exists an element T_0 ($\in \mathcal{A}$) such that $T = T_0^* T_0$ where T_0^* is the adjoint element of T_0. Let $\mathcal{A}^*$ be the dual Banach space of $\mathcal{A}$. That is, $\mathcal{A}^* \equiv \{\rho \ : \ \rho$ is a continuous linear functional on $\mathcal{A}\}$ with the norm $||\cdot||_{\mathcal{A}^*}$ ($\equiv \sup\{|\rho(T)|$: $||T||_{\mathcal{A}} \leq 1\}$). Define the *mixed state class* $\mathfrak{S}^m(\mathcal{A}^*)$ such that $\mathfrak{S}^m(\mathcal{A}^*) \equiv \{\rho \in \mathcal{A}^*$: $||\rho||_{\mathcal{A}^*} = 1$ and $\rho(T^*T) \geq 0$ for all $T \in \mathcal{A}$ $\}$. A mixed state ρ, i.e., $\rho \in \mathfrak{S}^m(\mathcal{A}^*)$, is called a *pure state* if it satisfies that "$\rho = \lambda\rho_1 + (1-\lambda)\rho_2$ for some $\rho_1, \rho_2 \in \mathfrak{S}^m(\mathcal{A}^*)$ and $0 < \lambda < 1$ " implies "$\rho = \rho_1 = \rho_2$". Define $\mathfrak{S}^p(\mathcal{A}^*) \equiv \{\rho^p \in \mathfrak{S}^m(\mathcal{A}^*) : \rho^p$ is a pure state $\}$, which is called a *state space.*

As a natural generalization of Davies' idea in quantum mechanics (cf. [2]), a C^*-*observable* (or in short, *observable, fuzzy observable*) $\mathbf{O} \equiv (X, \mathcal{F}, F)$ in a C^*-algebra $\mathcal{A}$ is defined such that it satisfies that

(i) X is a set, and $\mathcal{F}$ is the subfield of the power set $\mathcal{P}(X)$ ($\equiv \{\Xi : \Xi \subseteq X\}$),

(ii) for every $\Xi \in \mathcal{F}$, $F(\Xi)$ is a positive element in $\mathcal{A}$ such that $F(\emptyset) = 0$ and $F(X) = I$ (where 0 is the 0-element in $\mathcal{A}$),

(iii) for any countable decomposition $\{\Xi_1, \Xi_2, ..., \Xi_n, ...\}$ of Ξ, ($\Xi, \Xi_n \in \mathcal{F}$), it holds that $\rho\Big(F(\Xi)\Big) = \lim_{N\to\infty} \rho\Big(\sum_{n=1}^N F(\Xi_n)\Big)$ $\quad (\forall \rho \in \mathfrak{S}^m(\mathcal{A}^*))$.

Note that by Hopf extension theorem we can get the probability measure space $\big(X, \overline{\mathcal{F}}, \rho(F(\cdot))\big)$ where $\overline{\mathcal{F}}$ is the smallest σ-field that contains $\mathcal{F}$ (cf. [4]).

With any *system* S, a C^*-algebra $\mathcal{A}$ can be associated in which the fuzzy measurement theory of that system can be formulated. A *state* of the system S is represented by a pure state ρ^p ($\in \mathfrak{S}^p(\mathcal{A}^*)$), an *observable* is represented by a C^*-observable $\mathbf{O} \equiv (X, \mathcal{F}, F)$ in the C^*-algebra $\mathcal{A}$. Also, the *measurement of the observable* $\mathbf{O}$ *for the system* S *with the state* ρ^p is represented by $\mathbf{M}_{\mathcal{A}}(\mathbf{O}, S_{[\rho^p]})$ in the C^*-algebra $\mathcal{A}$.

The axiom presented below is analogous to (or, a kind of generalizations of) Born's probabilistic interpretation of quantum mechanics.

AXIOM 1. *Consider a measurement* $\mathbf{M}_{\mathcal{A}}(\mathbf{O} \equiv (X, \mathcal{F}, F),\ S_{[\rho^p]})$ *in a* C^**-algebra* $\mathcal{A}$. *Assume that* x ($\in X$) *is the measured value obtained by the measurement* $\mathbf{M}_{\mathcal{A}}(\mathbf{O}, S_{[\rho^p]})$.

Then, the probability that the x $(\in X)$ belongs to a set Ξ $(\in \overline{\mathcal{F}})$ is given by $\rho^p(F(\Xi))$.

For simplicity, in this paper we always assume that X is finite, and $\mathcal{F} = \mathcal{P}(X)$, i.e., the powers set of X.

Remark 1.1. We believe that this axiom dominates all measurements, i.e., classical and quantum measurements. In fact, as consequences of Axiom 1, in [3] and [4] we clarified several fundamental facts, for example, the justification of "standard syllogism", ergodic problem (i.e., the principle of equal weight in statistical mechanics), the foundation of Shannon's entropy, the errors in Heisenberg's uncertainty relation and so on. Also, the relation between Kolmogorov's probability theory and Axiom 1 was well discussed in [4]. In one word, the probability measure space $(X, \overline{\mathcal{F}}, \rho^p(F(\cdot)))$, for the first time, acquires a reality under Axiom 1.

Remark 1.2. A system S always has its state ρ^p $(\in \mathfrak{S}^p(\mathcal{A}^*)$). However, in usual cases we do not know the state ρ^p of the system S. If we know it, we may not need to measure it. Hence, when we want to emphasize that the state ρ^p is unknown, we often denote $\mathbf{M}_{\mathcal{A}}(\mathbf{O}, S_{[\rho^p]})$ by $\mathbf{M}_{\mathcal{A}}(\mathbf{O}, S_{[*]})$. And furthermore, $\mathbf{M}_{\mathcal{A}}(\mathbf{O}, S_{[*]})$ is identified with $\mathbf{O}$.

