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Complete space-like surfaces in the Minkowski 3-space with constant mean curvature

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ABSTRACT. In the Minkowski 3-space, the pseudospheres and hyperbolic cylinders are the typical complete space-like surfaces with constant mean curvature. In this paper, we characterize the hyperbolic cylinder among such surfaces.

§0. Introduction.

Let L^3 be the Minkowski 3-space, that is, $\mathbf{R}^3$ with the indefinite metric $\langle\ ,\ \rangle = (dx^1)^2 + (dx^2)^2 - (dx^3)^2$. A surface in L^3 is called *space-like* if the induced metric of $\langle\ ,\ \rangle$ is positive definite. On a space-like surface, the notions of the first fundamental form, the second fundamental form, and the mean curvature are defined in the same way as on a surface in the euclidean space.

The existence of the space-like surface in L^3 with constant mean curvature H has been studied by Calabi [**Ca**] and Cheng-Yau [**CY**], who established the Bernstein-type theorem for such surfaces in the case $H = 0$, *maximal* space-like surface.

In this paper, we investigate *complete* space-like surfaces with non-zero constant mean curvature H. The most well-known example of such a surface is the *pseudosphere*:

$$S(H) = \{(x^1, x^2, x^3) \in L^3 :\ (x^1)^2 + (x^2)^2 - (x^3)^2 = -\frac{1}{H^2},\ x^3 > 0\}, \tag{0.1}$$

which is the only complete, totally umbilic space-like surface with constant mean curvature H. Note that $S(H)$ is isometric to the Poincaré disc with constant curvature $-H^2$.

Among non-umbilical space-like surfaces, the following *hyperbolic cylinder* is the implest one:

$$C(H) = \{(x^1, x^2, x^3) \in L^3 :\ (x^1)^2 - (x^3)^2 = -\frac{1}{4H^2},\ x^3 > 0\}. \tag{0.2}$$

This is the only complete, flat space-like surface with non-zero constant mean curvature H.

Though many other constant mean curvature surfaces are constructed by Treibergs [**Tr**] as entire graphs on the x^1x^2-plane which solve his asymptotic Dirichlet problem, $S(H)$ and $C(H)$ are distinctive among such surfaces.

For example, Choquet-Bruhat [**CB**] characterized $S(H)$ as the only constant mean curvature slices in L^3 with some assumptions, and Goddard [**Go**] showed that any perturbation of $S(H)$ with constant mean curvature must be translation of L^3.

In this paper, we characterize the hyperbolic cylinder $C(H)$ as the complete space-like surface with constant mean curvature which is "uniformly" non-umbilic. More precisely, our main theorem is the following:

Theorem. *The hyperbolic cylinder $C(H)$ is the only complete space-like surface in L^3 with non-zero constant mean curvature H whose principal curvatures λ_1 and λ_2 satisfy*

$$(\lambda_1 - \lambda_2)^2 \geq \varepsilon^2 \tag{0.3}$$

for some positive number ε.

In §**1**, the fundamental equations for space-like surfaces are reviewed. Using these equations, we show in §**2** that the second fundamental form of a space-like surface satisfying the assumptions of the theorem is determined when the surface is conformal to $\mathbf{R}^2$. In this case, the Gauss equation shows that there exists an entire solution of the equation $\Delta\rho = \lambda \sinh\rho$ on $\mathbf{R}^2$, where λ is a positive constant. As a consequence of he maximum principle, we prove in §3 that the only entire solution of this equation is the trivial one, which gives $C(H)$. The proof of the theorem follows immediately this fact.

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§1. Space-like surfaces with constant mean curvature.

