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# Harmonic non-holomorphic maps of 2-tori into the 2-sphere

Dedicated to Professor Shingo Murakami on his 60th birthday.

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**Abstract**

A harmonic map of a compact Riemann surface into the unit 2-sphere $S^2$ is necessarily holomorphic or anti-holomorphic if its degree is sufficiently large. In particular, a harmonic map of a torus into $S^2$ is neither holomorphic nor anti-holomorphic if and only if its degree is 0. In this paper, such harmonic maps of tori with arbitrary conformal structures are constructed. To construct these, the relationship between Gauss maps of surfaces with constant mean curvature and harmonic maps of surfaces into $S^2$ is applied.

## 0 Introduction.

The purpose of this paper is to investigate harmonic maps of Riemann surfaces of genus 1 into the unit sphere $S^2$. Holomorphic or anti-holomorphic maps of a Riemann surface $\Sigma$ into $S^2$ are trivial harmonic maps.

Consider a Riemann surface $\Sigma$ of genus $\gamma$. In [2], Eells and Wood proved that a harmonic map of $\Sigma$ into $S^2$ with degree $d$ must be holomorphic or anti-holomorphic if $|d| > \gamma - 1$. Hence the homotopy classes of harmonic maps which are neither holomorphic nor anti-holomorphic are restricted. So, it is meaningful to find harmonic maps of Riemann surfaces into $S^2$ which are neither holomorphic nor anti-holomorphic.

There are some examples of such surfaces. For a surface of genus $\gamma > 2$, Lemaire [5] showed that there exists a Riemann surface which admits a harmonic non-holomorphic and non anti-holomorphic map into $S^2$ with given degree $d \leq \gamma - 1$. In the case of genus one, that is torus, a harmonic map into $S^2$ is neither holomorphic nor anti-holomorphic if and only if its degree is 0. In this case, there exists a Riemann surface which admits surjective harmonic map into $S^2$ of degree 0 [6].

The notion of harmonic map of a Riemann surface $\Sigma$ depends only on its conformal structure. In this paper, harmonic maps of tori with *arbitrary* conformal structure into $S^2$ is constructed. More precisely, the main theorem is the following.

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**Theorem.** *There exist countably many "equivalent" classes of surjective harmonic maps of an arbitrary torus into $S^2$ which are neither holomorphic nor anti-holomorphic.*

The meaning of *equivalent* is described in Section 1.

Related to this problem, Eells and Wood [3] showed that there exist harmonic non-holomorphic, non anti-holomorphic maps with arbitrary degree of any conformal structure of torus into the complex projective space $CP^n$ provided $n \geq 2$. Our main result is the case of $n = 1$ of their theorem.

A Gauss map of a conformal immersion of a Riemann surface into $\mathbf{E}^3$ with constant mean curvature is a harmonic map of the surface into $S^2$. Immersions of certain tori with constant mean curvature are constructed by Wente [8,9] and Abresch [1]. To prove our theorem, we apply their results and construct conformal immersions of the universal cover $\mathbf{C}$ of torus into $\mathbf{E}^3$ with constant mean curvature whose Gauss maps are invariant under the deck transformations. These immersions may not induce tori with constant mean curvature.

## 1 Harmonic maps of tori into the 2-sphere.

Let $\Sigma$ be a compact Riemann surface and $g$ its riemannian metric associated with the conformal structure of it. The energy of a smooth map $f$ of $\Sigma$ into the unit 2-sphere $S^2$ is defined as

$$E(f) = \frac{1}{2}\int_\Sigma |df|^2\, dv_g, \tag{1.1}$$

where $|\cdot|$ and $dv_g$ are the norm and the volume element of $g$ respectively. Note that the definition of $E(\cdot)$ depends only on the conformal structure of $\Sigma$, not on the choice of a riemannian metric $g$.

A critical point of the functional $E(\cdot)$ is called a *harmonic map.* A map $f : \Sigma \to S^2$ is harmonic if and only if it satisfies the Euler-Lagrange equation

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} - \frac{2\bar{f}}{1+|f|^2}\frac{\partial f}{\partial z}\frac{\partial f}{\partial \bar{z}} = 0 \tag{1.2}$$

with respect to the complex coordinate $z$ of $\Sigma$. Here we identify $S^2$ with $\mathbf{C} \cup \{\infty\}$ by the stereographic projection at the north pole.