Remark 1.3. When $\mathcal{A}$ is a commutative C^*-algebra, i.e., $T_1T_2 = T_2T_1$ $(\forall T_1, T_2 \in \mathcal{A})$, by Gelfand theorem (cf. [6], [3]) we can put $\mathcal{A} = C(\Omega)$, the algebra composed of all complex-valued continuos functions on a compact space Ω. Thus, we have the identification: $\Omega \ni \omega \longleftrightarrow \delta_\omega \in \mathfrak{S}^p(C(\Omega)^*)$ where δ_ω is a point measure at ω, i.e., $\delta_\omega(f) = f(\omega)$ $(\forall \omega \in \Omega, \forall f \in C(\Omega))$. Under this identification, the Ω is also called a *state space*. An observable $(X, \mathcal{P}(X), F)$ in a commutative C^*-algebra $C(\Omega)$ is usually denoted by $(X, \mathcal{P}(X), f_{(\cdot)})$, where $f_\Xi = F(\Xi)$ $(\forall \Xi \in \mathcal{P}(X))$. Here, note that f_Ξ, $\Xi \in \mathcal{P}(X)$, is the membership function on Ω. Also, it clearly holds that $f_\Xi(\omega) = \sum_{x \in \Xi} f_{\{x\}}(\omega)$ $(\forall \omega \in \Omega)$ for all Ξ $(\in \mathcal{P}(X))$. Note that a bi-continuous map $\phi : \Omega \to \Omega$ is equivalent to a C^*-automorphism $\Phi : C(\Omega) \to C(\Omega)$ such that $(\Phi f)(\omega) = f(\phi(\omega))$ $(\forall f \in C(\Omega), \forall \omega \in \Omega)$. Thus any state equation on the state space Ω can be represented in terms of C^*-automorphisms as a C^*-

dynamical system. That is, our proposal (1.1) includes "usual dynamical system theory", if we put $\Omega = \mathbf{R}^n \cup \{\infty\}$, i.e., the one point compactification of $\mathbf{R}^n$

As mentioned in [3] and [4], we introduce the following classification in fuzzy measurement theory:

$$\text{fuzzy measurement theory}\begin{cases} \text{classical measurement theory (for classical systems)} \\ \\ \text{quantum measurement theory (for quantum systems)} \end{cases}$$

where a C^*-algebra $\mathcal{A}$ is commutative or non-commutative. Note that quantum measurement theory is well known as a principle of quantum mechanics. Our proposal (1.1) asserts that Axiom 1 is most fundamental for classical systems as well as quantum systems. It is surprising that measurement theory was first discovered in quantum mechanics. The reason will be mentioned later. In this paper we focus on classical measurements. However, it should be noted that all arguments in this paper can be easily applied to quantum systems. Cf. Remark 3.2 mentioned later.

Now we shall study a typical example of classical measurements, which will promote an understanding of our arguments in the next sections.

Consider a classical system S formulated by a commutative C^*-algebra $C(\Omega)$. A *physical quantity* (or in short, *quantity*) is represented by a real-valued (or more generally, $\mathbf{R}^m$-valued) continuous function on the state space Ω. Let $T : \Omega \to \mathbf{R}$ be a quantity on Ω. For example, assume that the value of the $T(\omega)$ represents the temperature of a room S with the state δ_ω ($\leftrightarrow \omega \in \Omega$, cf. Remark 1.3).

First we consider a precise measurement for the quantity T. Put $T_{\min} = \min_{\omega\in\Omega} T(\omega)$ and $T_{\max} = \max_{\omega\in\Omega} T(\omega)$. Let N be a (sufficiently large) natural number. Define the set X_N by $\left\{x_n \equiv \frac{nT_{\max}+(N-n)T_{\min}}{N} : n = 0,1,...,N\right\}$. Consider the *fuzzy numbers observable* $\mathbf{O}_N$ ($\equiv (X_N, \mathcal{P}(X_N), \zeta_{(\cdot)})$) in $C([T_{\min}, T_{\max}])$ such that

$$\zeta_{\{x_n\}}(\lambda) = \max\left\{0, \min\{L_n^+(\lambda), L_n^-(\lambda)\}\right\} \quad (\forall\lambda \in [T_{\min}, T_{\max}],\ n = 0,1,...,N), \qquad (1.3)$$

where

$$L_n^+(\lambda) = \frac{N\lambda - (n-1)T_{\max} - (N-n+1)T_{\min}}{T_{\max} - T_{\min}},$$
$$L_n^-(\lambda) = \frac{-N\lambda + (n+1)T_{\max} + (N-n-1)T_{\min}}{T_{\max} - T_{\min}}.$$

Thus, for any set $\Xi \in \mathcal{P}(X_N)$, the membership function ζ_Ξ on the closed interval $[T_{\min}, T_{\max}]$ is determined by $\sum_{x_n \in \Xi} \zeta_{\{x_n\}}(\lambda)$.

Using the fuzzy numbers observable $\mathbf{O}_N$, we define the observable $\mathbf{O}_N^T$ in $C(\Omega)$ such that $\mathbf{O}_N^T = \big(X_N, \mathcal{P}(X_N), \zeta_{(\cdot)} \circ T\big)$ where $(\zeta_\Xi \circ T)(\omega) = \zeta_\Xi(T(\omega))$ $(\forall \Xi \in \mathcal{P}(X), \forall \omega \in \Omega)$. Let ω_0 be any element in Ω. Thus we have the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}_N^T, S_{[\delta_{\omega_0}]})$ in $C(\Omega)$, i.e., the measurement of the observable $\mathbf{O}_N^T$ for the room (or system) with the state δ_{ω_0}. According to Axiom 1, the probability that the measured value obtained by the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}_N^T, S_{[\delta_{\omega_0}]})$ is equal to x_n is given by $\delta_{\omega_0}(\zeta_{\{x_n\}} \circ T) = \zeta_{\{x_n\}}(T(\omega_0))$. Here note, by (1.3), that $\zeta_{\{x_n\}}(T(\omega_0)) = 0$ if $|x_n - T(\omega_0)| \geq (T_{\max} - T_{\min})/N$. Therefore, if we get the measured value x_n $(\in X_N)$ by the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}_N^T, S_{[\delta_{\omega_0}]})$, then we can almost surely expect that

$$|T(\omega_0) - x_n| \leq (T_{\max} - T_{\min})/N.$$

Thus, under the hypothesis that N is sufficiently large, that is, a precise measurement is taken, we can consider the identification: $T \leftrightarrow \mathbf{O}_N^T$. The hypothesis is usually assumed in physics. Therefore, in most cases we do not need a fuzzy observable but a quantity in physics. In other words, Axiom 1 may be not needed if we are concerned with a precise measurement for T. However, it may be worth while mentioning that this fact is due to the peculiarity of classical measurements, that is, Axiom 1 is always needed for quantum measurements. We assume that this is the reason that Axiom 1 was first discovered in quantum mechanics.

