Szegö Operators and a Paley-Wiener Theorem on SU(1,1)

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§1. Introduction. In 1934 Paley and Wiener [PW] showed that the Fourier transform \( f \rightarrow f^* \) on \( \mathbb{R} \) is a bijection of \( C_c^\infty(\mathbb{R}) \) onto the set of holomorphic functions of exponential type. Let \( G \) be a reductive Lie group with a maximal compact subgroup \( K \) of \( G \). Then the analogous theorem to characterize the image of \( C_c^\infty(G,K) \), \( K \)-finite functions in \( C_c^\infty(G) \), under the Fourier transform was finally solved by Arthur [A] in 1983. During the these 50 years a number of authors had proved the Paley-Wiener theorem for particular classes of groups.

Some difficulties arise in the proof of the surjectivity, especially, of showing compactness of the support of a function whose Fourier transform is holomorphic of exponential type, and there are some directions to obtain the fact. The first one is, as in the case of \( \mathbb{R} \), the way of changing of contours of integration in the Fourier inversion formula. Ehrenpreis and Mautner [EM] solved the case of \( SU(1,1) \) and Johnson [J] rephrased the result in terms of Harish-Chandra's generalized c-functions. The main problems in this direction were (1) how to obtain a sharp estimate for Harish-Chandra expansion which allows us to change the contours of integration and (2) how to treat residues which appear during the contour change. For the \( K \)-biinvariant or right \( K \) invariant functions on general groups \( G \) the residues don't appear. Then the main problem (1) was solved by
Helgason [H1], Gangolli [G] for $C^\infty(K\backslash G/K)$ and by Helgason [H2] for $C^\infty(G/K)$. Roughly speaking in these cases the image is characterized by holomorphic functions of exponential type satisfying functional equations related with the small Weyl group of $G$.

When we treat $K$-finite functions on $G$, we encounter the residues during the contour change, so the problem (2) is essential. This was solved by noting a relation between the residues and matrix coefficients of nonunitary principal series of $G$, especially the discrete series of $G$. For the real rank one groups this was done by C.ampoli [C] and for arbitrary groups by Arthur [A]. In his proof W. Casselman's theory of a realization of $(g,k)$ modules played an important role to treat the residues. Then the image of $C^\infty(G,K)$ is characterized by holomorphic functions of exponential type satisfying functional equations that matrix coefficients of nonunitary principal series of $G$ satisfy.

The second direction of proving the compactness is completely different from the first one and is algebraic in nature. For complex semisimple Lie groups the Paley-Wiener theorem was solved by Zelobenko [Z] and for any groups with one conjugacy classes of Cartan subgroups of $G$ was done by Delorme [D].

The aim of this paper is to offer a third direction of proving the Paley-Wiener theorem. Especially, we shall give a new approach to obtain the theorem for right $K$-finite functions on $G = SU(1,1)$. The Plancherel formula for $L'(G)$ indicates that $L'$ functions on $G$ consist of wave packets and cusp forms on $G$. Although any functions in $C^\infty(G,K)$ are uniquely determined by the integral part - the sum of wave packets - in the Fourier inversion formula, the result stated above the image satisfies functional equations that matrix
coefficients of nonunitary principal series of $G$ satisfy does not express clearly the relation between wave packets and cusp forms. So, we shall characterize simultaneously the two parts of the right $K$-finite functions on $G$. As mentioned above, the residues, which appear in the contour change, are real obstacles in the proof of the surjectivity. Therefore, we want to avoid using the Harish-Chandra expansion from which the singularities arise. Actually, reducing the theorem to the one for right $K$-invariant functions, we won't use the theory of $c$-functions. In this approach the theory of Szegö operators will play an important role.

We shall treat right $n$-type functions on $G$; $n \in \mathbb{Z}$ and the left $K$-type is of free. In §3, as generalization of the classical Szegö projection defined on the unit circle (cf. [R], p.178), the Szegö operators $S_{\nu \epsilon \mathbb{R}}, (\epsilon = 0, \frac{1}{2} \text{ and } \nu \in \mathbb{R})$ will be defined (see (3.11)). They are deeply related with the principal series and the discrete series of $G$, and some properties will be investigated in §4 and §5. Then, in §6, we shall rephrase the Plancherel formula for $L^p(G), L^q$ functions on $G$ with right $K$-type $n$, by using the Szegö operators (see Theorem 6.10). Actually, wave packets can be written as an integral of $S_{\nu \epsilon \mathbb{R}}, \mu, (\lambda) d \lambda$, where $\mu, \nu$ is the Plancherel measure and $\nu = \frac{1}{2} + i \lambda$, and the discrete part - $L^q$ sum of cusp forms - as a finite sum of $S_{\nu \epsilon \mathbb{R}}, (1 \leq m \leq n, m \in \frac{1}{2} \mathbb{Z} \text{ and } 2m=2n \mod(2))$. This new phrase of the Plancherel formula is useful to express the relation between wave packets and the discrete part of compactly supported, $C^\infty$ functions on $G$ (see Lemma 7.1), and moreover, it makes easy to see the fact that the formula can be reduced to the one for right $K$-invariant functions on $G$ by applying a suitable differential operator on $G$ (see Corollary 6.4 and Remark 6.5). This indicates that the Paley-Wiener theorem for right $n$-type functions
will be reduced to the one for right K-invariant functions which has
no discrete part (see Remark 6.5). In this direction the Paley-Wiener
theorem will be proved in §7. Especially, we don’t use the Harish-
Chandra expansion for K-finite spherical functions and we don’t need
to treat singularities of generalized c-functions, only we pay
attention to the ones of \( P_{\nu}(\lambda)^{-1} \) (see Corollary 6.4 []). By the same
way, this direction is also applicable to the characterization of \( L' \)
Schwartz space on \( G \) with right K-type \( n \) (see Theorem 7.4).

§2. Notation. Let \( G \) be SU(1,1), the group of all \( C \)-linear
transformations of \( C' \) which are of determinant one, and \( G = KAN \) an
Iwasawa decomposition of \( G \), where \( K, A \) and \( N \) are, respectively the
maximal compact, vector and unipotent subgroups of \( G \) consisting of
all matrices in \( G \) of the form:

\[
k_s = \begin{pmatrix} e^{i\theta/2} & 0 \\ e^{-i\theta/2} & 0 \end{pmatrix} \quad (0 \leq \theta < 4\pi),
\]

\[
a_t = \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{pmatrix} \quad (t \in \mathbb{R})
\]

and

\[
n_\xi = \begin{pmatrix} 1+i\xi/2 & -i\xi/2 \\ i\xi/2 & 1-i\xi/2 \end{pmatrix} \quad (\xi \in \mathbb{R}).
\]

Let \( g = k + a + n \) denote the corresponding Iwasawa decomposition of
the Lie algebra \( g \) of \( G \). Let \( A' = \{ a_t ; t > 0 \} \) and \( M = \{ \pm 1 \} \), the
centralizer of \( A \) in \( K \). Then the Cartan decomposition of \( G \) is given by
\( G = KCL(A')K \). For \( x \in G \) we define \( H(x) \) as the unique element in \( a \) such
that \( x \in K \text{exp} H(x) N \) and \( \sigma(x) \) as the unique positive number such that
\( x \in \mathbb{K}_{n,m} \). Let \( \mathfrak{u}^* \) denote the complexification of an algebra \( \mathfrak{u} \) and 
\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
\( \mathfrak{u}^* \) the dual space of \( \mathfrak{u}^* \). Then \( \mathfrak{a}^* = \mathbb{C}[\mathfrak{g}] \) and \( \mathfrak{h}^* \cong \mathbb{C}^n \) are 
Cartan subalgebras of \( \mathfrak{g}^* \). We define \( \rho_0 \in \mathfrak{a}^* \) and \( \rho \in \mathfrak{h}^* \) as follows.

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
\( \rho_0([\mathfrak{g}]) = 1 \) and 
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
\( \rho([\mathfrak{g}]) = i \).

Let \( D \) be the open unit disk \( |z| < 1 \) in \( \mathbb{C} \) and \( T \) the boundary of \( D \).
Then each element \( g \) in \( G \) acts transitively as analytic automorphism of \( D \) under

\[
z \to g \cdot z = (\beta z + \alpha)^{-1}(\alpha z + \beta); \quad g = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \quad \text{and} \quad z \in D.
\]

This action is naturally extended to the boundary \( T \). Then \( K \) and \( M \) are respectively the subgroups of \( G \) fixing \( 0 \) in \( D \) and \( 1 \) in \( T \), so we have the identifications:

\[
D = G/K \quad \text{and} \quad T = K/M.
\]

Let \( dk = (4\pi)^{-1}d\theta \) denote the normalized Haar measure on \( K \) and \( dg \)
the one on \( G \) normalized as the following integral formula holds:

\[
\int_0^* f(g)dg = 2\pi (4\pi)^{-1} \int \int \int f(k,\mathfrak{a},\mathfrak{k}^*) \text{sh} \theta d\theta d\tau \tau \quad (2.1)
\]

whenever the integral exists. For each measurable space \((X,dx) \mathcal{L}^p(X)
(1 \leq p < \infty)\) denotes the space consisting of all the functions \( f \) on \( X \) for which 
\( \int |f(x)|^p dx < \infty \) with obvious norm.