Let Σ be a space-like surface in L^3 with constant mean curvature H. Then the first fundamental form, *i.e.*, the induced metric $g = \langle\ ,\ \rangle|_\Sigma$ gives a riemannian metric on Σ. So, we can take isothermal parameters (u,v) as local coordinates of Σ in which g is written as

$$g = e^\sigma(du^2 + dv^2) \tag{1.1}$$

with some smooth function $\sigma(u,v)$. Using a complex parameter $z = u + \sqrt{-1}v$, we can also write

$$g = e^\sigma\, dz d\bar{z}.$$

Take the unit normal vector field of Σ, *i.e.*, a vector field ν along Σ which satisfies $\langle\nu,\nu\rangle = -1$. So, the second fundamental form h of Σ is defined as a symmetric 2-tensor on Σ by

$$h(X,Y) = -\langle\bar{\nabla}_X\nu, Y\rangle \quad \text{for } X,Y \in T_p\Sigma$$

at each point p on Σ, where $\bar{\nabla}$ is the canonical connection of L^3. Since the mean curvature $H = \frac{1}{2}\operatorname{trace}_g h$, h is written as

$$h = Ldu^2 + 2Mdudv + (2e^\sigma H - L)dv^2$$

in the present isothermal coordinates.

Let λ_1 and λ_2 be principal curvatures of Σ, *i.e.*, the eigenvalues of h with respect to the metric g. So, the Gaussian curvature K and the mean curvature H are written as

$$K = -\lambda_1\lambda_2 = e^{-2\sigma}\{M^2 - L(2e^\sigma H - L)\},$$
$$H = \frac{1}{2}(\lambda_1 + \lambda_2),$$

and

(1.2) $$(\lambda_1 - \lambda_2)^2 = 4(H^2 + K) = 4e^{2\sigma}\{(L - e^\sigma H)^2 + M^2\}$$

holds.

Define a function Φ on Σ locally as

1.3) $$\Phi(z) = (L - e^\sigma H) - \sqrt{-1}M.$$

So,

(1.4) $$(\lambda_1 - \lambda_2)^2 = 4|\Phi|^2 e^{-2\sigma}.$$

Note that a point p on Σ with a complex coordinate z is an umbilical point if and only if $\Phi(z) = 0$.

In the present coordinates, the fundamental equations of Σ become as the following:

Lemma 1.1. *Let Σ be a space-like surface in L^3 with constant mean curvature H, and (u, v) its isothermal coordinates in which the first fundamental form g is written as* (1.1). *Then,*

(1) (Codazzi equation) *The locally defined function $\Phi(z)$ in* (1.3) *is holomorphic.*

(2) (Gauss equation) *The Gaussian curvature K of Σ is the intrinsic sectional curvature of (Σ, g), i.e.,*

$$K = -\frac{1}{2}e^{-\sigma}\Delta\sigma = (H^2 e^\sigma - |\Phi|^2 e^{-\sigma}), \quad \textit{where} \quad \Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}.$$

For example, let $\Sigma = C(H)$, the hyperbolic cylinder defined in (0.2). Putting $u = H^{-1}\tanh^{-1}(x^1/x^3)$ and $v = x^2$, we have the global isothermal coordinates (u, v) on Σ in which g, h and Φ are written as the following:

(1.5) $$\begin{cases} g = du^2 + dv^2 \\ h = 2H du^2 \\ \Phi = H = \text{constant} . \end{cases}$$

In particular, $C(H)$ is isometric to the euclidean plane $\mathbf{R}^2$.

Conversely, a complete flat space-like surface with non-zero constant mean curvature H is congruent to $C(H)$.

By (1.2), the Gaussian curvature K of Σ satisfies $-H^2 \leq K$. On the other hand, K must be non-positive if Σ is closed with respect to the euclidean topology [**CY**]. In particular, if $H = 0$, Σ must be flat. This proves the Bernstein-type theorem for space-like surfaces in L^3.

§2. Complete space-like surface conformal to $\mathbf{R}^2$.

In this section, a complete space-like surface Σ is assumed to be conformal to the euclidean plane $\mathbf{R}^2$. So, we can take the standard coordinates (u, v) of $\mathbf{R}^2$ as the *global* isothermal coordinates of Σ in which the first fundamental form g has the form

$$g = e^{\sigma}(du^2 + dv^2) = e^{\sigma}\, dz d\bar{z} \tag{2.1}$$

with some smooth function σ on $\mathbf{R}^2$. So, the complex valued function $\Phi(z)$ is defined on the whole plane $\mathbf{C} = \mathbf{R}^2$, and holomorphic because of Lemma 1.1(1). That is, Φ is an entire holomorphic function on $\mathbf{R}^2$. Though there are many entire function on $\mathbf{C}$, Φ must be constant under the assumptions of our theorem.