On the other hand, a fuzzy observable is essential for a rough measurement of the quantity T. As the particular case of the above $\mathbf{O}_N$, i.e., $N = 1$, consider the fuzzy numbers observable $\mathbf{O}_\psi = (\{\mathrm{w}, \mathrm{c}\}, \mathcal{P}(\{\mathrm{w}, \mathrm{c}\}), \psi_{(\cdot)})$ in $C([T_{\min}, T_{\max}])$ such that

$$\psi_{\{\mathrm{w}\}}(\lambda) = \frac{1}{T_{\max} - T_{\min}}(\lambda - T_{\min}) \qquad (\forall \lambda \in [T_{\min}, T_{\max}])$$

and $\psi_{\{\mathrm{c}\}}(\lambda) = 1 - \psi_{\{\mathrm{w}\}}(\lambda)$. Here 'w' and 'c' may mean "warm" and "cool" respectively. Thus, we have the (rough) measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}_\psi^T, S_{[\delta_{\omega_0}]})$ in $C(\Omega)$ where $\mathbf{O}_\psi^T = \big(\{\mathrm{w}, \mathrm{c}\}, \mathcal{P}(\{\mathrm{w}, \mathrm{c}\}), \psi_{(\cdot)} \circ T\big)$. By Axiom 1, the probability that we get the measured value x ($\in \{\mathrm{w}, \mathrm{c}\}$) by the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}_\psi^T, S_{[\delta_{\omega_0}]})$ is given by

$$\delta_{\omega_0}(\psi_{\{x\}} \circ T) = \psi_{\{x\}}(T(\omega_0)) = \begin{cases} (T(\omega_0) - T_{\min})/(T_{\max} - T_{\min}) & \text{if } x = \mathrm{w} \\ (T_{\max} - T(\omega_0))/(T_{\max} - T_{\min}) & \text{if } x = \mathrm{c}. \end{cases} \tag{1.4}$$

Remark 1.4. Note that the above arguments are completely physical. Therefore, the satement S_p: *"The probability that the measured value 'w' is obtained by the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}_\psi^T, S_{[\delta_{\omega_0}]})$ is greater than 0.7."* is a physical one. Now consider the statement S_n: *"This room is warm."*, which is of course assumed to be a statement in our usual conversation. Therefore, the statement S_n is a "fuzzy" one. Here note the following fact.

($\sharp_1$) We can see whether the physical statement S_p is true or not. On the other hand, strictly speaking we can not say "yes" or "no" to the question "Is this room warm or not ?".

Thus there is an essential difference between "natural language" and "physical language". However we may think that these two S_p and S_n are not so different. If so, the correspondence: $S_n \mapsto S_p$ may be regarded as the translation from "natural language" into "physical language". Also, under the identification: $S_n \leftrightarrow S_p$, we can ascertain whether this room is warm or not. Though the translation: $S_n \mapsto S_p$ may not be good and will be improved in the following sections, it should be noted that even the translation: $S_n \mapsto S_p$ can not be presented in the conventional frame of physics, i.e., without Axiom 1. This seems to show something of the great power of expression. However, we must add that our original motivation of the proposal "fuzzy measurement theory" is not for a translation but to derive fundamental scientific facts (cf. Remark 1.1). Since system theory does not necessarily deal with only physical phenomena, the word "physical language" will be called "system theoretical language" in the following sections. Cf. Remark 3.1 later.

2. FUZZY LOGIC IN MEASUREMENTS.

The idea mentioned in Remark 1.4 is essential throughout this paper. In this and the next sections we shall try to formulate and develop the idea in the system theory (1.1), which is modeled on mechanics. Consider the following measurement $\mathbf{M}$ formulated in a commutative C^*-algebra $C(\Omega)$:

$$\mathbf{M} \ := \ \mathbf{M}_{C(\Omega)}\Big(\mathbf{O} \equiv (X, \mathcal{P}(X), f_{(\cdot)}), S_{[*]}\Big). \tag{2.1}$$

Now consider the statement such as

($\sharp_2$) *the measured value obtained by the measurement* $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[\delta_\omega]})$ *belongs to* Ξ.

This ($\sharp_2$) is called a *statement in* $\mathbf{M}$, and is denoted by $P_{\mathbf{M}}(\Xi, \omega)$.

The measurement $\mathbf{M}$ ($\approx \mathbf{O}$, cf. Remark 1.2) clearly determines the following correspondence:

$$\mathcal{P}(X) \ni \Xi \quad \mapsto \quad f_\Xi \in C(\Omega). \tag{2.2}$$

Here note that f_Ξ is a membership function on Ω. Thus, if we are allowed to use the term "fuzzy set operation" in Zadeh's theory, we can say that the formula (2.2) determines the rule of "fuzzy set operation" (cf. [4]). For example, $\Xi_1 \cap \Xi_2 \mapsto f_{\Xi_1 \cap \Xi_2}$, $\Xi_1 \cup \Xi_2 \mapsto f_{\Xi_1 \cup \Xi_2}$, $X \setminus \Xi \mapsto 1 - f_\Xi$ and so on. $\Big($ In Section 3, we study "fuzzy set operation for grade quantities", which is induced by (2.2). $\Big)$

Since the $P_{\mathbf{M}}(\Xi; \omega)$ is the notation of the above statement ($\sharp_2$), we see, by usual way, that, for any Ξ, Ξ_1, $\Xi_2 \in \mathcal{P}(X)$ and any $\omega \in \Omega$,

$$\begin{aligned}
&\neg P_{\mathbf{M}}(\Xi; \omega) = P_{\mathbf{M}}(X \setminus \Xi; \omega), \\
&P_{\mathbf{M}}(\Xi_1; \omega) \wedge P_{\mathbf{M}}(\Xi_2; \omega) = P_{\mathbf{M}}(\Xi_1 \cap \Xi_2; \omega), \\
&P_{\mathbf{M}}(\Xi_1; \omega) \vee P_{\mathbf{M}}(\Xi_2; \omega) = P_{\mathbf{M}}(\Xi_1 \cup \Xi_2; \omega), \\
&P_{\mathbf{M}}(\Xi_1; \omega) \to P_{\mathbf{M}}(\Xi_2; \omega) = P_{\mathbf{M}}((X \setminus \Xi_1) \cup \Xi_2; \omega).
\end{aligned}$$

That is, for any fixed ω ($\in \Omega$), the operations (i.e., "$\neg$", "$\wedge$" and so on) are closed in the class $\{P_{\mathbf{M}}(\Xi;\omega) : \Xi \in \mathcal{P}(X)\}$.