Let \( K^* \) and \( M^* \) denote the sets of equivalence classes of irreducible
unitary representations of $K$ and $M$ respectively, which are parametrized as

$$K^\infty = \{ r_n; n \in \frac{1}{2} \mathbb{Z} \} \text{ and } M^\infty = \{ \sigma_a; \varepsilon = 0, \frac{1}{2} \}.$$  

Actually, they are defined by $r_n(k) = e^{i\pi n}$ and $\sigma_a(\pm 1) = (\pm 1)\sigma_a$. 

Last for $\varepsilon = 0$, $\frac{1}{2}$ we let

$$Z_\varepsilon = \{ n \in \frac{1}{2} \mathbb{Z}; 2n \equiv 0 \mod(2) \}.$$

### § 3. Szegő operators

For $\sigma, \epsilon \in M^\infty$ and $r_n \in K^\infty$ let

$$C^\infty(K, \epsilon) = \{ f \in C^\infty(K); f(\sigma_a (x) f(k) \text{ for } \sigma_a, k \in K)$$

and

$$C^\infty(G, \tau_n) = \{ f \in C^\infty(G); f(\tau_n (k) g) = \tau_n (k) f(g) \text{ for } k \in K, g \in G \}.$$  

Obviously, if let

$$I_\epsilon : C^\infty(T) \rightarrow C^\infty(K, \epsilon)$$

denote the operator defined by $I_\epsilon(F)(k) = e^{i\pi \epsilon F(e^{i\pi})}$, we can identify $C^\infty(T)$ with $C^\infty(K, \epsilon)$, especially, $I_\epsilon$ is an isometry between $L'(T)$ and $L'(K, \epsilon)$, the $L'$ completion of $C^\infty(K, \epsilon)$.

For $\nu \in C$ the Szegő operator

$$S_{\nu} : C^\infty(K, \epsilon) \rightarrow C^\infty(G, \tau_n)$$

is defined by
\[
S_{\nu,n}(f)(x) = \int_{\nu H(x^{-1}k)} e^{i\lambda(x^{-1}k)} f(k) dk
\]

\[
= e^{i\nu n (\epsilon - \epsilon^*)} (1 - |w|/z)^{-\nu}
\]

\[
x \int_{0}^{2\pi} \left. \frac{|1 - e^{-i\theta}w|^{2\nu}}{(1-e^{-i\theta}w)^{2\nu} \varphi} \right| \varphi, \nonumber
\]

where \(x = k_{\nu} a_{\nu} k_{\nu} \in G \) and \(w = z \cdot 0 = \text{ht} / 2 e^{i\theta} \in D \) (see [KW], p.178). Clearly, \(S_{\nu,n}(f)(x) \equiv 0\) except \(n \in \mathbb{Z}_{\nu}\), and when \(\epsilon = \frac{1}{2}, \nu = -\frac{1}{2}\) and \(n = \pm \frac{1}{2}\), the integral of \(S_{\nu,n}(f)(x)\) coincides with the classical Szegö projection operator on \(L^1(T)\) (cf. [R], p.178). Actually, for \(f \in L^1(T)\) with the Fourier series \(\sum_{\nu} a_{\nu} e^{i\theta}\), if we let

\[
F_{\nu}(w) = \sum_{\nu = 0}^{\infty} a_{\nu} e^{i\theta}\quad (w \in D),
\]

then

\[
S_{\nu,n}(f)(x) = e^{i\nu n (\epsilon - \epsilon^*)} (1 - |w|/z)^{\nu} F_{\nu}(w)
\]

and

\[
S_{\nu,n}(f)(x) = e^{-i\nu (\epsilon - \epsilon^*)} (1 - |w|/z)^{\nu} F_{\nu}(w). \quad (3.2)
\]

§ 4. Principal series and \(S_{\nu,n}(\cdot)\). For \(\epsilon \in \{0, \frac{1}{2}\}\) and \(\nu \in \mathbb{C}\) let \((\pi_{\nu}, L^1(T))\) denote the principal series representation of \(G\), that is defined by, for \(f \in L^1(T)\)

\[
\pi_{\nu}(f)(\xi) = |\beta^{-\xi + \alpha^-}|^{-2\nu} \left( \frac{\beta^{-\xi + \alpha^-}}{\beta^{-\xi + \alpha^-}} \right)^{2\nu} \int_{\beta^{-\xi + \alpha^-}} F(\xi + \beta^{-\xi + \alpha^-}). \quad (4.1)
\]
where \( \zeta = e^{\xi} \in \mathbb{T} \) and \( g = \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} \in G \) (cf. [Su], p.207). Let \( \{ e_p(\zeta) ; p \in \mathbb{Z} \} \) denote the complete orthonormal system of \( L^2(\mathbb{T}) \) given by

\[
e_p(\zeta) = \zeta^{-p} = e^{-2\pi i p}.
\]

Then it follows from (4.1) that

\[
\pi_{\omega}(x)e_p(\zeta) = e^{i\omega_p(x)} \frac{(1-|\omega|^2)}{|1-\omega \zeta^{-1}|} \frac{|1-\omega \zeta^{-1}|}{|1-\omega \zeta^{-1}|^2} \zeta^{-p}, \quad (4.2)
\]

where \( x = k_0 a_c k_0 \in G \) and \( \omega = x \cdot 0 \in D \), so we see that

**Lemma 4.1.** There exists a positive constant \( C \) such that

\[
| \pi_{\omega}(x)e_p(\zeta) | \leq C e^{\varepsilon(x)+1} |e_p(\zeta)| \quad (x \in G).
\]

Moreover, by comparing the definition (3.1) of \( S_{\omega,x} \) with (4.1), we can deduce that

**Proposition 4.2.** Let \( \varepsilon \in (0,\nu) \), \( n \in \mathbb{Z} \), and \( \gamma \in \mathbb{C} \). Then for \( f \in L^2(\mathbb{T},\varepsilon) \)

\[
S_{\omega,x}(f)(x) = \int_{\mathbb{T}} \pi_{\omega}(x)e_{\gamma}(\zeta) l_{\gamma}^{-1}(f)(\zeta) d\zeta.
\]

Let \( \pi_{\omega}(x) \) \((p, q \in \mathbb{Z})\) denote the matrix coefficient of \( \pi_{\omega}(x) \)

\[
(x \in G) \text{ defined by}
\]

\[
\pi_{\omega}(x) = (\pi_{\omega}(x)e_p,e_q).
\]

(4.3)
Then, by substituting (4.2) for (4.3) the explicit form of \( \pi_{x,z}(a) \) is given by

\[
(1-r^2)^{r^{p+q}} F\left( \begin{array}{c}
-v-q+\varepsilon \\
p-q
\end{array} ; v+q+\varepsilon ; p-q+1; r^2 \right) \quad (p \geq q)
\]

and

\[
(1-r^2)^{r^{p+q}} F\left( \begin{array}{c}
-v+q+\varepsilon \\
-q-p
\end{array} ; v-p-\varepsilon ; q-p+1; r^2 \right) \quad (q \geq p),
\]

where \( r = \text{tht}/2 \) and \( F(a,b,c;z) \) is the hypergeometric function (cf. [Sa], p.74). Then, using this expression, we can easily deduce that the matrix coefficients satisfy the following relations (cf. [J], §4 and [B], p.26).