The automorphisms of $\Sigma$ and the isometries of $S^2$ act on the set of harmonic maps of $\Sigma$ into $S^2$.

**Lemma 1.1** *Let $\varphi : \Sigma \to \Sigma$ be a holomorphic or anti-holomorphic map and $\tau : S^2 \to S^2$ an isometry. Then if $f : \Sigma \to S^2$ is harmonic, so is $\tau \circ f \circ \varphi$.*

From now on, we call $\tau \circ f \circ \varphi$ in the above lemma an *associated harmonic mdp* of $f$.

DEFINITION. Two harmonic maps $f_1$, $f_2 : \Sigma \to S^2$ are called *equivalent* if there exists a harmonic map $f : \Sigma \to S^2$ such that both $f_1$ and $f_2$ are associated harmonic maps of $f$.

In particular, if $\Sigma$ is a torus, covering maps are holomorphic, and composing these with given harmonic map $f$, one can construct infinitely many associated harmonic maps of $f$. In this case, there are infinitely many equivalent harmonic maps with different energy value.

It is well-known that the Gauss map of a surface with constant mean curvature in the euclidean 3-space $\mathbf{E}^3$ is harmonic. In the rest of this section, we assume that $\Sigma$ is a torus and study the relationships between harmonic maps and immersions of constant mean curvature.

Let $\Sigma = T$ be a torus. Then $T$ is represented as a quotient $\mathbf{C}/\Gamma$, where $\Gamma$ is a lattice of $\mathbf{C}$.

Consider a conformal immersion $\mathbf{x} : \mathbf{C} \to \mathbf{E}^3$. In the complex coordinate $z = u + \sqrt{-1}\, v$ of $\mathbf{C}$, the normal vector $\xi(p)$ of $\mathbf{x}$ at $p \in \mathbf{C}$ is written as

$$\xi(p) = \frac{\mathbf{x}_u(p) \times \mathbf{x}_v(p)}{|\mathbf{x}_u(p) \times \mathbf{x}_v(p)|}.$$

The immersion $\mathbf{x}$ is called *orientation preserving* (*resp. reversing*) if the orientation of $\mathbf{E}^3$ determined by the frame $(\mathbf{x}_u, \mathbf{x}_v, \xi)$ coincides (resp. does not coincide) with the canonical orientation of $\mathbf{E}^3$.

The Gauss map $\tilde{\nu}_{\mathbf{x}}$ of $\mathbf{x}$ is a map of $\mathbf{C}$ to $S^2$ which maps $p \in \mathbf{C}$ to the unit normal vector $\xi(p)$ at $p$. If $\tilde{\nu}_{\mathbf{x}}$ is invariant under the action of the lattice $\Gamma$, it induces a map $\nu_{\mathbf{x}} : T = \mathbf{C}/\Gamma \to S^2$. The following lemma is a criterion for the equivalence of such harmonic maps.

**Lemma 1.2** *Let* $\mathbf{x}_i : \mathbf{C} \to \mathbf{E}^3\, (i = 1, 2)$ *be orientation preserving (resp. reversing) conformal immersions with constant mean curvature* $H = 1/2$ *whose Gauss maps are invariant under the action of a lattice* $\Gamma$. *Assume that the harmonic maps* $\nu_{\mathbf{x}_i} : T = \mathbf{C}/\Gamma \to S^2\, (i = 1, 2)$ *are equivalent. Then the image of* $\mathbf{x}_1$ *is congruent to that of* $\mathbf{x}_2$ *under the motions of* $\mathbf{E}^3$.