Next define a "truth function" of a statement $P_{\mathbf{M}}(\Xi;\omega)$ in $\mathbf{M}$. There is a very reason to consider that the "truth (= 1)" or "fault (= 0)" of the statement $P_{\mathbf{M}}(\Xi;\omega)$ is determined by the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[\delta_\omega]})$. That is, when we get the measured value x ($\in X$) by the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[\delta_\omega]})$, $\underline{\text{t.f.}}[P_{\mathbf{M}}(\Xi;\omega)]$, the (real) truth function of $P_{\mathbf{M}}(\Xi;\omega)$, is determined by

$$\underline{\text{t.f.}}[P_{\mathbf{M}}(\Xi;\omega)] = \begin{cases} 1 & (x \in \Xi) \\ 0 & (x \notin \Xi). \end{cases}$$

Though it is quite natural, in this paper we consider another "truth function". That is, we are interested in the (probabilistic) truth function of the statement $P_{\mathbf{M}}(\Xi;\omega)$ in $\mathbf{M}$ such that

$$\text{t.f.}[P_{\mathbf{M}}(\Xi;\omega)] = \delta_\omega(f_\Xi) \ \ (= f_\Xi(\omega)). \tag{2.3}$$

This is, of course, due to Axiom 1. Hence the "truth function" is the same as "probability". We believe that this truth function (2.3) is just fit to Aristotle's spirit. Cf. [5].

Remark 2.1. It should be noted that the statement "$P_{\mathbf{M}}(\Xi_1;\omega_1) \wedge P_{\mathbf{M}}(\Xi_2;\omega_2)$", $\omega_1 \neq \omega_2$, cannot be regarded as a statement *in* $\mathbf{M}$. In order to define the truth function of this kind of statement, we must prepare the repeated measurement $\mathbf{M}_{C(\Omega^2)}(\mathbf{O} \otimes \mathbf{O}, S_{[\delta_{\omega_1} \otimes \delta_{\omega_2}]})$. (For the repeated measurement, see [4] or the formula (2.7) mentioned later.) That is, we must start from $\mathbf{M}_{C(\Omega^2)}(\mathbf{O} \otimes \mathbf{O}, S_{[*\otimes *]})$ instead of $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]})$ in (2.1).

Now we shall prepare the following proposition, which is of course a statement in the system theory (1.1).

Proposition 2.2. (fuzzy modus ponens). *Let* $\mathbf{M}_{C(\Omega)}(\mathbf{O} \equiv (X, \mathcal{P}(X), f_{(\cdot)}), S_{[*]})$ *be a measurement in* $C(\Omega)$. *Let* Ξ_1 *and* Ξ_2 *be elements in* $\mathcal{P}(X)$. *Let* $\omega_0 \in \Omega$, $0 \leq \epsilon \leq \alpha' \leq 1$ *and* $0 \leq \alpha \leq \alpha' \leq 1$.

(i). *Assume that*

$$1 - \epsilon \leq \text{t.f.}[P_{\mathbf{M}}(\Xi_1;\omega_0) \rightarrow P_{\mathbf{M}}(\Xi_2;\omega_0)] \tag{2.4}$$

and

$$\alpha' \le \text{t.f.}[P_{\mathbf{M}}(\Xi_1;\omega_0)]. \tag{2.5}$$

Then, it holds that

$$\alpha' - \epsilon \le \text{t.f.}[P_{\mathbf{M}}(\Xi_2;\omega_0)]. \tag{2.6}$$

(ii). *Assume (2.5) and*

$$1 - \epsilon \le \text{t.f.}[P_{\mathbf{M}}(\Xi_1;\omega) \to P_{\mathbf{M}}(\Xi_2;\omega)] \tag{2.4'}$$

for all $\omega(\in\Omega)$ *such that* $\text{t.f.}[P_{\mathbf{M}}(\Xi_1;\omega)] \ge \alpha$. *Then, (2.6) holds.*

Proof. Since (ii) is the special case of (i), it suffices to show (i). We see that

$$\begin{aligned}\text{t.f.}[P_{\mathbf{M}}(\Xi_2;\omega_0)] &\ge \text{t.f.}[P_{\mathbf{M}}(\Xi_1\cap\Xi_2;\omega_0)] = \text{t.f.}[P_{\mathbf{M}}(\Xi_1\setminus(\Xi_1\setminus\Xi_2);\omega_0)] \\ &= \text{t.f.}[P_{\mathbf{M}}(\Xi_1;\omega_0)] - \big(1-\text{t.f.}[P_{\mathbf{M}}(\Xi_1;\omega_0)\to P_{\mathbf{M}}(\Xi_2;\omega_0)]\big) \ge \alpha' - \epsilon.\end{aligned}$$

This completes the proof.

As a corollary of Proposition 2.2 (ii), we have the following statement in the system theory (1.1), which may be regarded as the translation of the fuzzy modus ponens (1.2). However, we do not consider that it is a good translation. Cf. Remark 2.3 later.