**Lemma 4.3.** Let \( \varepsilon \in \{0, \frac{1}{2}\} \), \( v \in \mathbb{C} \) and \( p, q \in \mathbb{Z} \). Then for \( x \in G \)

\begin{align*}
(1) \quad & \pi_{x,z}(x) = \pi_{x,z}(-x^{-1}) \\
(2) \quad & \pi_{x,z}(x) = \omega_{x,z} \pi_{x,z}(x),
\end{align*}

where \( \omega_{x,z} \) is given by

\[
\frac{-v-q+\varepsilon}{p-q} \quad \frac{-v+q+\varepsilon}{q-p} \quad \frac{-v+q+\varepsilon}{q-p} \quad \frac{-v-q+\varepsilon}{p-q} \quad (p \geq q) \quad \text{and} \quad (q \geq p).
\]

We regard \( X=\frac{1}{2}[-1,0] \) and \( Y=\frac{1}{2}[0,1] \) as left invariant differential operators on \( G \) and put \( E_x = \pm X + Y \). Then, since \( E_x = \pm d\pi_{x,z}(X) + \text{id} \pi_{x,z}(Y) \) make a shift of \( K \)-types according to

\[
E_x \cdot e_p = (p+\varepsilon + v)e_{p+1}
\]

and

\[
E_x \cdot e_p = (p+\varepsilon - v)e_{p-1}
\]

(4.5)
(cf. [Su], p.216), it follows from Proposition 4.2 that

**Lemma 4.4.** Let the notation be as above. Then

\[ E_n s_{r,*}(f) = (n^r \nu) s_{r,*} n_1(f). \]

§5. **Discrete series and \( S_{r,*}. \)** For \( n \in \frac{1}{2} \mathbb{Z} \) and \( |n| \geq 1 \) let \((T_n, A_{r,*}(D))\) denote the discrete series representation of \( G \), where \( A_{r,*}(D) \) is the \( L^1 \) weighted Bergman space on the unit disc \( D = \{ z \in \mathbb{C}; |z| < 1 \} \) defined by, for \( n \geq 1 \)

\[ A_{2,n-1}(D) = \{ F: D \to \mathbb{C}; F \text{ is holomorphic on } D \text{ and} \]

\[ \| F \|_{2,n-1} = \{ (2n-1)^{-1} \int_D |F(z)|^2 (1-|z|^2)^{2n-2} |dz|^2 \}^{\frac{1}{2}} < \infty \}

and for \( n \leq 1 \), it is made up of conjugate holomorphic functions on \( D \) with the norm given by replacing \( n \) with \( |n| \). Then \( T_n(g)F \) (\( g \in G \) and \( F \in A_{r,*}(D) \)) is defined by, for \( n \geq 1 \)

\[ T_n(g)(F)(z) = (\alpha z + \beta)^{-2n} \eta(z), \quad (5.1) \]

where \( z \in D \) and \( g' = [\begin{array}{cc} \alpha & \beta \\ \beta & \alpha \end{array}] \in G; \) for \( n \leq -1 \), it is defined by \( T_n(g)(F) = \text{conj}(T_{n+1}(g)(\text{conj}(F))) \), where \( \text{conj} \) is the operator taking the complex conjugation (cf. [Su], p.229). Let \( \{ e_p(z); p \in \mathbb{N} \} \) denote the complete orthonormal system of \( A_{r,*}(D) \) defined by

-10-
\[ e \mathcal{P}(z) = \lambda_2 z^m (n \geq 1) \quad \text{and} \quad \lambda_2 z^n (n \leq 1). \]

where \((\lambda_2)^n = \Gamma(p+2|n|)/\Gamma(p+1) \Gamma(2|n|)\). Let \(T^*(g)\) \((p, q \in \mathbb{N})\) denote the matrix coefficient of \(T_n(g) \ (g \in G)\) defined by

\[ T^*(g) = (T_n(g)e^q, e^p). \quad (5.2) \]

Then, \(\|T^*\| = 4\pi(2|n| - 1)^{-1} \) (cf. [Su], p. 326), and comparing (4.1) with (5.1) and (4.3) with (5.2), we can deduce that

\textbf{Lemma 5.1.} Let \(\varepsilon \in \{0, \frac{1}{2}\}, m, n \in \mathbb{Z}, n \geq m \geq 1\) and \(p, q \in \mathbb{Z}, q \geq m - \varepsilon\).

\[ \begin{align*}
1 & \quad \pi_\varepsilon, m(g) e^{m-\varepsilon}(\xi) = (\lambda_{2, m})^{-1} \pi_{m, m}(g) e^{-m}(z) \mid z = e^{-\varepsilon}, (\xi). \\
2 & \quad \pi_{\varepsilon, 2}(g) = (\lambda_{2, m})^{\varepsilon} \lambda_{1, m}^{-1} \lambda_{2, m}^{-1} \pi_{m, m}(g).
\end{align*} \]

Let \(\Omega\) be the Casimir operator in \(U(g_c)\) given by \(-H^2 - \frac{i}{2}(X^2 + Y^2)\), where \(-2H = [X, Y]\). Then it is well known (cf. [Su], p. 288) that \(Z(g_c) = C \Omega\) and

\textbf{Lemma 5.2.} Let \(\varepsilon \in \{0, \frac{1}{2}\}, n \in \mathbb{Z},\) and \(|n| \geq 1\). Then

\[ \Omega \pi_{\varepsilon, n}(x) = \Omega \pi_m(x) = n(1-n) \quad (x \in G). \]

In what follows we shall investigate the relation between the discrete series \(T_n\) and the Szegö operators \(S_{\varepsilon, n}\).

Let \(V_\varepsilon\) \((\ell \in \mathbb{N})\) denote the set of all homogeneous polynomials of degree \(\ell\) with variables \(z_1\) and \(z_2\). Then the finite dimensional representation \((\pi_\varepsilon, V_\varepsilon)\) of \(G_c = SL(2, \mathbb{C})\) is defined by
\[ \pi_s(g)P(z) = P(z \cdot g) \quad \text{for} \quad z = (z_1, z_2), \]

where \( P \in V_s \) and \( z \cdot g = (az_1 + cz_2, bz_1 + dz_2) \) if \( g = \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in G_s \). Especially, when \( g = k, a, k^*, z_1 \cdot g \) and \( z_2 \cdot g \) are respectively given as follows.

\[ e^{\frac{ic\varepsilon \sigma^3}{2} \text{cht}/2} z_1 + e^{-\frac{ic\varepsilon \sigma^3}{2} \text{sht}/2} z_2 \]

and

\[ e^{\frac{ic\varepsilon \sigma^3}{2} \text{sht}/2} z_1 + e^{-\frac{ic\varepsilon \sigma^3}{2} \text{cht}/2} z_2. \]  

\[ (5.3) \]

Let \( d = d = \dim V_s = \ell + 1 \) and \( J_s = \{ 1, 2, \ldots, d \} \). Then \( v_i = [(i-1)!((\ell - i + 1)!)^3 \]

\[ z_1^{i-2}z_2^{i-1} \quad (i \in J_s) \]

is a \((2i-\ell-2)\rho_s\)-weight vector with respect to \( h_s \) and \( v_{i-1} = [(i-1)!((\ell - i + 1)!)^3(\ell-z_1)^{i-1}(z_1+z_2)^{\ell-i} \quad (i \in J_s) \]

is a \(-(2i-\ell-2)\rho_s\)-weight vector with respect to \( g_s \). Especially, we shall equip \( V_s \) with the inner product for which \( \{v_i; i \in J_s\} \) is an orthonormal system of \( V_s \). If we put \( C_{ii} = (v_i, v_i) \) and \( [D_0] = [C_{ii}] \),

we see that \( C_{ii} = 2^i[(i-1)!((\ell - i + 1)!)^3, \quad \| v_i \|^2 = \ell + 2 \) and

\[ v_i = \sum_{i \in J_s} C_{ii} v_i \quad \text{and} \quad v_i = \sum_{i \in J_s} B_{ii} v_i^{\circ}. \]  

\[ (5.4) \]

Let \( \pi_s^i(x) \) \( (i,j \in J_s) \) denote the matrix coefficient of \( \pi_s(x) \) \( (x \in G) \)

defined by

\[ (\pi_s(x)v_i, v_i). \]  

\[ (5.5) \]

**Lemma 5.3.** Let \( a, b \in \mathbb{Z}_s \) \( (\varepsilon = 0, \frac{\ell}{2}) \) and \( |b| \leq a \). Then

\[ e^{2\pi i \varepsilon h(x)} \tau_b(A(x))^{-1} = C_{a-b+1} \tau_1^{a-1} \sum_{i \in J_{2a}} C_{a-b+1} \pi_s^{\alpha}(x), \]

where \( d = 2a + 1 \).
Proof. Noting the Iwasawa decomposition of $x \in G$, we easily see that the left hand side is equal to $C_{w^{-1}} e^i (\pi_m(x)v_*, v_*)$. Then, substituting with $v_* = \Sigma C_{w^{-1}} v_*$, we have the desired result.

Q.E.D.