PROOF. Suppose Gauss maps $\nu_{\mathbf{x}_1}$ and $\nu_{\mathbf{x}_2}$ are equivalent. Then we can assume that there exist a holomorphic map $\varphi : T \to T$ and an isometry $\tau : S^2 \to S^2$ which satisfy

$$\nu_{\mathbf{x}_2} = \tau \circ \nu_{\mathbf{x}_1} \circ \varphi.$$

Lifting this to the universal cover $\mathbf{C}$ of $T$, we have

$$\tilde{\nu}_{\mathbf{x}_2} = \tau \circ \tilde{\nu}_{\mathbf{x}_1} \circ \tilde{\varphi},$$

where $\tilde{\varphi}$ is the lift of $\varphi$. Since $\mathbf{x}_1$ and $\mathbf{x}_2$ preserve (resp. reverse) the orientation, $\tau$ is extended to the motion $\tilde{\tau}$ of $\mathbf{E}^3$ as an element of $SO(3)$. Then,

$$\tilde{\nu}_{\mathbf{x}_2} = \tilde{\nu}_{\tilde{\tau}\circ\mathbf{x}_1} \circ \tilde{\varphi}.$$

On the other hand, the immersions $\mathbf{x}_1$ and $\mathbf{x}_2$ are determined by their Gauss maps and mean curvatures up to a translation [4]. Hence

$$\mathbf{x}_2 = \tilde{\tau} \circ \mathbf{x}_1 \circ \tilde{\varphi} + c,$$

where $c$ is a vector in $\mathbf{E}^3$. □

REMARK. For a given harmonic non-holomorphic map $f : T \to S^2$, one can construct a conformal branched immersion $\mathbf{x}$ of the universal cover of $T$ to $\mathbf{E}^3$ with constant mean curvature whose Gauss map $\nu_{\mathbf{x}} = f$ [4].

Moreover, if $f$ is not anti-holomorphic, the degree of $f$ is 0. In this case, $f_{\bar{z}}$ have no zeroes because of the index formula of Eells-Wood [2], and then, the immersion $\mathbf{x}$ have no branched points.

In particular, the set of harmonic maps of $T$ to $S^2$ with degree 0 corresponds bijectively to the set of conformal immersions of $\mathbf{C}$ to $\mathbf{E}^3$ with non-zero constant mean curvature whose Gauss map is $\Gamma$-invariant.

## 2 Construction of harmonic maps.

In this section, we construct harmonic maps of tori into $S^2$ using the construction of twisted tori with constant mean curvature by Wente [9]. More precisely, we shall prove the following theorem mentioned in Section 0.

**Theorem 2.1** *Let $T$ be a torus with arbitrary conformal structure. Then there exist countably many equivalent classes of surjective harmonic maps of $T$ into $S^2$ which are neither holomorphic nor anti-holomorphic.*

To begin with, we review the relationship between sinh-Gordon equation and surface theory.

Let $\Omega_{a_0b_0} = (-a_0, a_0) \times (-b_0, b_0)$ be a rectangular domain of $\mathbf{R}^2$. Then the Dirichlet problem of the sinh-Gordon equation

$$\Delta\omega + \cosh\omega \sinh\omega = 0 \tag{2.1}$$

on $\Omega_{a_0b_0}$ has the unique positive solution $\omega_0$ when $a_0^{-2} + b_0^{-2} > 4\pi^{-2}$ [8,1,7]. This solution can be extended to the doubly periodic solution $\tilde{\omega}_0$ of (2.1) by the odd reflections about $\partial\Omega_{a_0b_0}$. The fundamental domain of $\tilde{\omega}_0$ is a rectangle.

To construct harmonic maps, doubly periodic solutions of (2.1) with "skew" lattices which are constructed by Wente [9] are used.

**Lemma 2.2** ([9], Theorem 1) *For sufficiently small positive $a_0$ and $b_0$, there exist a neighborhood $U$ of $(a_0, b_0, 0) \in \mathbf{R}^3$ and a smooth function $\omega(u, v;\ a, b, c)$ on $\mathbf{R}^2 \times U$ which satisfy the following conditions.*

(1) *For each $(a, b, c) \in U$, $\omega(u, v;\ a, b, c)$ is a solution of (2.1) on $\mathbf{R}^2$.*

(2) *Let* $\mathrm{p}_1 = (2a, 0)$ *and* $\mathrm{p}_2 = (2c, 2b)$. *Then*

$$\omega(\mathrm{u} + \mathrm{p}_1;\, \mathrm{a}) = \omega(\mathrm{u} + \mathrm{p}_2;\, \mathrm{a}) = \omega(-\mathrm{u};\, \mathrm{a}) = -\omega(\mathrm{u};\, \mathrm{a}).$$

(3) $\omega(u, v;\, a_0, b_0, 0) = \omega_0(u, v)$.