Statement I. *Consider a finite set of tomatos. Let* Ω *be the state space with the discrete topology, in which each tomato is represented. Thus we can identify the set of tomatos with the state space* Ω. *And consider a measurement* $\mathbf{M}_{C(\Omega)}(\mathbf{O}\equiv(X,\mathcal{P}(X),f_{(\cdot)}),S_{[*]})$ *in* $C(\Omega)$. *Put* Ξ_1 = "RD" *and* Ξ_2 = "RP" *where "RD" and "RP" means "red" and "ripe" respectively. For example, put* $\epsilon = 0.05$, $\alpha = 0.7$ *and* $\alpha' = (\alpha)^{1/4} = 0.914\cdots$. $\Big($ *The reason that we put* $\alpha' = (\alpha)^{1/4}$ *will be mentioned in Statement II. However, this is not essential.* $\Big)$ *Assume that*

$$0.95\ (=1-\epsilon) \le \text{t.f.}[P_{\mathbf{M}}(\text{RD};\omega)\to P_{\mathbf{M}}(\text{RP};\omega)]$$

for all $\omega(\in\Omega)$ *such that* $\text{t.f.}[P_{\mathbf{M}}(\text{RD};\omega)] \ge 0.7\ (=\alpha)$. *And assume that, for some* ω_0 $(\in\Omega)$,

$$0.914\cdots(=\alpha') \le \text{t.f.}[P_{\mathbf{M}}(\text{RD};\omega_0)].$$

Then we see that

$$0.864\cdots(=\alpha'-\epsilon)\leq \text{t.f.}[P_{\mathbf{M}}(\text{RP};\omega_0)].$$

Remark 2.3. The fuzzy modus ponens (1.2) in Section 1 seems to be a statement concerning "grade". On the other hand, the above statement relates to "probability" since "truth function" is essentially equal to "probability". Therefore, we assume that there is a gap between "fuzzy modus ponens (1.2)" and Statement I. In other words, the translation : (1.2) $\mapsto$ Statement I is not good. In the next section, we clarify the relation between "probability" and "grade". And the translation will be improved as Statement III, which is essentially equivalent to Statement I.

Before we proceed to the next section, let us mention the system theoretical formulation of the word "very" in the fuzzy modus ponens (1.2). Let $\mathbf{M}^{\otimes n}\equiv \mathbf{M}_{C(\Omega^n)}\Big(\mathbf{O}^{\otimes n}\equiv (X^n,\mathcal{P}(X^n),\ f^{\otimes n}_{(\cdot)}),\ S_{[\otimes^n *]}\Big)$ be a repeated measurement of $\mathbf{M}_{C(\Omega)}(\mathbf{O}\equiv(X,\ \mathcal{P}(X),\ f_{(\cdot)}),\ S_{[*]})$. Here, the observable $\mathbf{O}^{\otimes n}$ in $C(\Omega^n)$ $(=\otimes^n_{k=1}C(\Omega))$ is defined by

$$f^{\otimes n}_{\Xi^1\times\Xi^2\times\cdots\times\Xi^n}(\omega_1,\omega_2,...,\omega_n)=f_{\Xi^1}(\omega_1)\cdot f_{\Xi^2}(\omega_2)\cdots f_{\Xi^n}(\omega_n) \tag{2.7}$$

for all $\Xi^k\in\mathcal{P}(X)$ and $\omega_k\in\Omega$, $k=1,2,\cdots n$. Also, the $\otimes^n\delta_{\omega_k}$ ($\equiv\otimes^n_{k=1}\delta_{\omega_k}$) in $S_{[\otimes^n *]}$ is the point measure at $(\omega_1,\omega_2,...,\omega_n)$ $(\in\Omega^n)$. For example put $n=4$. And put "$\overline{\text{RD}}$" $=\Xi_1\times X\times X\times X$, "$\overline{\text{RP}}$" $=\Xi_2\times X\times X\times X$, "$\overline{\text{VRD}}$" $=\Xi_1\times\Xi_1\times\Xi_1\times\Xi_1$ and "$\overline{\text{VRP}}$" $=\Xi_2\times\Xi_2\times\Xi_2\times\Xi_2$. And furthermore, put $\omega^{\otimes 4}=(\omega,\omega,\omega,\omega)$ $(\in\Omega^4)$. Here we see, by (2.7), that

$$\begin{aligned}&\text{t.f.}[P_{\mathbf{M}^{\otimes 4}}(\overline{\text{VRD}};\omega^{\otimes 4})]=f^{\otimes 4}_{\Xi_1\times\Xi_1\times\Xi_1\times\Xi_1}(\omega,\omega,\omega,\omega)=\big(f_{\Xi_1}(\omega)\big)^4\\&=\big(\text{t.f.}[P_{\mathbf{M}}(\text{RD};\omega)]\big)^4=\big(f^{\otimes 4}_{\Xi_1\times X\times X\times X}(\omega,\omega,\omega,\omega)\big)^4=\big(\text{t.f.}[P_{\mathbf{M}^{\otimes 4}}(\overline{\text{RD}};\omega^{\otimes 4})]\big)^4\end{aligned}$$

(where t.f.$[P_{\mathbf{M}}(\text{RD};\omega)]$ is defined in Statement I), and similarly, t.f.$[P_{\mathbf{M}^{\otimes 4}}(\overline{\text{VRP}};\omega^{\otimes 4})]=\big(\text{t.f.}[P_{\mathbf{M}^{\otimes 4}}(\overline{\text{RP}};\omega^{\otimes 4})]\big)^4=\big(\text{t.f.}[P_{\mathbf{M}}(\text{RP};\omega)]\big)^4$. Note that "taking a measurement $\mathbf{M}_{C(\Omega^4)}$ $(\mathbf{O}^{\otimes 4},\ S_{[\otimes^4\delta_\omega]})$" is of course the same as "taking measurements $\mathbf{M}_{C(\Omega)}(\mathbf{O},S_{[\delta_\omega]})$ four times". Therefore, there is a reason to consider that "$\overline{\text{VRD}}$" and "$\overline{\text{VRP}}$" respectively implies "very red" and "very ripe".

Here, we have the following statement in the system theory (1.1), which is essentially equal to Statement I.