Lemma 2.1. *Let Σ be a complete surface as above whose principal curvatures λ_1 and λ_2 satisfy*

$$(\lambda_1 - \lambda_2)^2 \geq \varepsilon^2 > 0 \tag{2.2}$$

for some positive ε. Then the function $\Phi(z)$ in (1.2) *must be constant.*

Proof: Substituting (2.1) into (2.3), we have

$$2\varepsilon^{-1}|\Phi| \geq e^{\sigma}. \tag{2.3}$$

Consider a riemannian metric

$$\hat{g} = 2\varepsilon^{-1}|\Phi|(du^2 + dv^2) = 2\varepsilon^{-1}|\Phi| dz d\bar{z}$$

on $\mathbf{R}^2 = \mathbf{C}$. Then, (2.3) shows $\hat{g} \geq g$ as quadratic forms on $T\mathbf{R}^2$. So, by the completeness of g, $\hat{g}$ is also a complete metric on $\mathbf{R}^2$.

On the other hand, the Gaussian curvature of $\hat{g}$ is

$$K_{\hat{g}} = -\frac{\varepsilon}{4}|\Phi|^{-1}\Delta \log|\Phi| = 0$$

since Φ is holomorphic.

Hence, $\hat{g}$ is the flat complete metric on $\mathbf{R}^2$. Then, there exists an isometry

$$\mu : (\mathbf{C}, \hat{g}) \longrightarrow (\mathbf{C}, g_0),$$

where g_0 is the standard metric of $\mathbf{C}$. The isometry μ can be considered as an entire holomorphic function which maps $\mathbf{C}$ onto $\mathbf{C}$ injectively, since it is conformal. So, μ must have a pole of order 1 at ∞. Thus μ is linear, *i.e.*,

$$\mu(z) = az + b$$

for some constant $a \neq 0$ and b.

Hence

$$2\varepsilon^{-1}|\Phi| dz d\bar{z} = \hat{g} = \mu^* g_0 = |a|^{-2} dz d\bar{z},$$

and then, Φ must be constant. ∎

Substituting this into the Gauss equation, Lemma 1.1(2), and putting $\lambda = 4|H/\Phi|$, we have the following equation.

Corollary 2.2. *Let Σ be as in* Lemma 2.1 *and $\rho = \sigma + \log|H/\Phi|$. Then ρ satisfies the equation*

$$\Delta\rho = \lambda \sinh\rho \quad \text{on } \mathbf{R}^2, \tag{2.4}$$

where $\Delta = \dfrac{\partial^2}{\partial u^2} + \dfrac{\partial^2}{\partial v^2}$, *and* $\lambda = 4|H/\Phi|$, *a positive constant.*

The trivial solution $\rho \equiv 0$ gives the flat metric on Σ, and hence, it corresponds to the hyperbolic cylinder $C(H)$.

§3. Non existence of non-trivial solutions of (2.4).

In this section, we shall prove the following proposition, the maximum principle for the equation (2.4).

Proposition 3.1. *Let λ be a positive number. Then the equation*

$$\Delta\rho = \lambda \sinh\rho \qquad \text{on } \mathbf{R}^2 \tag{3.1}$$

has no entire solutions except $\rho \equiv 0$.

To prove this, we look at radially symmetric solutions of (3.1). Consider the ordinary differential equation

$$(3.2) \qquad \begin{aligned} &\text{(a)} \qquad \varphi''(r) + \frac{1}{r}\varphi'(r) = \lambda \sinh\varphi(r) \quad \text{for } r \geq 0, \\ &\text{(b)} \qquad \varphi(0) = a > 0, \qquad \varphi'(0) = 0, \end{aligned}$$

where $'$ is the derivation with respect to r. So, the solution of (3.2) is a radially symmetric solution of (3.1) with $r = \sqrt{u^2 + v^2}$. First, we claim the local existence of a solution of (3.2).

Lemma 3.2. *There exists a local solution of* (3.2)(a) *and* (3.2)(b).

Proof: Write (3.2) as

$$\varphi(r) = a + \int_0^r \frac{ds}{s} \int_0^s t \sinh\varphi(t)\,dt,$$

and use a usual iteration argument. ∎

Nevertheless, there exist no global solutions of (3.2) except the trivial solution $\rho \equiv 0$.