*Here* $\mathrm{u}$ (*resp.* $\mathrm{a}$) *means* $(u, v)$ (*resp.* $(a, b, c)$).

The solution $\omega(u, v) = \omega(u, v;\, a, b, c)$ of (2.1) determines a surface with constant mean curvature. That is, the Frenet equation

$$\begin{aligned} \mathbf{x}_{uu} &= \omega_u \mathbf{x}_u - \omega_v \mathbf{x}_v + L\xi \\ \mathbf{x}_{uv} &= \omega_u \mathbf{x}_v + \omega_v \mathbf{x}_u + M\xi \\ \mathbf{x}_{vv} &= \omega_v \mathbf{x}_v - \omega_u \mathbf{x}_u + N\xi \\ \xi_u &= -e^{-2\omega}(L\mathbf{x}_u + M\mathbf{x}_v) \\ \xi_v &= -e^{-2\omega}(M\mathbf{x}_u + N\mathbf{x}_v) \end{aligned} \tag{2.2}$$

is integrable and its solution $\mathbf{x} : \mathbf{R}^2 = \mathbf{C} \to \mathbf{E}^3$ determines an immersion with constant mean curvature $1/2$. Here, $L$, $M$ and $N$ are the coefficients of the second fundamental form of $\mathbf{x}$:

$$\begin{aligned} L &= L(\beta) &&= e^{\omega}(\sinh\omega \cos^2\beta + \cosh\omega \sin^2\beta), \\ M &= M(\beta) &&= -\sin\beta\cos\beta, \\ N &= N(\beta) &&= e^{\omega}(\cosh\omega \cos^2\beta + \sinh\omega \sin^2\beta). \end{aligned} \tag{2.3}$$

The fundamental forms of the immersion $\mathbf{x}$ with respect to the unit normal vector $\xi$ are

$$\begin{aligned} |dx|^2 &= e^{2\omega}(du^2 + dv^2), \\ -dx \cdot d\xi &= L\, du^2 + 2M\, dudv + N\, dv^2. \end{aligned} \tag{2.4}$$

The principal curvatures of $\mathbf{x}$ are $\lambda_1 = \cosh\omega$ and $\lambda_2 = \sinh\omega$. The lines of curvature with respect to $\lambda_1$ and $\lambda_2$ are straight lines and $\beta$ is the angle between $\lambda_1$-curvature lines and the $u$-axis.

Since the immersion $\mathbf{x}$ is determined by $a$, $b$, $c$, $\beta$ up to a motion of $\mathbf{E}^3$, we write $\mathbf{x}(u, v) = \mathbf{x}(u, v;\, a, b, c, \beta)$.

The Gauss map $\tilde{\nu}$ of $\mathbf{x}$ determines a harmonic map of $\mathbf{R}^2 = \mathbf{C}$ into $S^2$. Using this, we construct harmonic maps of the torus of given conformal structure into $S^2$.

To construct harmonic maps of tori into $S^2$, it is enough to find $(a, b, c, \beta)$ such that the normal vector $\xi$ of $\mathbf{x}(\mathrm{u};\, \mathrm{a}, \beta)$ is doubly periodic with respect to $\mathrm{u}$.

By (2) in Lemma 2.2 and (2.4), there exist motions $E_i$ $(i = 1, 2)$ of $\mathbf{E}^3$ which satisfies

$$\mathbf{x}(\mathrm{u} + 2\mathrm{p}_i;\, a, b, c, \beta) = E_i \circ \mathbf{x}(\mathrm{u};\, a, b, c, \beta).$$

Decompose $E_i$ to their linear parts and parallel translations as

$$E_i = A_i + c_i \qquad (i = 1, 2),$$

where $A_i \in SO(3)$ and $c_i \in \mathbf{R}^3$ (cf. [9]). Then $A_i$ $(i = 1, 2)$ are normalized by orthogonal matrices $P_i$ as

$$ {}^tP_iA_iP_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_i & -\sin\theta_i \\ 0 & \sin\theta_i & \cos\theta_i \end{pmatrix} \qquad (i = 1, 2). $$

The rotational angles $\theta_i$ $(i = 1, 2)$ depend on $a$, $b$, $c$ and $\beta$. If both $\theta_i \in 2\pi\mathbf{Q}$ $(i = 1, 2)$, then the Gauss map $\tilde{\nu}_{\mathbf{x}}$ is invariant under a certain lattice.