Let $\varepsilon \in \{0, \frac{1}{2}\}$, $m \in \mathbb{Z}$, and $i \in J_m$. Then for $f \in L'(K, \varepsilon)$ we define

$$S_{m,n}(f)(x) = \sum_{p \in J_{m+1}} S_{m-n+1} \left( f \pi_{m}^{d_1} \right)(x) \pi_{m+1}^{d_1}(x^{-1})$$  \hspace{1cm} (5.6)

where $d = 2m+2$ and $f \pi_{m}^{d_1}$ is the function on $K$ given by $f(k) \pi_{m}^{d_1}(k)$ $(k \in K)$. Then it follows that

**Proposition 5.4.** Let $\varepsilon \in \{0, \frac{1}{2}\}$ and $m$, $n \in \mathbb{Z}$. Suppose that $m \geq -\frac{1}{2}$ and $-m \leq n \leq m+1$. Then for $f \in L'(K, \varepsilon)$

$$S_{m,n}(f)(x) = \sum_{i \in J_{m+1}} C_{m+1} d_i^{-1} \sum_{i \in J_{m+1}} S_{m,n}(f)(x)$$

where $x \in G$ and $d = 2m+2$.

Proof. We can rewrite the integral in the definition of $S_{m,n}$ as

$$S_{m,n}(f)(x) = \int_{x^*}^{-\frac{1}{2}H(x^{-1}k)} \tau_{m}(\kappa(x^{-1}k))^{-1}$$

$$\times e^{(m+\frac{1}{2})H(x^{-1}k)} \tau_{n}(\kappa(x^{-1}k))^{-1} f(k) dk.$$

Then, noting the assumption on $m$ and $n$, we can apply Lemma 5.3 for $a = m+\frac{1}{2}$ and $b = n-\frac{1}{2}$ to the right hand side. Then it follows that
\[-\frac{1}{2} \mathcal{H}(x^{-1}k) - \sum_{i \in I^{2\omega \setminus 1}} \mathcal{C}_{m-n}^{-1} \mathcal{H}_{m-n} \sum_{k \in \mathbb{N}} \tau_{m}(x^{-1}k)^{-1} x \mathcal{J}_{m \setminus 1}(x^{-1}k)f(k)dk.\]

Then, since \( \pi \mathbb{A}(x^{-1}k) = \sum_{\nu} \pi \mathbb{A}(x^{-1}) \pi \mathbb{A}(k) \), the desired result follows from the definition of \( S_{m, \mu, \nu} \).

Q.E.D.

**Theorem 5.5.** Let \( \varepsilon \in \{0, \frac{1}{2}\} \), \( m \in \mathbb{Z} \), and \( m \geq -\frac{1}{2} \). Let \( f(\xi) = \sum_{p \in \mathbb{Z}} a_{p} \xi^{p} \) be a function in \( L^{1}(K, \varepsilon) \) satisfying \( |a_{p}| = 0 \) for \( |p| \leq m \). Then

\[
S_{m, m, \nu}(f)(x) = (1 - |x|^2)^{-\frac{m}{2}} e^{i\nu \lambda(x^{-1})} S_{m, -\lambda, \nu}(fe^{-i\lambda(x^{-1})}x)
\]

\[
(1 - |x|^2)^{-\frac{m}{2}} e^{i\nu \lambda(x^{-1})} (I_{\nu}f)(x)w^{-\nu(x^{-1})},
\]

where \( x = k_{a}a_{k} \cdot x \in G \) and \( w = x \cdot 0 \in D \).

**Proof.** Since \( v_{s}^{*} (s \in J_{m, \nu}) \) are weight vectors with respect to \( h_{m, \nu} \), we see from (5.4) and (5.5) that

\[
\pi \mathbb{F}_{m+1}(k_{a}) = \sum_{s, p \in J_{m, \nu}} D_{s}C_{p}S_{m, -\lambda, \nu}(fe^{-i\lambda(x^{-1})}x)e^{i\nu(x^{-1})}x^{-1}
\]

Therefore, we can rewrite (5.6) as follows.

\[
S_{m, m}(f)(x) = \sum_{s, p \in J_{m+1}} D_{s}C_{p}S_{m, -\lambda, \nu}(fe^{-i\lambda(x^{-1})}x)e^{i\nu(x^{-1})}x^{-1}
\]

-14-
\[ = S_{\mu, \nu, \lambda}(f e^{-i(\omega \cdot x) \cdot \theta}) \sum_{\rho} D_{\mu \nu} \delta_{\rho \nu} \pi_{2m+1}(x^{-1}). \]

Here we used (3.2) and the assumption that \( a_p = 0 \) for \( |p| \leq m \) to obtain the last equation. We note that

\[ \sum_{\rho} D_{\mu \nu} \delta_{\rho \nu} \pi_{2m+1}(x^{-1}) \]

\[ = \sum_{\rho} D_{\mu \nu} (\pi_{2m+1}(x^{-1})v_\rho^{-1}, v_\rho) \]

\[ = (\pi_{2m+1}(x^{-1})(w_{z_1+z_2})^{d-1}, v_\rho). \]

Then, it follows from Proposition 5.4 that

\[ S_{\mu, \nu, \lambda}(f)(x) = C_{2d-1} S_{\mu, \nu, \lambda}(f e^{-i(\omega \cdot x) \cdot \theta}) \]

\[ \times \sum_{i \in \mathcal{J}_{2m+1}} C_{11} (\pi_{2m+1}(x^{-1})(w_{z_1+z_2})^{d-1}, v_\rho). \]

\[ = C_{2d-1} S_{\mu, \nu, \lambda}(f e^{-i(\omega \cdot x) \cdot \theta}) \]

\[ \times (\pi_{2m+1}(x^{-1})(w_{z_1+z_2})^{d-1}, v_\rho). \]

We recall that \( \pi_{2m+1}(x^{-1}) \) transforms \( w_{z_1+z_2} \) to

\[ (1 - |w|^2)^{k} e^{i(\omega \cdot x - z_1 \cdot x/2)} z_2 \]  \hspace{1cm} (5.7)

(see (5.3)). Therefore, since \( v_\rho = (l^!)^m z_{l^1} \) and \( C_{11} \| v_\rho \|_2 = (l^!)^m \), we can deduce that \( S_{\mu, \nu, \lambda}(f)(x) \) must be equal to

\[ S_{\mu, \nu, \lambda}(f e^{-i(\omega \cdot x) \cdot \theta})(1 - |w|^2)^{k} e^{i(\omega \cdot x - z_1 \cdot x/2)} . \]
The second equation in the statement easily follows from (3.2)

Q.E.D.

We retain the notation and the assumption in Theorem 5.5. Then the theorem and (2.1) implies that if \( m \geq 0 \), \( S_{\alpha,\mu}(f) \in L^2(G) \) and thus, by Lemma 4.4, \( S_{\alpha,\mu}(f) (0 \leq m \leq n-1) \) also belongs to \( L^2(G) \). Then substituting the decomposition of \( f: f(\xi) = \Sigma a_p \xi^p \), where \( p \in \mathbb{Z} \), and \( |p| > m \), with Proposition 4.2 and using (4.5), we see that \( S_{\alpha,\mu}(f) \) can be written as an \( L^2 \) linear combination of the matrix coefficients \( \pi_{n-1}^\xi \), where \( q \geq m+1 \) and so \( T_{\alpha,\mu}^\pi \) by Lemma 4.3 (2) and Lemma 5.1 (2). This fact also follows from the left \( K \)-type decomposition of \( S_{\alpha,\mu}(f) \), say \( \Sigma_n S(f) \). In fact, each \( S(f) \) is an \( L^2 \) function on \( G \) with \( K \)-type \( (\ell, n) \) and, by Proposition 4.2 and Lemma 5.2, it is also a \( Z(g_0) \)-eigenfunction with eigenvalue \(-m(m+1)\). Therefore, \( S(f) \) must be a cusp form on \( G \), and thus a scalar multiplication of the matrix coefficient \( T_{\alpha,\mu}^\pi \) of \( T_{\alpha,\mu} \). Clearly, \( \ell \geq m+1 \). So, we obtained

**Proposition 5.6.** We keep the notation and the assumption in Theorem 5.5 and suppose that \( n \in \mathbb{Z} \), and \( 0 \leq m \leq n-1 \). Then \( S_{\alpha,\mu}(f) \) can be written as an \( L^2 \) linear combination of \( T_{\alpha,\mu}^\pi \) (\( p \geq 0 \)).

Next theorem will not be used in the argument below. However, it is an important and interesting property that expresses the relation among the Szegő operators \( S_{\alpha,\mu} (0 \leq m \leq n-1) \).