Lemma 3.3. *There exists no entire, radially symmetric solution* $\varphi(r)$ *of* (3.1) *with* $\varphi(0) > 0$.

Proof: Suppose $\varphi(r)$ be an entire radially symmetric solution of (3.1) with $\varphi(0) = a > 0$. So, φ satisfies (3.2).

Write the equation (3.2) as

$$(r\varphi')' = r\sinh\varphi. \tag{3.3}$$

By (3.2)(b) and (3.3),

$$\varphi'(r) > 0 \qquad \text{for } r > 0 \tag{3.4}$$

holds, and then, φ is a simply increasing function of r. In particular, $\sinh\varphi(r) \geq \sinh a$ for $r \geq 0$. Substituting this into (3.3), we have

$$(r\varphi')' \geq \lambda \sinh a.$$

Integrating this twice, the inequality

$$\varphi - a \geq \frac{r^2}{4}\lambda \sinh a \tag{3.5}$$

holds, and hence φ tends to $+\infty$ as $r \to \infty$.

On the other hand,

$$\lambda \sinh\varphi \leq \varphi''$$

because of (3.2)(a) and (3.4). Integrating this,

$$\begin{aligned}
\{\varphi'(r)\}^2 &= \int_0^r \{\varphi'(s)^2\}'\,ds = 2\int_0^r \varphi''(s)\,\varphi'(s)\,ds \\
&\leq 2\lambda \int_0^r \sinh\varphi(r)\,\varphi'(r)\,dr = 2\lambda \int_a^{\varphi(r)} \sinh x\,dx \\
&\leq 2\lambda(\cosh\varphi(r) - \cosh a) \\
&\leq 2\lambda(\cosh^2\varphi(r) - 1) = 2\lambda \sinh^2\varphi(r),
\end{aligned}$$

since $\cosh\varphi(r) \geq 1$. Then,

$$\frac{\varphi'}{r} \leq \frac{\sqrt{2\lambda}}{r}\sinh\varphi$$
$$\leq \frac{1}{2}\lambda\sinh\varphi \quad \text{for } r > r_1,$$

where $r_1 = \sqrt{2/\lambda}$. Substituting this into (3.2)(a), we have

$$\varphi'' \geq \frac{1}{2}\lambda\sinh\varphi \qquad \text{for } r > r_1.$$

Thus,

$$\begin{aligned}\{\varphi'(r)\}^2 - \{\varphi'(r_1)\}^2 &= 2\int_{r_1}^{r} \varphi'(s)\,\varphi''(s)\,ds \\ &= \lambda\int_{\varphi(r_1)}^{\varphi(r)} \sinh x\,dx \\ &= \lambda\{\cosh\varphi(r) - \cosh\varphi(r_1)\}.\end{aligned}$$

Hence, there exists a positive number r_2 such that for $r \geq r_2$,

$$\begin{aligned}\varphi'(r) &\geq \sqrt{\lambda\{\cosh\varphi(r) - \cosh\varphi(r_1)\}} \\ &\geq C_1\exp(\frac{\varphi(r)}{2}),\end{aligned}$$

where C_1 is a positive constant. Integrating this inequality, we have

$$C_1^{-1}\exp(-\frac{\varphi(r)}{2}) \geq r + C_2 \quad \text{for } r > r_2 \tag{3.6}$$

with some constant C_2. Here, $\lim_{r\to\infty}\varphi(r) = +\infty$ because of (3.5). Then, the left-hand side of (3.6) tends to 0 when $r \to +\infty$. This shows that r is bounded, and contradicts the assumption. ∎

Corollary 3.4. *Let φ be a non-trivial radially symmetric solution of* (3.1) *with* $\varphi(0) >$ 0. *Then, there exists a positive number R for which* $\lim_{r\to R}\varphi(r) = +\infty$.

Proof: By (3.4), $\varphi(r)$ is an increasing function of r.

On the other hand, φ is a solution of (3.2)(a) in a finite interval $[0, R)$ because of Lemma 3.2. Hence, φ tends to $+\infty$ as $r \to R$. ∎

Proof of Proposition 3.1: Let ρ be an entire solution of (3.1) which is not identically 0. So, we can suppose $\rho(0) \neq 0$. Assume $\rho(0) = 2a > 0$ and take a radially symmetric solution of (3.1) with $\varphi(0) = a$. So, there exists a positive number R such that $\lim_{r\to R}\varphi(r) = +\infty$ because of Corollary 3.4.