Fix $(a, b, c) \in U$, where $U$ is the neighborhood in Lemma 2.2. For this triplet, we define a map from a neighborhood of $(1, 0)$ to $\mathbf{R}^2$ as

$$ \Phi_{abc} : (t, \beta) \mapsto (\theta_1(at, bt, ct, \beta), \theta_2(at, bt, ct, \beta)). $$

Then the following lemma holds.

**Lemma 2.3** *For a certain* $(a_0, b_0)$, *the following assertions hold.*

(1) *The derivative* $d\Phi_{a_0b_00}$ *does not degenerate at* $(t, \beta) = (1, 0)$.

(2) *The image of the regular points of the Gauss map* $\tilde{\nu} : \mathbf{C} \to S^2$ *of* $\mathbf{x}(a_0, b_0, 0, 0)$ *is* $S^2$.

We prove this in the next section.

The proof of Theorem 2.1 follows this fact.

PROOF OF THEOREM 2.1. Take $a_0$ and $b_0$ as in Lemma 2.3. Then there exist a neighborhood $V$ of $(a_0, b_0, 0)$ in $\mathbf{R}^3$ and a positive number $\varepsilon$ which satisfy:

(1) If $(a, b, c) \in V$, then $d\Phi_{abc}$ does not degenerate at $(t, \beta) = (1, 0)$.

(2) If $(a, b, c) \in V$ and $|\beta| < \varepsilon$, the Gauss map of the immersion $\mathbf{x}(u, v;\ a, b, c, \beta)$ is surjective.

We denote the lattice generated by the periods $2\mathrm{p}_1 = 4(a, 0)$ and $2\mathrm{p}_2 = 4(b, c)$ of $\tilde{\omega}(a, b, c)$ as $\Gamma(2\mathrm{p}_1, 2\mathrm{p}_2)$ and define $T(a, b, c) = \mathbf{C}/\Gamma(2\mathrm{p}_1, 2\mathrm{p}_2)$.

Not that for a given torus $T$, the set of $(a, b, c)$ for which there exists a conformal map of $T$ to $T(a, b, c)$ is dense in $V$ (see Appendix). Take $(a, b, c)$ from this set and consider a holomorphic map

$$ \varphi : T \to T(a, b, c). \tag{2.5} $$

Let $\mu_{0i} = \theta_i(a, b, c, 0)$ $(i = 1, 2)$. Then by (1), $\Phi_{abc}$ gives a diffeomorphism from a neighborhood of $(1, 0) \in \mathbf{R}^2$ to that of $(\mu_{01}, \mu_{02}) \in \mathbf{R}^2$. Take a pair $(\mu_1, \mu_2)$ in the neighborhood of $(\mu_{01}, \mu_{02})$ as

$$ \mu_i = \frac{n_i}{m_i} \cdot 2\pi \quad (i = 1, 2), \tag{2.6} $$

where $n_i/m_i$ $(i = 1, 2)$ are irreducible fractions. Then there exists a pair $(t, \beta)$ in the neighborhood of $(1, 0)$ such that

$$\theta_i(at, bt, ct, \beta) = \mu_i \quad (i = 1, 2). \tag{2.7}$$

The Gauss map $\tilde{\nu}(u, v;\ at, bt, ct, \beta)$ of the immersion $\mathbf{x}(u, v;\ at, bt, ct, \beta)$ is invariant under the action of $\Gamma(2m_1\mathrm{p}_1, 2m_2\mathrm{p}_2)$ at $(t, \beta)$ in (2.7). Then this induces a harmonic map

$$\nu = \nu(u, v;\ at, bt, ct, \beta) : \mathbf{C}/\Gamma(2m_1\mathrm{p}_1, 2m_2\mathrm{p}_2) \to S^2.$$

Composing $\nu$ and the natural covering projection

$$\rho : T(a, b, c) = \mathbf{C}/\Gamma(2\mathrm{p}_1, 2\mathrm{p}_2) \to \mathbf{C}/\Gamma(\frac{2}{m_2}\mathrm{p}_1, \frac{2}{m_1}\mathrm{p}_2),$$

we have a harmonic map

$$\nu \circ \rho \circ \varphi : T \to S^2. \tag{2.8}$$

This map is surjective because of (2).