**Theorem 5.7.** Let \( m \in \mathbb{Z} \), \((\varepsilon = 0, \frac{1}{2})\) and \( m \geq -\frac{1}{2} \). Then for \( f \) in \( L^2(K, \varepsilon) \)

\[
\sum_{n \in \mathbb{Z}, \mu = 1}^{2m-1} \left( e^{i \xi \cdot \langle \varepsilon, \pi \rangle} \right)^{n-1} S_{\alpha,\mu - 2 - n}(f)(x)
\]
\[=(1-|w|^2)^{-\infty} e^{i\langle x, y \rangle}\left(S_{h_{-1}} S_{h_{1}}(f)\right)_{+1}\]
\[=(1-|w|^2)^{-\infty} e^{i\langle x, y \rangle}\left(I_{+1}(f)\right)_{+1},\]

where \(x=v \cdot x, y \cdot x \in G\) and \(w=x \cdot 0 \in D\).

Proof. We keep the notation in (3.1). Then we note that
\[
\sum_{n \in J_{2m+1}} \frac{(1-e^{-ix\cdot y})^2}{n! |1-e^{-ix\cdot y}|^2} e^{i\langle x, y \rangle} |w|^{n-1}
\]
\[=(1+\frac{(1-e^{-ix\cdot y})^2}{|1-e^{-ix\cdot y}|^2} e^{i\langle x, y \rangle} |w|)^{2m+1}
\]
\[=\frac{(1-|w|^2)^{2m+1}}{1-e^{ix\cdot y}}.
\]

Therefore, the desired relation follows from (3.1) and (3.2).

Q.E.D.

§ 6. Plancherel formula. In this section we shall rewrite the Plancherel formula for \(L'(G)\) (cf. [Su], p.344 and p.346) by using the Szegö operators \(S_{\pm \infty}\).

The Plancherel formula implies that each \(L'\) function \(f\) on \(G\) can be written as \(f = f' + f^*\), where \(f'\) is the sum of wave packets, the integral part of the formula, and \(f^*\) is a linear combination of cusp forms, the discrete part of the formula, so \(L'(G)\) has a direct sum decomposition:

-17-
\[ L^2(G) = \mathbb{F} L^2(G) \oplus \mathbb{F} L^2(G). \]  \hspace{1cm} (6.1)

For \( f \) in \( L'(G) \) we denote by \( f = \Sigma f_n \) the K-type decomposition of \( f \), where \( m, n \in \frac{1}{2} \mathbb{Z} \) and the K-type of \( f_n \) is \((m,n)\). When we restrict our attention to \( L' \) functions on \( G \) with right K-type \( n \), we denote the decomposition (6.1) as

\[ L^2_n(G) = \mathbb{F} L^2_n(G) \oplus \mathbb{F} L^2_n(G) \]

By the same way we denote the decomposition of compactly supported \( C^\infty \)
functions on \( G \) with right K-type \( n \) as

\[ C^\infty_n(G) = \mathbb{F} C^\infty_n(G) \oplus \mathbb{F} C^\infty_n(G). \]

For \( R \geq 0 \) let \( G(R) \) denote the compact set in \( G \) defined by \( \sigma(x) \leq R \)
when \( x \in G(R) \). Then \( C^\infty_n(G;R) \) denotes the set of all \( C^\infty \)
functions on \( G \) whose supports are contained in \( G(R) \) and \( C^\infty_n(G;R) \) the subspace with
right K-type \( n \). Let \( \cdot C_n(G) \) denote the space of cusp forms on \( G \) with
right K-type \( n \). Then the following proposition will play an important role in \( \S 7 \).

**Proposition 6.1.** For each \( R > 0 \)

\[ \cdot C^\infty_n(G;R) = \cdot C_n(G). \]

**Proof.** By the definition it is clear that \( \cdot C^\infty_n(G) \subset \cdot C_n(G) \), so
we shall prove the reverse. Let \( f \) be in \( \cdot C_n(G) \). First we assume that
the left K-type of \( f \) is \( q \) (\( q > \varepsilon \)). Then, as stated before Proposition
5.6, the discrete part of \( L'(G) \) with K-type \((q,n)\) is an \( L' \) span of a

-18-
finite number of cusp forms on \( G \), say \( \phi_s (1 \leq s \leq N) \) that are linearly independent and real analytic on \( G \). Therefore, for an arbitrary open subset \( S \) in \( G(\mathbb{R}) \) we can choose compactly supported, \( C^\infty \) functions \( h_s (1 \leq s \leq N) \) on \( G \) such that \( (h_s, \phi_t) = \delta_{st} \) \((1 \leq s, t \leq N)\) and \( \text{supp}(h_s) \subset \text{CL}(S) \). Obviously, we may assume that the K-type of \( h_s \) is \((q, n)\). Let

\[
g = \sum_{1 \leq s \leq N} (f, \phi_s) h_s.
\]

Then, \( g \in C_c^\infty(G) \), \( \text{supp}(g) \subset \text{CL}(S) \) and \( \ast g = f \), because

\[
(f, \phi_s) = \sum_{1 \leq t \leq N} (f, \phi_s)(h_t, \phi_t) = (f, \phi_s).
\]

Therefore, we see that \( f \in \ast C_c^\infty(G) \).

Next we shall consider the case of an arbitrary \( f \) in \( \ast C_c(G) \). Let \( f = \sum f_s \) denote the left K-type decomposition of \( f \) and \( S_m (m \in \mathbb{Z}) \) the open subsets in \( G(\mathbb{R}) \) such that \( \text{CL}(S_p) \cap \text{CL}(S_q) = \emptyset \) if \( p \neq q \). Then, as proved above, for each \( m \) there exists a compactly supported, \( C^\infty \) function \( g \) with K-type \((m, n)\) such that \( \ast g = f \) and \( \text{supp}(g) \subset S_m \). Therefore, if we put \( g = \sum g_s \), we see that \( g \in C_c^\infty(G; \mathbb{R}) \) and \( \ast g = f \), so \( f \in \ast C_c(G; \mathbb{R}) \).

This completes the proof of the reverse: \( \ast C_c^\infty(G; \mathbb{R}) \supset \ast C_c(G) \).

Q.E.D.

Let \( \varepsilon \in \{0, \frac{1}{2}\} \), \( n \in \mathbb{Z} \), and \( \nu = \frac{1}{2} + i \lambda \in \mathbb{C} \). For \( f \) in \( C_c(G) \) we define the Fourier transform \( f^\wedge(\lambda, \zeta) \) \((\lambda, \zeta) \in \mathbb{R} \times \mathbb{T} \) associated with the principal series \( \pi_{\nu, \ast} \) by

\[
f^\wedge(\lambda, \zeta) = \int_G f(\gamma) \text{conj}(\pi_{\nu, \ast}(\gamma) \theta_{\nu, \ast}(\zeta)) d\gamma \tag{6.2}
\]
and moreover, for \( m \in \mathbb{Z} \), we define

\[
\tilde{f}^m(\lambda) = \int_{\mathbb{R}} f(g) \text{conj}(\pi_{\mathcal{K}} \circ \gamma_{-m-n}^*(g)) dg. \tag{6.3}
\]

Let \( \alpha(\lambda, \xi) \) be a function on \( \mathbb{R} \times \mathbb{T} \). Then we define \( A_n(\alpha)(\lambda, x) \) on \( \mathbb{R} \times \mathbb{G} \) by

\[
A_n(\alpha)(\lambda, x) = S_n \gamma_{-n}^*(\mathcal{I}_1 \alpha(\lambda, \cdot))(x) \tag{6.4}
\]

whenever this integral exists. When \( \alpha(\lambda, \xi) \) is integrable in \( \xi \) for a fixed \( \lambda \in \mathbb{R} \), the integral exists for the \( \lambda \) (see Lemma 4.1 and Proposition 4.2). We call \( \alpha(\lambda, \xi) \) a holomorphic function of uniform exponential type \( \mathbb{R} \) if it is holomorphic in \( \lambda \) and if there exists a constant \( R \geq 0 \) such that for each \( N \geq 0 \)

\[
\sup_{\lambda \in \mathbb{C}, \xi \in \mathbb{T}} e^{-N|\Im(\lambda)|} (1 + |\lambda|)^N |\alpha(\lambda, \xi)| < \infty.
\]

Then, as noted above, it follows from Lemma 4.1 that, if \( \alpha(\lambda, \xi) \) is a holomorphic function of uniform exponential type, \( A_n(\alpha)(\lambda, x) \) is well defined for \( (\lambda, x) \in \mathbb{C} \times \mathbb{G} \) and holomorphic in \( \lambda \). We also define antiholomorphic functions of uniform exponential type by the same way.

**Lemma 6.2.** Let \( f \) be in \( C_G(\mathbb{R}; \mathbb{R}) \).

1. \( \tilde{f}^m(\lambda) (\lambda \in \mathbb{C}) \) is an antiholomorphic function of exponential type \( \mathbb{R} \) and \( \tilde{f}^m(\lambda) = (\tilde{f}_n)^m(\lambda) \).