Let $f = \varphi - \rho$, a function defined on $B_R = \{(u,v) : r = \sqrt{u^2+v^2} < R\}$ with $\lim_{r\to R} f = +\infty$. Then, f takes a minimum at some point p in B_R. Assume $f(p) < 0$. So,

$$\begin{aligned}\Delta f(p) &= \Delta\varphi(p) - \Delta\rho(p)\\ &= \lambda\{\sinh\varphi(p) - \sinh\rho(p)\}\\ &= 2\lambda\cosh\frac{\varphi(p)+\rho(p)}{2}\sinh\frac{f(p)}{2}\\ &< 0.\end{aligned}$$

This is a contradiction to that f takes its minimum at p. Hence $f = \varphi - \rho \geq 0$ in B_R. In particular, $f(0) = \varphi(0) - \rho(0) = a - 2a = -a < 0$. This is impossible. Thus there exists no entire solution ρ of (3.1) which takes a positive value.

When $\rho(0) < 0$, we have the same conclusion by considering $-\rho$ instead of ρ. ∎

Remark. In [**Os**], Ossermann showed the non-existence of entire solutions of $\Delta u \geq f(u)$, where f is a positive, increasing function with large growth rate. Though our equation (2.4) does not satisfy his assumptions, almost all parts of his proof are valid for Proposition 3.1.

§4. Proof of the main theorem.

Let Σ be a complete space-like surface satisfying the assumptions of the theorem. Note that a complete space-like surface can be represented an entire graph on the x_1x_2-plane in L^3. In particular, Σ must be simply connected. Thus, Σ is conformal to either the Poincaré disc $\mathbf{H}^2$ or the euclidean plane $\mathbf{R}^2$, since it is non-compact.

By the assumptions,

$$2\varepsilon^{-1}|\Phi| \geq e^{\sigma} \tag{4.1}$$

holds in the isothermal coordinates as in §**2**.

Assume Σ is conformal to $\mathbf{H}^2 = (D, g_0)$, where $D = \{z \in \mathbf{C} : |z| < 1\}$ and $g_0 = 4dzd\bar{z}/(1-|z|^2)$. So, (Σ, g) is isometric to $(D, g = e^{\sigma}\,dzd\bar{z})$ for some function σ on D. Here, the completeness of g implies

$$\lim_{(u,v)\to\partial D} e^{\sigma} = +\infty.$$

So, the function Φ is a non-vanishing holomorphic function on D which satisfies

$$\lim_{(u,v)\to\partial D} |\Phi| = +\infty \tag{4.2}$$

because of (4.1). Let $\Psi = \Phi^{-1}$. Then, Ψ is holomorphic in D and continuous on $\bar{D}$ with $\Psi|_{\partial D} = 0$. Then, by Cauchy's formula,

$$\Psi(0) = -\frac{\sqrt{-1}}{2\pi}\int_{\partial D}\frac{\Psi(z)}{z}\,dz = 0.$$

This is impossible. Therefore Σ cannot be conformal to $\mathbf{H}^2$.

Hence Σ must be conformal to $\mathbf{R}^2$. Then we can take global coordinates (u, v) of Σ in which the first fundamental form g is written as (2.1). So, σ in (2.1) satisfies the equation (2.4) and then, must be constant because of Lemma 3.1. Thus g is the flat metric and hence, Σ is congruent to the hyperbolic cylinder $C(H)$. This completes the proof of the main theorem. ∎

Remark. If a non-trivial radially symmetric solution of (2.4) satisfies (4.1), $\rho(0)$ must be negative.

Take a radially symmetric solution ρ of (2.4) with $\rho(0) > 0$. So σ becomes a metric on a disc $B_R = \{(u,v) : \ u^2 + v^2 < R^2\}$ and (B_R, g) is a complete metric on B_R because of Lemma 3.4. This gives a complete space-like surface with constant mean curvature H which is conformal to $\mathbf{H}^2$, but it does not satisfy the assumption (0.3).

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