The Gauss map of a surface is holomorphic if and only if the surface is minimal, and anti-holomorphic if and only if the surface is totally umbilic. Hence the harmonic map (2.8) is neither holomorphic nor anti-holomorphic.

Since the set of $(a, b, c)$ admitting a conformal map (2.5) is dense in $V$, one can construct countably many harmonic maps of $T$ to $S^2$.

Now it is sufficient to show that there exist countably many equivalent classes of such harmonic maps.

Let $(a_1, b_1, c_1)$ and $(a_2, b_2, c_2)$ be two points of $V$ admitting conformal maps as (2.5) which are sufficiently close in $V$. Then $\nu_i = \nu(u, v;\ a_it_i, b_it_i, c_it_i, \beta_i)$ $(i = 1, 2)$ determine harmonic maps of $T$ into $S^2$ for some $(t_i, \beta_i)$. If $t_1 \neq t_2$, the functions $\omega_i = \omega(u, v;\ a_it_i, b_it_i, c_it_i)$ $(i = 1, 2)$ are distinct each other because their fundamental periods are independent. Hence $(\mathbf{C}, e^{2\omega_1}|dz|^2)$ is not isometric to $(\mathbf{C}, e^{2\omega_2}|dz|^2)$. Then by Lemma 1.2, $\nu_1$ and $\nu_2$ are not equivalent. □

## 3 Proof of Lemma 2.3.

PROOF OF THE FIRST PART OF LEMMA 2.3.

In the case of $c = \beta = 0$, $\omega$ and $\mathbf{x}$ are the solutions explained by Abresch [1]. By his calculation, $\theta_1(ta_0, tb_0, 0, 0) \equiv 0$. Then $\partial\theta_1/\partial t = 0$ at $t = 1$ and $\beta = 0$. Take $(a_0, b_0)$ such that $a_0 = b_0$. Observing the calculation in [1, Lemma 4.5], we have $\partial\theta_2/\partial t < 0$ at $t = 1$ and $\beta = 0$.

Now it is sufficient to prove $\partial\theta_1/\partial\beta \neq 0$. To prove this, we shall prepare some terminology. Let $e_1 = e^{-\omega}\mathbf{x}_u$, $e_2 = e^{-\omega}\mathbf{x}_v$, $e_3 = \xi$ be the Frenet frame of the immersion $\mathbf{x}(u, v;\ a_0, b_0, 0, \beta)$ and

$$X(u, \beta) = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

for fixed $v = 0$. We normalize $X(0, \beta) = I$. So, $X(u, \beta)$ is expanded with respect to $\beta$ as

$$X(u, \beta) = X_0(u) + \beta Y(u) + o(\beta),$$

where $X_0(u) = X(u, 0)$ and $Y(u) = \partial X/\partial\beta|_{\beta=0}$. Note that $X(4a_0, 0) = I$ because of [1]. So we have the following lemma by direct calculations.

**Lemma 3.1**

$$\left.\frac{\partial\theta_1}{\partial\beta}\right|_{\substack{t=1\\ \beta=0}} = 0 \text{ if and only if } Y(4a_0) = 0.$$

PROOF. Since $X \in SO(3)$, there exists an orthogonal matrix $P = P(u, \beta) \in SO(3)$ which normalize $X$:

$$X(u, \beta) = {}^tPDP, \tag{3.1}$$

where

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Here, $\theta$ is a function of $u$ such that $\theta(4a_0) = \theta_1$. Expand $P$ and $D$ with respect to $\beta$ as

$$\begin{aligned} P &= P_0 + \beta T + o(\beta), \\ D &= I + \beta H + o(\beta), \end{aligned}$$

where

$$H = \left.\frac{\partial\theta_1}{\partial\beta}\right|_{\substack{t=1\\ \beta=0}} \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since ${}^tPP = I$, we have

$${}^tP_0P_0 = I, \qquad {}^tP_0T - TP_0 = 0.$$

Then, by (3.1) and keeping in mind that $X_0(4a_0, 0) = I$,

$$\begin{aligned} X(4a_0, \beta) &= I + \beta\, {}^tPDP + o(\beta) \\ &= I + \beta\, {}^tP_0HP_0 + o(\beta). \end{aligned}$$

Hence the conclusion holds. □

To complete the proof, it is enough to show $Y(4a_0) \neq 0$.