2. \( f(\lambda, \xi) ((\lambda, \xi) \in \mathbb{C} \times \mathbb{T}) \) is an antiholomorphic function.
function of uniform exponential type $R$ and

$$f^*(\lambda, \xi) = \sum_{n \in \mathbb{Z}} \omega_n f(\lambda) e^{-i\lambda n}.$$  

(3) \quad \mathcal{A}_n(f^*) (\lambda, x) = \mathcal{A}_n(f^*) (-\lambda, x) \quad ((\lambda, x) \in \mathcal{R} \times \mathcal{G}).$

Proof. (1) and (2) are obvious from Lemma 4.1, (6.2) and (6.3), so we shall prove (3).

$$\mathcal{A}_n(f^*) (\lambda, x) = \int_{\mathcal{R} \times \mathcal{G}} \pi^{1/2} \xi \in \mathcal{R} \times \mathcal{G}, f^*(\lambda, \xi) d\xi$$

$$= \sum_{n \in \mathbb{Z}} \omega_n f(\lambda) \pi^{1/2} \xi \in \mathcal{R} \times \mathcal{G}$$

$$= \sum_{n \in \mathbb{Z}} \omega_n \int_{\mathcal{R} \times \mathcal{G}} f(g) \text{conj}(\pi^{1/2} \xi^{-1} (g) \pi^{1/2} \xi^{-1} (x^{-1})) dg$$

$$= \int_{\mathcal{R} \times \mathcal{G}} f(g) \text{conj}(\pi^{1/2} \xi^{-1} (x^{-1})) dg.$$

Then, since $\omega_n \xi^{1/2} \mathbb{Z} = 1$, (3) follows from Lemma 4.3 (2).

Q.E.D.

Now we shall consider the inversion formula of the Fourier transform defined by (6.2). Let

$$\mu_\varepsilon(\lambda) = \begin{cases} \lambda \pi \text{th}(\pi \lambda) & (\varepsilon = 0) \\ \lambda \pi \text{ch}(\pi \lambda) & (\varepsilon = \frac{1}{2}) \end{cases}.$$  

(6.5)

Then it is well known (cf. [Su], Ch. V, § 8 and [B], § 10) that for $f \in \mathcal{F}(\mathbb{G})$

-21-
\[ \hat{\alpha}(x) = \int \hat{\alpha}(\lambda) \pi_{\tau(x)} \mu_\tau(\lambda) d\lambda \]

and

\[ \int |\alpha(x)|^2 dx = \int |\hat{\alpha}(\lambda)|^2 \mu_\tau(\lambda) d\lambda. \] (6.6)

Let

\[ L^2_n(G \times T) = \{ \alpha(\lambda, \xi) \in L^2(G \times T, \mu_\tau(\lambda) d\lambda d\xi) : \]

\[ A_{\alpha}(\lambda, \xi) = A_{\alpha}(-\lambda, \xi) \text{ for } (\lambda, \xi) \in \mathbb{R} \times \mathbb{E}. \]

Then, for \( \alpha \in L^2(G \times T) \), if we define

\[ \hat{\alpha}(x) = \int |\hat{\alpha}(\lambda) \pi_{\tau(x)} \mu_\tau(\lambda) d\lambda \quad (x \in \mathbb{E}), \] (6.7)

we see the following

**Proposition 6.3.** The Fourier transform \( f(x) \rightarrow f^*(\lambda, \xi) \) is an isometry of \( L^2(G) \) onto \( L^2(G \times T) \) and the inversion formula is given by

\[ f(x) = \sum_{\lambda \in \mathbb{Z}} \int \hat{\alpha}(\lambda) \pi_{\tau(x)} \mu_\tau(\lambda) d\lambda \]

\[ = (f^*)^*(x). \]

Proof. Except the last equation the assertions are obvious from Lemma 6.2 (2) and (6.6), so we shall prove the last equation. Clearly, it is enough to prove it for \( f \in C_0(G) \). Then it follows from (4.3) and Lemma 6.2 (2) that

\[ \sum_{\lambda \in \mathbb{Z}} \int \hat{\alpha}(\lambda) \pi_{\tau(x)} \mu_\tau(\lambda) d\lambda \]

-22-
\[
\sum_{\mu \in \mathbb{Z}_+} \int_{\mu} f^\gamma(\lambda)(\pi_\mu(x) e_{\mu-\cdot}, e_{\mu-\cdot}) \mu_\gamma(\lambda) d\lambda
\]

\[
\int_{\mu} (\pi_\mu(x) e_{\mu-\cdot}, \text{conj}(f^\gamma(\lambda, \cdot))) \mu_\gamma(\lambda) d\lambda.
\]

This integral is nothing but \((f^\gamma)^*(x)\) by Proposition 4.2 and (6.7).

Q.E.D.

**Corollary 6.4.** Let \(f\) be in \(\mathcal{C}_\infty(G)\). Then

\[\begin{align*}
1) & \quad f(x) = (E_{\infty})^{n-k} \int_{\mu} s_{\mu-\cdot-\cdot}(1, f^\gamma(\lambda, \cdot) P_\mu(\lambda)^{-1}) \mu_\gamma(\lambda) d\lambda, \\
\text{where} & \quad P_\mu(\lambda) = (n-\frac{1}{2} + i\lambda)(n-3/2 + i\lambda) \cdots (\varepsilon + \frac{1}{2} + i\lambda). \\
2) & \quad A_\times(f^\gamma P_\mu^{-1})(\lambda, x) = A_\times(f^\gamma P_\mu^{-1})(-\lambda, x) \quad ((\lambda, x) \in \mathbb{R} \times \mathbb{G}).
\end{align*}\]

**Proof.** (1) follows from the inversion formula in Proposition 6.3 and Lemma 4.4. We shall prove (2). By the same argument in Lemma 6.1 (3) we see that

\[A_\times(f^\gamma P_\mu^{-1})(\lambda, x) = \int_{\mu} f(g) \text{conj}(\pi_\mu P_\mu^{-1}(x^{-1}g)) dg.\]

Then, since \(\omega^\mu_{f^\gamma} = P_\mu(\lambda) / P_\mu(-\lambda)\) and \(\text{conj}(P_\mu(\lambda)) = P_\mu(-\lambda)\) by the definition, it follows from Lemma 4.3 (2) that

\[P_\mu(\lambda)^{-1} \pi_\mu \omega^\mu_{f^\gamma} = P_\mu(-\lambda)^{-1} \pi_\mu \omega^\mu_{f^\gamma}.\]

Therefore, the desired relation is obtained.

Q.E.D.
Remainder. We note that the integral of the formula in Corollary 6.4 (1) is nothing but apply the inversion formula for \( L'_r(G) \) to the function \( f^\varepsilon(\lambda, \xi)P_r(\lambda) \) satisfying (2). The formula for \( L'_r(G) \) is simpler than one for \( L'_r(G) \), because it is made up only of wave packets, that is, the discrete part does not appear. Actually, the following theorem is well known for \( \varepsilon = 0 \) by [H2] and \( \varepsilon = \frac{1}{2} \) by the same way.

**Theorem.** 1) \( L'_r(G) = \mathcal{L}'(G) \) and the Fourier transform \( f \to f^\varepsilon \) is an isometry of \( L'_r(G) \) onto \( L'_r(R \times T) \).

2) The Fourier transform \( f \to f^\varepsilon \) is a bijection of \( \mathcal{C}_r(G; R) \) onto the set of holomorphic functions \( \alpha(\lambda, \xi) \) of uniform exponential type \( R \) satisfying \( A_r(\alpha)(\lambda, x) = A_r(\alpha)(-\lambda, x) \).

The reduction formula in Corollary 6.4 will play an important role in §7. In fact, it reduces the proof of the Paley-Wiener theorem for \( \mathcal{C}_r(G) \) to the one for \( \mathcal{C}_r(G) \) stated in Theorem (2).

Next we shall consider the Fourier transform associated with the discrete series \( T_m \) \((m \in \frac{1}{2} \mathbb{Z} \text{ and } |m| \geq 1)\) and the inversion formula, which are investigated in [K].

Let \( n \in \mathbb{Z} \) and \( I_n = \{ k \in \mathbb{Z} : 1 \leq k \leq n \} \). Then for \( m \in I_n \) and \( f \in \mathcal{C}_\varepsilon(G) \) we define the Fourier transform \( F_m(f)(z) \) \((z \in D)\) associated with the discrete series \( T_m \) by

\[
F_m(f)(z) = \sum_{k \in I_n} f(k) \text{conj}(T_m(g)e_{m}(x))dg
\]

(see §5 and [K]). When we express the dependence on \( n \), we use the notation \( F_n \) instead of \( F_m \). Let \( f^\varepsilon(z) \) denote a vector of functions
on $D$ given by

$$f^\natural(z) = (F_{-\pi}(f)(z) : m \in \mathbb{N}). \quad (5.9)$$

Then we see the following

**Proposition 6.6.** Let the notation be as above.

1. $F_m^\#(L^2(G)) = F_m^\#(L^2(G)) = A_{m+1}(D)$.