Statement II. *Assume the above notations. Let* $\mathbf{M}^{\otimes 4} \equiv \mathbf{M}_{C(\Omega^4)}(\mathbf{O}^{\otimes 4}, S_{[\otimes^4 *]})$ *be the repeated measurememt of* $\mathbf{M}_{C(\Omega)}$ $(\mathbf{O},\, S_{[*]})$. *For example, put* $\epsilon = 0.05$, $\alpha = 0.7$. *Assume that*

$$0.95\ (= 1 - \epsilon) \le \text{t.f.}[P_{\mathbf{M}^{\otimes 4}}(\overline{\text{RD}}; \omega^{\otimes 4}) \to P_{\mathbf{M}^{\otimes 4}}(\overline{\text{RP}}; \omega^{\otimes 4})]$$

for all $\omega^{\otimes 4}$ $\big(\equiv (\omega, \omega, \omega, \omega) \in \Omega^4 \big)$ *such that* $\text{t.f.}[P_{\mathbf{M}^{\otimes 4}}(\overline{\text{RD}}; \omega^{\otimes 4})] \ge 0.7\ (= \alpha)$. *And assume that, for some* $\omega_0^{\otimes 4} \equiv (\omega_0, \omega_0, \omega_0, \omega_0) \in \Omega^4$,

$$0.7\ \big(= (0.914\cdots)^4 \big) \le \text{t.f.}[P_{\mathbf{M}^{\otimes 4}}(\overline{\text{VRD}}; \omega_0^{\otimes 4})].$$

Then we see that

$$0.559\cdots \big(= (0.864\cdots)^4 \big) \le \text{t.f.}[P_{\mathbf{M}^{\otimes 4}}(\overline{\text{VRP}}; \omega_0^{\otimes 4})].$$

3. GRADE AND PROBABILITY

As mentioned in Remark 2.3, in this section we shall clarify the relation between "probability" and "grade". This is essential for a better translation of the fuzzy modus ponens (1.2).

A *grade quantity* (or, *normalized quantity*) G on a state space Ω is defined by a quantity on Ω such that its range is included in the closed interval $[0, 1]$ (or more generally, $[0, 1]^m$). For example, define the $T^g : \Omega \to \mathbf{R}$ such that $T^g = \psi_{\{\text{w}\}} \circ T$ in (1.4), i.e., $T^g(\omega) = (T(\omega) - T_{\min})/(T_{\max} - T_{\min})$ $(\forall \omega \in \Omega)$. Then the $T^g : \Omega \to \mathbf{R}$ is a grade quantity. We know, of course, of many grade quantities in physics, for example, "coefficient of restitution", "[relative humidity]/100", "[refractive index]", etc.

In mathematics, we can define a lot of "operations for grade quantities", for example, $1 - G(\omega)$, $\min\{G_1(\omega), G_2(\omega)\}$, $G_1(\omega) \cdot G_2(\omega)$ and so on. However, it should be noted that these are usually meaningless in physics. For example, min $\{$ T^g, [relative humidity]/100

$\}$ is clearly meaningless in itself. Nevertheless, we shall demonstrate that "operation for grade quantities" is meaningful in certain measurements.

Let $G\ (\equiv (G_1, G_2)) : \Omega \to [0,1]^2$ be a grade quantity, i.e., a quantity on a state space Ω such that $0 \le G_k(\omega) \le 1$ ($\forall \omega \in \Omega,\ k = 1,2$). For example, we may consider that $G_1 = T^g$ and $G_2 =$ [relative humidity]/100.

First we consider a precise measurement for the grade quantity $G\ (\ \equiv (G_1, G_2))$. Let N be a (sufficiently large) natural number. Define the set X_N by $\Big\{x_n \equiv \frac{n}{N} : n = 0, 1, ..., N\Big\}$. Let $\mathbf{O}_{N}\ (\ \equiv (X_{N}, \mathcal{P}(X_{N}), \zeta_{(\cdot)})\ )$ be the fuzzy numbers observable in $C([0,1])$ as defined by (1.3) for $T_{\min} = 0$ and $T_{\max} = 1$. And, for each $k = 1,2$, put $\mathbf{O}_{N}^{G_k} = \big(X_{N}, \mathcal{P}(X_{N}),\ \zeta_{(\cdot)} \circ G_k\big)$, which is an observable in $C(\Omega)$. And furthermore, define the observable $\mathbf{O}_{N}^{G_1} \times \mathbf{O}_{N}^{G_1}\ (\equiv \mathbf{O}_{N}^{G})$ in $C(\Omega)$ by $\Big(X_{N}^{\ 2}, \mathcal{P}(X_{N}^{\ 2}), (\zeta_{(\cdot)} \circ G_1) \times (\zeta_{(\cdot)} \circ G_2)\Big)$ where

$$[(\zeta_{\Xi_1} \circ G_1) \times (\zeta_{\Xi_2} \circ G_2)](\omega) = \zeta_{\Xi_1}(G_1(\omega)) \cdot \zeta_{\Xi_2}(G_2(\omega)) \quad (\forall \Xi_1 \times \Xi_2 \in \mathcal{P}(X_{N}^{\ 2}), \forall \omega \in \Omega).$$

By the same arguments in Section 1, we can consider the identification: $\mathbf{O}_{N}^{G} \leftrightarrow G\ (\ \equiv (G_1, G_2)\ )$ if N is sufficiently large. Thus, if we are concerned with the precise measurement of G, we do not need Axiom 1. As mentioned in Section 1, this is the reason that Axiom 1 has been overlooked in the conventional description of classical mechanics.

Next we consider a rough measurement for the grade quantity $G\ (\ \equiv (G_1, G_2))$. For example, consider a fuzzy numbers observable $\mathbf{O}_\phi\ (\ \equiv (\{0,1\}^2, \mathcal{P}(\{0,1\}^2), \phi_{(\cdot)})\ )$ in $C([0,1]^2)$ such that

$$\phi_{\{1\}\times\{0,1\}}(\lambda_1, \lambda_2) = \lambda_1, \qquad \phi_{\{0,1\}\times\{1\}}(\lambda_1, \lambda_2) = \lambda_2,$$
$$(\forall (\lambda_1, \lambda_2) \in [0,1]^2). \tag{3.1}$$

$\Big($ In this paper, an observable in $C(K)$ is called a *fuzzy numbers observable* where K is a compact subset of $\mathbf{R}^m \cup \{\infty\}$. $\Big)$ The existence is of course guaranteed, for example, we can define the $\mathbf{O}_{\phi'}$ such as ${\phi'}_{\{1\}\times\{1\}}(\lambda_1, \lambda_2) = \lambda_1 \cdot \lambda_2$, ${\phi'}_{\{1\}\times\{0\}}(\lambda_1, \lambda_2) = \lambda_1 \cdot (1 - \lambda_2)$, ${\phi'}_{\{0\}\times\{1\}}(\lambda_1, \lambda_2) = (1 - \lambda_1) \cdot \lambda_2$ and ${\phi'}_{\{0\}\times\{0\}}(\lambda_1, \lambda_2) = (1 - \lambda_1) \cdot (1 - \lambda_2)$. Also, we can define the $\mathbf{O}_{\phi''}$ such as ${\phi''}_{\{1\}\times\{1\}}(\lambda_1, \lambda_2) = \min\{\lambda_1, \lambda_2\}$, ${\phi''}_{\{1\}\times\{0\}}(\lambda_1, \lambda_2) = \lambda_1 - \min\{\lambda_1, \lambda_2\}$, ${\phi''}_{\{0\}\times\{1\}}(\lambda_1, \lambda_2) = \lambda_2 - \min\{\lambda_1, \lambda_2\}$ and ${\phi''}_{\{0\}\times\{0\}}(\lambda_1, \lambda_2) = 1 - \max\{\lambda_1, \lambda_2\}$. We may call the $\mathbf{O}_{\phi''}$ a *Lukasiewicz fuzzy numbers observable* (cf. the formula (3.5) later).