The Frenet equation (2.2) shows that $e_2 \equiv (0, 1, 0)$ along the curve $\gamma(u) = \mathrm{x}(u, 0; a_0, b_0, 0, \beta)$. Then the curve $\gamma(u) = (x, 0, z)$ lies in $xz$-plane. Using this, we have

$$Y(4a_0) = X_0(4a_0) \begin{pmatrix} 0 & -z(4a_0) + z(0) & 0 \\ z(4a_0) - z(0) & 0 & -x(4a_0) + x(0) \\ 0 & x(4a_0) - x(0) & 0 \end{pmatrix}$$

by Wente's calculation [9, equation (3.6)].

Now we assume $a_0 = b_0$. In this case, the curve $\gamma$, the $\lambda_1$-curvature line of $x(a_0, b_0, 0, 0)$ never closes up [1]. Then $x(4a_0) - x(0)$ or $z(4a_0) - z(0)$ does not vanish. Hence $Y(4a_0) \neq 0$, and then $\partial\theta_1/\partial\beta \neq 0$ at $t = 1$ and $\beta = 0$. □

PROOF OF THE SECOND PART OF LEMMA 2.3.

Here, we shall prove the surjectivity of the Gauss map $\nu$ when $c = \beta = 0$. In this case, the solutions $\omega$ and $\mathbf{x}$ are those of Abresch [1]. Consider the curves

$$\begin{aligned} \gamma_1(u) &= \mathbf{x}(u, b_0;\ a_0, b_0, 0, 0) \\ \gamma_2(v) &= \mathbf{x}(a_0, v;\ a_0, b_0, 0, 0), \end{aligned}$$

which are the curvature lines of the immersion $\mathbf{x} = \mathbf{x}(u, v;\ a_0, b_0, 0, 0)$. Both $\gamma_1$ and $\gamma_2$ are plane curve in $\mathbf{E}^3$. Moreover the first normal vector of $\gamma_i$ coincides with the normal vector $\xi$ of $\mathbf{x}$ along $\gamma_i$. The behavior of normal vectors along $\gamma_1$ and $\gamma_2$ is analyzed by Abresch [1]. First, the Gauss image of $\gamma_2$ is a great circle in $S^2$. We assume this circle is the equator.

Then the Gauss image of $\gamma_1$ contains a meridian as a regular set of the Gauss map [1, page 185]. Hence, there exists an open strip $F$ containing this meridian in the regular set of the Gauss image of the surface.

In case $\theta_2 = (n/m)2\pi$, the equator intersects $m$-times with copies of the Gauss image of $\gamma_1$.

Now we perturb $a_0$ and $b_0$ such that the denominator $m$ of $\theta_2/(2\pi)$ is sufficiently large. Then $m$copies of the strip $F$ cover whole $S^2$. Hence the image of the regular points of the Gauss map is $S^2$. □

## A Appendix.

The purpose of this section is to prove the following proposition referred in Section 2.

Throughout this section, we denote a lattice of $\mathbf{C} = \mathbf{R}^2$ generated by $\mathbf{p}_1$ and $\mathbf{p}_2$ by $\Gamma(\mathbf{p}_1, \mathbf{p}_2)$.

**Propostion A.1** *Let $T_0 = \mathbf{C}/\Gamma(\mathbf{p}_1, \mathbf{p}_2)$ be a given torus. Then for any torus $T$, the set of lattices $\Gamma$ which admit holomorphic maps from $T$ to $\mathbf{C}/\Gamma$ is dense in the neighborhood of $\Gamma(\mathbf{p}_1, \mathbf{p}_2)$.*

In the proof of this proposition, the following lemma is essential.