2. For each $\beta \in A_{m+1}(D)$ we define

$$\beta^\natural(x) = (4\pi)^{-1}(2m-1)(1-\tau^2)^{-\frac{1}{2}}e^{ikx} \beta(u),$$

where $x = k_xk_z \in G$ and $\tau = \tau \in \mathbb{R} \in D$. Then

$$F_m^\#(\beta^\natural) = \beta.$$

3. We keep the notation in (2). Then

$$F_m^\#((\Gamma(2m)/\Gamma(n-m)) E_{m} \beta^\natural) = \beta.$$

Proof. See [K], Theorem 4.1 and Theorem 5.5. Here we shall give the proofs of (2) and (3). Obviously, it is enough to prove the assertion for each $\beta(z) = e_m(z) = \lambda_m^*z^p$ ($p \in \mathbb{N})$. Then it easily follows from (5.1) and (5.2) that $\beta^\natural(x) = c_m^2 T_{m}^\#(x)$, where $c_m^2 = 4\pi(2m-1)^{-1}$, and moreover, since $T_{m}(g)e^\natural = \Sigma_s T_{m}^\#(g)e^\natural$,

$$F_m^\#(c_m^{-2} T_{m}^\#(z)) = \int c_m^{-2} F_m^\#(g) \text{conj}(T_m(g)e^\natural(z)) dg$$

$$= e_m(z).$$
Therefore, (2) is obtained. We recall that $T_{m}^{n} = (\lambda_{m}^{\leq n}/\lambda_{m-n}) \times 
abla_{m}^{n}$ (see Lemma 5.1 [2]). Then, applying $E_{-m}$ to the right hand side (see (4.5)) and using Lemma 5.1 [2] again, we see that

$$E_{m}^{n} = T_{m}^{n} = \left( \frac{\Gamma(n-m+1)}{\Gamma(n+m)} \right)^{\ast} T_{m}^{-n}.$$ 

Then, repeating the argument in the proof of (2), we can obtain (3).

Q.E.D.

Let

$$A_{n}^{\ast}(D) = \oplus_{n \in I_{n}} A_{k,n-1}(D)$$

be the direct sum of the weighted Bergman spaces $A_{k,n-1}(D)$ ($m \in I_{n}$) with the norm given by the sum of $|| \cdot ||_{k,n-1}$ ($m \in I_{n}$). Then for each $\beta = (\beta_{m}; m \in I_{n}) \in A_{n}^{\ast}(D)$ we let

$$\beta^{\gamma}(x) = \sum_{m \in I_{n}} \left( \frac{\Gamma(2m)}{\Gamma(n-m+1) \Gamma(n+m)} \right)^{k} \times E_{m}^{n}((4\pi)^{-\frac{1}{2}}(2m-1)(1-r^{2}))^{\ast m \in c^{k+r}} \beta_{w}(w),$$

where $x = k_{a}a_{k}$ and $w = x \cdot 0 = re^{\theta}$. Here we note the fact that the set of the discrete series $T_{m}$ has an element with $K$-type $n$ in the representation space $A_{k,n-1}(D)$ is just given by $T_{m}$ ($m \in I_{n}$). Then, applying Proposition 6.6, we can deduce the following

**Proposition 6.7.** The Fourier transform $f(x) \rightarrow f^{\gamma}(x)$ is an isometry of $L_{1}^{\gamma}(G)$ onto $A_{n}^{\ast}(D)$ and the inversion formula is given

$$f(x) = (f^{\gamma})^{\gamma}(x).$$
We say that $\beta_\ast(\beta_\ast) \in A^m(D)$ has a bounded boundary value if each $\beta_\ast \in A_{a_{\ast}}(D)$ has a bounded boundary value function on $T$. Then we have the following

Lemma 6.8. Let $\beta_\ast \in A_{a_{\ast}}(D)$ and suppose that it has a bounded boundary value function on $T$. Then

$$\beta_\ast(x) = (4\pi)^{-1}((m-1))S_{a_{\ast}, m-1}(I, (\lambda_\ast m / \lambda_\ast m)^2 \lambda_\ast m^{-1} \beta_\ast \lambda_\ast m^{-1})(x).$$

Proof. Since $\beta_\ast$ is bounded on $T$, the right hand side is well defined (see Lemma 4.1 and Proposition 4.2), and so the equation holds if it holds to each $e_\ast^p(z^-) = \lambda_\ast m^p (p \in \mathbb{N})$. In fact, it follows from Proposition 4.2 and Lemma 5.1 (2) that

$$S_{a_{\ast}, m-1}(I, (\lambda_\ast m / \lambda_\ast m)^2 \lambda_\ast m^{-1} (e_\ast^p) \lambda_\ast m^{-1})(x)$$

$$= \lambda_\ast m^{-1} \lambda_\ast m^{-1} \lambda_\ast m^{-1} \pi_\ast m^{-1} \ast m^{-1}(x)$$

$$= \lambda_\ast m^{-1} \lambda_\ast m^{-1} \lambda_\ast m^{-1} \pi_\ast m^{-1} \ast m^{-1}(x^-)$$

$$= \text{conj}(\pi_\ast m^{-1} \ast m^{-1}(x^-))$$

$$= \text{conj}(T_{\ast m}^{-1}(x^-))$$

$$= T_{\ast m}^{-1}(x).$$

Then, by the same argument in the proof of Proposition 6.6 (2) and (3), the desired equation for $e_\ast^p$ follows.

Q.E.D.
Corollary 6.9. If $\beta \in A'_n(D)$ has a bounded boundary value,

$$\beta^\gamma(x) = \sum_{m \in I_n} \frac{1}{\Gamma(2m+1)\Gamma(n+1)} \left( \lambda_{n-m}/\lambda_{n+1} \right)^m S_{n-m}(\lambda \otimes x^{-1}) \beta_{m+n+1}(x).$$

Last, for $f \in L'_n(G)$, we let

$$f^\wedge = (f^\wedge(\lambda, \xi), f^\wedge(z)) \quad ((\lambda, \xi, z) \in R \times T \times D).$$

We (6.2), (6.8) and (6.9)). Then Proposition 6.3 and Proposition 6.7 imply that

Theorem 6.10. The Fourier transform $f \mapsto f^\wedge$ is an isometry of $L'_n(G)$ onto $L'_n(R \times T)$ $\in A'_n(D)$ and the inversion formula is given by

$$f(x) = f^\wedge(\cdot, \cdot)^\wedge + f^\wedge(\cdot)^\wedge$$

$$= \int_{R \times T} f^\wedge(\lambda, \cdot)(x) \mu, (\lambda) d\lambda$$

$$+ \sum_{m \in I_n} \frac{1}{\Gamma(2m+1)\Gamma(n+1)} \left( \lambda_{n-m}/\lambda_{n+1} \right)^m E_{n+m} F_{m+n}(f^\wedge(x).$$

§7. Paley-Wiener theorem. We retain the notations in the previous sections. In this section we shall give a characterization of Fourier transforms $f^\wedge$ of compactly supported, $C^\infty$ functions $f$ on $G$.

Let $f$ be in $C^\infty(G)$. Then, by Lemma 6.2 (2), $f^\wedge(\lambda, \xi)$ is an anti-holomorphic function of uniform exponential type, and by Lemma 5.1 (1) and Lemma 4.1, $F_n(f)(z)$ $(m \in I_n)$ has a bounded boundary value on $T$. Especially, we can obtain the following relation.

-28-
Lemma 7.1.

\[ f^\lambda(-n-\frac{1}{2}i, \xi) = \lambda_{n+1} F_{n}(f)(\xi) e_{n-1}(\xi). \]

Proof. It follows from Lemma 5.1 (1) that

\[ f^\lambda(-n-\frac{1}{2}i, \xi) = \int f(g) \text{conj}(\pi_{n}(g)e_{n-1}(\xi)) dg \]

\[ = \lambda_{n+1} \int f(g) \text{conj}(T_{n}(g)e_{n}(z)) dg \bigg|_{z = e_{n-1}(\xi)} \]

\[ = \lambda_{n+1} F_{n}(f)(\xi) e_{n-1}(\xi). \]

Q.E.D.