Using the above observable $\mathbf{O}_\phi$, we have the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}_\phi^G \equiv (\{0,1\}^2, \mathcal{P}(\{0,1\}^2), \phi_{(\cdot)} \circ G), S_{[*]})$ in $C(\Omega)$. Let $\mathbf{O} = (X, \mathcal{P}(X), f_{(\cdot)})$ be as in (2.1). And put $\mathbf{O} = \mathbf{O}_\phi^G$. Hence, $X = \{0,1\}^2$, $f_\Xi(\omega) = \phi_\Xi(G_1(\omega), G_2(\omega))$ ($\forall \Xi \subseteq \{0,1\}^2$, $\forall \omega \in \Omega$). And furthermore, put $\Xi_1 = \{1\} \times \{0,1\}$ and $\Xi_2 = \{0,1\} \times \{1\}$. Thus, we see, by (2.3) and (3.1), that

$$\text{t.f.}[P_{\mathbf{M}}(\Xi_1; \omega)] = f_{\Xi_1}(\omega) = \phi_{\{1\}\times\{0,1\}}(G_1(\omega), G_2(\omega)) = G_1(\omega) = \text{"grade"} \tag{3.2}$$

and similarly, $\text{t.f.}[P_{\mathbf{M}}(\Xi_2; \omega)] = f_{\Xi_2}(\omega) = G_2(\omega) =$ "grade". Therefore, in this circumstance, we see that "probability (= truth function)" = "grade".

Also, recall (2.2). Then, we have the following correspondence:

$$\mathcal{P}(X) \ni \Xi \quad \mapsto \quad f_\Xi(\omega) = \phi_\Xi(G_1(\omega), G_2(\omega)) \in C(\Omega). \tag{3.3}$$

For example, if $\phi = \phi''$ (i.e., Lukasiewicz type), we see that

$$\Xi_1 \cap \Xi_2 \; (\equiv \{1\} \times \{1\}) \mapsto \min\{G_1(\omega), G_2(\omega)\}, \tag{3.4}$$

$$X \setminus \Xi_1 \; (\equiv \{0\} \times \{0,1\}) \mapsto 1 - G_1(\omega),$$

$$(X \setminus \Xi_1) \cup \Xi_2 \; (\equiv \{0,1\}^2 \setminus (\{1\} \times \{0\})) \mapsto \min\{1, 1 - G_1(\omega) + G_2(\omega)\} \tag{3.5}$$

and so on. Note that these are operations for grade quantities G_1 and G_2. Thus operations for grade quantities are meaningful in this circumstance. Using the word "fuzzy set operation" in Zadeh's theory, we may say that "fuzzy set operation" can be found in (3.3) as well as (2.2). That is, grade quantities G_1 and G_2 behave like "sets" in (3.3). For example, putting $G_1 = T^g$, $G_2 =$ [relative humidity]/100 and $\phi = \phi''$, we see that the (system theoretical) meaning of *"warm and humid"* is given by $\min\{\, T^g$, [relative humidity]/100 $\}$ as in (3.4). However note that it is meaningless in itself, i.e., without the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}^G_{\phi''},\ S_{[*]}\ )$. Therefore, we may say that "logic" is produced by "measurement".

As an immediate consequence of Satement I and (3.2), now we have the following statement in the system theory (1.1), which is our recommendable translation of the fuzzy modus ponens (1.2).

Statement III. *Let Ω be a state space, in which states of tomatos are represented. Thus we identify the set of tomatos with the state space Ω. Let $G\ (\equiv (G_1, G_2)) : \Omega \to [0,1]^2$ be a grade quantity on Ω. Here assume that the value of $G_1(\omega)$ [resp. $C_2(\omega)$] represents the grade of the "redness" [resp. "ripeness"] of a tomato ω. Consider the fuzzy numbers observable $\mathbf{O}_\phi \equiv (\{0,1\}^2, \mathcal{P}(\{0,1\}^2),\ \phi_{(\cdot)})$ in $C([0,1]^2)$ that satisfies (3.1). And consider a measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O} \equiv (X, \mathcal{P}(X), f_{(\cdot)}), S_{[*]})$ in $C(\Omega)$ such that $\mathbf{O} = \mathbf{O}^G_\phi$, i.e., $X = \{0,1\}^2$, $f_\Xi(\omega) = \phi_\Xi(G_1(\omega), G_2(\omega))$ $(\forall \Xi \subseteq \{0,1\}^2,\ \forall \omega \in \Omega)$. And furthermore, put "RD" $= \Xi_1 = \{1\} \times \{0,1\}$ and "RP" $= \Xi_2 = \{0,1\} \times \{1\}$. Also, for example, put $\epsilon = 0.05$, $\alpha = 0.7$ and $\alpha' = 0.914\cdots$. Assume that*

$$0.95 \leq \text{t.f.}[P_{\mathbf{M}}(\text{RD}; \omega) \to P_{\mathbf{M}}(\text{RP}; \omega)]$$

for all $\omega (\in \Omega)$ such that $G_1(\omega) \big(=$ "the redness of a tomato ω" $\big) \geq 0.7$. And assume that

$$0.914\cdots \leq G_1(\omega_0)\ \big(= \text{"the redness of a tomato } \omega_0\text{"}\big).$$

Then we see that

$$0.864\cdots \leq G_2(\omega_0)\ \big(= \text{"the ripeness of the tomato } \omega_0\text{"}\big).$$

Note that this statement, as well as the previous statements I and II, is system theoretical. That is, the grade quantity G_k (="redness", or ="ripeness") is assumed to be well defined by a certain quantitative formula. Thus the above result is the translation from "natural language" into "system theoretical language", i.e.,

$$\text{"fuzzy modus ponens (1.2)"} \quad \xrightarrow{\text{translation}} \quad \text{"Statement III"}. \tag{3.6}$$

We believe that this translation is fairly good. Also, note that Statement III is a mathematical theorem, or a corollary of Proposition 2.2, in the system theory (1.1). Thus, under the translation (3.6), for the first time we can say that "fuzzy modus ponens (1.2)" is true.