**Lemma A.2** *Let $\Gamma(\mathbf{p}_1, \mathbf{p}_2)$ be a lattice of $\mathbf{C}$. Then for any rational numbers $\lambda$ and $\mu$ ($\mu \neq 0$), there exists a holomorphic map from a torus $\Gamma(\mathbf{p}_1, \mathbf{p}_2)$ to $\mathbf{C}/\Gamma(p_1, \lambda\mathbf{p}_1 + \mu\mathbf{p}_2)$.*

PROOF. First, we claim that there exists a holomorphic map

$$\varphi : \mathbf{C}/\Gamma(\mathbf{p}_1, \mathbf{p}_2) \to \mathbf{C}/\Gamma(\mathbf{p}_1, \frac{n}{m}\mathbf{p}_2),$$

where $m$ and $n$ are integers. In fact, composing the natural projection

$$\varphi_1 : \mathbf{C}/\Gamma(\mathbf{p}_1, \mathbf{p}_2) \to \mathbf{C}/\Gamma(\frac{1}{n}\mathbf{p}_1, \frac{1}{m}\mathbf{p}_2)$$

and the homothety

$$\varphi_2 : \mathbf{C}/\Gamma(\frac{1}{n}\mathbf{p}_1, \frac{1}{m}\mathbf{p}_2) \to \mathbf{C}/\Gamma(\mathbf{p}_1, \frac{n}{m}\mathbf{p}_2),$$

we have a desired holomorphic map.

If $\lambda = 0$, the conclusion follows the above fact immediately.

Assume $\lambda \neq 0$. Since $\Gamma(\mathbf{p}_1, \mathbf{p}_1 + (\mu/\lambda)\mathbf{p}_2) = \Gamma(\mathbf{p}_1, (\mu/\lambda)\mathbf{p}_2)$, we have a holomorphic map

$$\psi_1 : \mathbf{C}/\Gamma(\mathbf{p}_1, \mathbf{p}_2) \to \mathbf{C}/\Gamma(\mathbf{p}_1, \mathbf{p}_1 + \frac{\mu}{\lambda}\mathbf{p}_2)$$

using the above fact. Moreover, there exists a holomorphic map

$$\psi_2 : \mathbf{C}/\Gamma(\mathbf{p}_1, \mathbf{p}_2) \to \mathbf{C}/\Gamma(\mathbf{p}_1, \lambda(\mathbf{p}_1 + \frac{\mu}{\lambda}\mathbf{p}_2)).$$

Then a holomorphic map $\psi_2 \circ \psi_1$ from $\mathbf{C}/\Gamma(\mathbf{p}_1, \mathbf{p}_2)$ to $\mathbf{C}/\Gamma(\mathbf{p}_1, \lambda\mathbf{p}_1 + \mu\mathbf{p}_2)$ is obtained.
□

PROOF OF PROPOSITION A.1. We can assume one of the generator of the lattice of $T$ coincides with that of $T_0$ without any loss of generality. That is, we assume

$$T = \mathbf{C}/\Gamma(\mathbf{p}_1, \mathbf{r}).$$

Then, to prove the proposition, it is enough to show that the following claim.

For any $\varepsilon > 0$, there exists a vector $\mathbf{q}$ such that

(1) $|\mathbf{p}_2 - \mathbf{q}| < \varepsilon$, and

(2) the lattice $\Gamma(\mathbf{p}_1, \mathbf{q})$ admits a holomorphic map from $\mathbf{C}/\Gamma(\mathbf{p}_1, \mathbf{r})$ to $\mathbf{C}/\Gamma(\mathbf{p}_1, \mathbf{q})$.

Since $\mathbf{p}_1$ and $\mathbf{r}$ are linearly independent, there exist numbers $\alpha$ and $\beta$ ($\beta \neq 0$) such that

$$\mathbf{p}_2 = \alpha\mathbf{p}_1 + \beta\mathbf{r}.$$

Take rational numbers $\lambda$ and $\mu$ which satisfy

$$|\lambda - \alpha| < \frac{\varepsilon}{2|\mathbf{p}_1|} \quad \text{and} \quad |\mu - \beta| < \frac{\varepsilon}{2|\mathbf{r}|}.$$

Then $\mathbf{q} = \lambda\mathbf{p}_1 + \mu\mathbf{r}$ satisfies the conditions (1) and (2). □

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