Let \( PW \) be the subspace of \( L^2(R \times \mathbb{T}) \oplus A^2(D) \) defined by

\[ PW = \{ \gamma = (\alpha(\lambda, \xi), \beta(z)) \in L^2(R \times \mathbb{T}, \mu_\lambda(\lambda) d\lambda d\xi) \oplus A^2(D) : \]

1. \( \alpha(\lambda, \xi) \) is an antiholomorphic function of uniform exponential type,

2. \( A_{n}(\alpha)(\lambda, x) = A_{n}(\alpha)(-\lambda, x) \quad ((\lambda, x) \in R \times S) \),

3. \( \alpha(-n-\frac{1}{2}i, \xi) = \lambda_{n+1} \beta_{n}(\xi) e_{n-1}(\xi) \quad (\xi \in \mathbb{T}) \),

where \( \beta(z) = (\beta_{n}(z); n \in I_n) \).

and \( PW(R) \) \( (R > 0) \) the subspace of \( PW \) consisting of \( \gamma = (\alpha, \beta) \) such that the exponential type of \( \alpha \) is \( R \). In particular, the condition (3)
of PW implies that

$$\beta \text{ has a bounded boundary value,} \quad (7.1a)$$

and

$$\alpha((-m-\frac{1}{2})i, \zeta) \text{ has zero at } \zeta=0 \text{ of order } m-\varepsilon \quad (7.1b)$$

and has a holomorphic extension on D.

Then the main theorem can be stated as

**Theorem 7.2.** (Paley-Wiener Theorem on SU(1,1)) The Fourier transform $f \mapsto \hat{f}$ is a bijection of $C_{0}^{\infty}(G;R)$ onto PW(R).

Proof. Except the surjectivity, the assertion follows from Theorem 6.10, Lemma 6.2 and Lemma 7.1, so we shall prove that if $\gamma \in \text{PW}(R)$, then $\gamma^\vee \in C_{0}^{\infty}(G;R)$. It follows from Theorem 6.10, (7.1a), Corollary 6.9 and (3) of PW that $\gamma^\vee$ can be written as

$$\gamma^\vee(x) = \alpha^\vee(x) + \beta^\vee(x)$$

$$= \int S_{-m-\frac{1}{2}}(I, \alpha(\lambda, \cdot))(x) \mu_x(\lambda) d\lambda$$

$$+ \sum_{\lambda \in I} (\lambda^{-m}/\lambda^{-m-\frac{1}{2}}) S_{-m-\frac{1}{2}}(I, \alpha((-m-\frac{1}{2})i, \cdot))(x).$$

**Lemma 7.3.** If $\beta \equiv 0$, then $\gamma^\vee \in C_{0}^{\infty}(G;R)$.

Proof. Clearly, $\beta \equiv 0$ implies that each $S_{-m-\frac{1}{2}}(I, \alpha((-m-\frac{1}{2})i, \cdot)) \equiv 0$ ($m \in I$) and thus, by applying E.**", $S_{-m-\frac{1}{2}}(I, \alpha((-m-\frac{1}{2})i, \cdot)) \equiv 0$ (see Lemma 4.4). Then, since $I, \alpha((-m-\frac{1}{2})i, \zeta)$ has zero at $\zeta=0$ of order $m$ and has a holomorphic extension on $D$ (see (7.1b)), it
follows from Theorem 5.5 that

$$a((x-\frac{1}{2})i, \zeta) \equiv 0 \ (x \in \mathbb{L}).$$

that is, $a(\lambda^-, \zeta)$ is a holomorphic function of uniform exponential type $R$, that has zero at $\lambda=(m-\frac{1}{2})i \ (m \in \mathbb{L})$. Then, comparing the zero points of $P_\zeta(\lambda)$ (see Corollary 6.4), we see that $a(\lambda^-, \zeta)P_\zeta(\lambda)^{-1}$ is a holomorphic function of uniform exponential type $R$. Then, noting Corollary 6.4 (2), we can apply Theorem (2) in Remark 6.5 to $a(\lambda^-, \zeta)P_\zeta(\lambda)^{-1}$ and thus, by Corollary 6.4 (1), we can conclude that $\gamma \in C^\infty_{\mathbb{L}}(G; R)$. This completes the proof of Lemma.

Q.E.D.

Now we return to the proof of the theorem. Since $\beta \in \mathcal{C}(G)$, Proposition 6.1 implies that there exists $g \in C^\infty_{\mathbb{L}}(G; R)$ such that $\gamma = \beta$, that is, $g(z) = \beta(z) \ (z \in \mathbb{D})$. Therefore, if we let

$$h = \gamma - g,$$

we see that $h \in \mathcal{F}_\mathbb{D}(R)$ and $h(z) = 0$. Then, applying Lemma 7.1 to $h^\wedge$, we can deduce that $h = (h^\wedge)^{\wedge} \in C^\infty_{\mathbb{L}}(G; R)$, and so $\gamma = h + g \in C^\infty_{\mathbb{L}}(G; R)$.

This completes the proof of Theorem.

Q.E.D.

Let $C^\infty_{\mathbb{L}}(G) \ (0 < p \leq 2)$ denote the $L^p$ Schwartz space with right $K$-type $n$, that is, the space of all $C^\infty$ functions $f$ in $L^p_{\mathbb{L}}(G)$ such that for any $x \in \mathbb{N}$ and $g, g' \in U(g_c)$

$$\sup_{x \in \mathbb{G}} |f(x; g; g')| e^{-x\sigma(x)(1+\sigma(x))} < \infty$$
(cf. [EK], p. 146). Let $R[p]$ ($1 < p \leq 2$) denote the strip in $C$ defined
by $\{ z \in C; |\text{Im}(z)| \leq (1/p - 1/2) \}$ and $R(p)$ the interior of $R[p]$. Then,
we define

$$L^S = \{ y = (a(\lambda, \zeta), b(z)) \in L^2(\mathbb{R} \times T, \mu, (\lambda) d\lambda d\zeta) \otimes A^s_\infty(D) \};$$

(1) $a(\lambda, \zeta)$ is, as a function of $\lambda$, an antiholomorphic function
on $R[p]$ and for any $p, q, r \in \mathbb{N}$

$$\sup_{\lambda \in R(p), \zeta \in T} |(d/d\lambda)^q (d/d\zeta)^r a(\lambda, \zeta) | (1 + |\lambda|)^r < \infty,$$

(2) $A_n(a)(\lambda, x) = A_n(a)(-\lambda, x)$ ($((\lambda, x) \in \mathbb{R} \times \mathbb{Q})$.

(3) If $m \in \mathbb{N}$ satisfies $m \leq l/p$, then

$$a(-m, 1/2) = \lambda^{m-1} \beta_m(\zeta)_{\theta_{-m}}(\zeta).$$

where $\beta = (\beta_m(z); m \in \mathbb{N})$.  

Then we can obtain the following

**Theorem 7.4.** Let $n \in \mathbb{Z}$, and $0 < p \leq 2$. The Fourier transform $f \rightarrow f^\wedge$ is
a bijection of $C^*_n(G)$ onto $L^S$.

**Proof.** When the right $K$-type $n$ is trivial, we know that the discrete
part $A^*_n(G)$ vanishes, and the theorem is obtained by [EK] for general
groups; also when $n = \frac{1}{2}$, it can be obtained by the same way. So, we
shall reduce the proof to the case of $n = \varepsilon$.

As in the case of $n = 0$ (cf. [EK], §4), the image $f^\wedge$ of $f$ in $C^*_n(G)$
satisfies (1) of L's, and moreover, as in Lemma 6.2 (3) and Lemma 7.1, \( f^* \) satisfies (2) and (3) of L's. Therefore, Theorem 6.10 implies that \( f \rightarrow f^* \) is an injection of \( C^*_\epsilon(G) \) into L's.

Let \( \gamma = (a, \beta) \in \text{L's}, \) and we shall show that \( \gamma^* \in C^*_\epsilon(G) \). As in the proof of Lemma 7.3, we can find a compactly supported C function \( g \) on \( G \) such that the right K-type is \( n \) and \( \gamma g = \beta g \). Moreover, if we put \( h = \gamma^* - g \), \( h \) has no discrete part and \( h^* \gamma = (m - \frac{1}{2})i, \xi \) \( \equiv 0 \) for all \( m \in \mathbb{I}_n \). Then, since \( h^* \gamma = \gamma^* - g \) is in L's, it satisfies (1) of L's and thus, \( h^* \gamma^* \) satisfies the conditions (1) and (2) of L's for \( n = \epsilon \) (see Corollary 6.4 (2)). Therefore, the result for \( n = \epsilon \) and Corollary 6.4 (1) deduce that \( h \in C^*_\epsilon(G) \), so, \( \gamma^* = h + g \in C^*_\epsilon(G) \).

Q.E.D.

Recently, Barker [B] removed completely the finite K-type restriction for \( C^*(G) \) and gave a characterization of \( C^*(G) \) under Fourier transform.

References