Return to Remark 1.4. Again define the grade quantity T^g such that $T^g = \psi_{\{w\}} \circ T$ in (1.4). And consider the statement S'_p : "$T^g(\omega_0) > 0.7$" in physics. It is clear that the S'_p is equivalent to the statement S_p in Remark 1.4. Thus we have the recommendable translation : *"This room is warm."* $\mapsto S'_p$. Also, under the translation, we can ascertain whether this room is warm or not.

The reader may be of the opinion that the translation from "natural language" into "system theoretical language" is somewhat unreasonable and subjective. We agree that the opinion is completely proper. However, we consider that the "subjectivity" is one of aspects of system theory. Though there may be other opinions for system theory, our opinion is presented below.

Remark 3.1. (Subjective aspect of system theory). We consider that system theory is a mathematical approach to an understanding of phenomena. Particularly, our proposal (1.1) is modeled on mechanics. Here, the word "phenomena" means "non-physical phenomena" as well as "physical phenomena", for example, economical phenomena, biological phenomena, complex physical phenomena and so on. Since system theory is not only physics, in general a system theoretical formulation of a phenomenon is not uniquely determined. In this sense, we consider that system theory is more or less "subjective". The modeling problem, i.e., how to obtain a proper system theoretical formulation of a phenomenon, is of course one of main fields of system theory. Note that Statement III can be regarded as a system theoretical formulation of "fuzzy modus ponens (1.2)". Thus we consider that the translation from "natural language" into "system theoretical language" is a kind of modeling problem in a broad sense. It is also obvious that the translation is not unique in general. For example, the quantitative definition of "redness" in Statement III is not unique, or some may propose another statement such as (1.2) in the system theory (1.1). However, this is not influential to our arguments. Again note that a system theoretical model is not unique in general. Thus, if a formulation (or, model) is not useful for one's purpose, he does not need to use it. This is the usual way of system theory. Still, system theory is a good approach to an understanding of phenomena.

Remark 3.2. (Quantum system theory). All arguments in this paper can be easily applied to quantum systems. In order to see this, it suffices to start from $\mathbf{M}_{\mathcal{A}}(\mathbf{O},\ S_{[*]})$ instead of $\mathbf{M}_{C(\Omega)}(\mathbf{O},\ S_{[*]})$ in (2.1). Here, the $\mathcal{A}$ is the non-commutative C^*-algebra, in which a quantum system S is described. Also, a quantum grade quantity $G\ (\equiv (G_1, G_2))$ in $\mathcal{A}$ is defined by a pair of positive elements G_1 and G_2 in $\mathcal{A}$ such that $0 \le G_1, G_2 \le I$ and $G_1 G_2 = G_2 G_1$. Then, from the commutativity of G_1 and G_2, we can define the "operation for quantum grade quantities" such as $\{0,1\}^2 \supseteq \Xi \mapsto \phi_\Xi(G_1, G_2) \in \mathcal{A}$. Also, the value of the quantum grade quantity G_k for a state ρ^p is defined by $\rho^p(G_k)$, $k = 1, 2$. Therefore, by a similar way, we can easily see that the fuzzy modus ponens (1.2) can be also translated into a quantum system theoretical statement. This is not surprising since we sometimes represent a quantum phenomenon by rough statement in a natural language.

4. CONCLUSIONS

Our main interest was to translate from "natural language" into "system theoretical language". We assume that the translation is a kind of modeling problem in a broad sense (cf. Remark 3.1). In order to obtain a good translation, "system theoretical language" must be rich. Thus we proposed the new frame (1.1) of system theory, which included the concepts of "measurement" as well as "state equation". We consider that the proposal (1.1) is merely the firm description of "usual system theory" and not a different kind of system theory. In other words, we believe that the system theory (1.1) is orthodox. And we translated "fuzzy modus ponens (1.2)" into Statement III in the system theory (1.1). The translation may be somewhat unreasonable. However, note that Statement III is a mathematical theorem, or a corollary of Proposition 2.2, in the system theory (1.1). Thus, under the identification : (1.2) $\leftrightarrow$Statement III, for the first time we can say that "fuzzy modus ponens (1.2)" is true. Though our result is merely one example of the translation from "natural language" into "system theoretical language", we believe that our method is fairly general. Thus we can expect that the system theory (1.1) has great power of expression.

Lastly again we must add that Axiom 1 was not introduced by a linguistic reason but a purely scientific reason (cf. Remark 1.1). We hope that the proposal (1.1) will be developed and examined from various viewpoints.

I am grateful to the anonymous referees who read the first draft of this paper. It is true that the words "objective" and "subjective" are rather misleading. Thus in this revised paper I did not use these words without explanation.

References

1. Bratteli, O. and Robinson, D.W., *Operator algebras and quantum statistical mechanics I,II*, Springer, Berlin., 1979, 1981.
2. Davies, E. B., *Quantum theory of open systems*, Academic Press, 1976.
3. Ishikawa, S., *Fuzzy inferences by algebraic method*, Fuzzy sets and Systems **87** (1997), 181 - 200.
4. Ishikawa, S., *A quantum mechanical approach to a fuzzy theory*, Fuzzy sets and Systems **90** (1997), 277 - 306.
5. Klir, G.J., Folger, T.A., *Fuzzy sets, Uncertainty, and Information*, Prentice-hall International Editions, 1992.
6. Sakai, S., *C^*-algebras and W^*-algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete (Band 60), Springer-Verlag Berlin Heisenberg New York, 1971.
7. Zadeh, L.A., *Fuzzy Sets*, Information and control, **8** (1965), 338-353.