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Discrete Part of L^p Functions on SU(1,1)

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Takeshi Kawazoe

Department of Mathematics
Faculty of Science and Technology
Keio University

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku, Yokohama, 223 Japan

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§1. Introduction. Let G be SU(1,1) and T_n ($n \in \frac{1}{2}Z$, $|n| \geq 1$) the holomorphic ($n > 0$) and the anti-holomorphic ($n < 0$) discrete series representation of G (cf. [Su], p.237). Let $f^n_{\ell m}$ (ℓ, $m \in \mathbf{N}$) denote the normalized matrix coefficient of T_n with K-type $(-(n\pm\ell),-(n\pm m))$, where "±" corresponds to the ± sign of n. Then the Fourier coefficients of f in $L^2(G)$ are defined by

$$\hat{f}(n;\ell,m)= \int_G f(g)\overline{f^n_{\ell m}}(g)dg \tag{1.1}$$

for $n \in \frac{1}{2}Z$, $|n| \geq 1$ and ℓ, $m \in \mathbf{N}$. In the previous manuscript [K] we define the operator Φ^n_m which maps the function f on G to the function $\Phi^n_m(f)$ on D, the open unit disk in $\mathbf{C}$, as follows.

$$\begin{aligned}\Phi^n_m(f)(z)&= \int_G \overline{f}(g)T_n(g)e^n_m(z)dg \quad (z \in D)\\ &=2\pi^{\frac{1}{2}}(2n-1)^{-\frac{1}{2}} \sum_{\ell=0}^{\infty} \overline{\hat{f}(n;\ell,m)}e^{\ell}_n(z),\end{aligned} \tag{1.2}$$

where $e^{\ell}_n(z)=(\Gamma(\ell+2n)/\Gamma(\ell+1)\Gamma(2n-1))^{\frac{1}{2}}z^{\ell}$ ($\ell \in \mathbf{N}$). Then a characterization of $\Phi^n_0(L^p(G))$ ($1\leq p\leq 2$) is obtained in [K], Theorem 8.1, actually, if $(n,p)\neq(1,1)$, it coincides with the weighted Bergman space $A_{p,\frac{1}{2}np-1}(D)$ on D (cf. [CR]), and if $(n,p)=(1,1)$, it is given by the subspace $H^1_0(D)$ of the classical Hardy space $H^1(D)$ which consists of all holomorphic functions F on D satisfying $F' \in A_{1,0}(D)$.

However, the approach and their proofs in [K] are complicated, and we treat only the case of m=0. Therefore, in this manuscript, we shall try to reform them and obtain the independence of m for $\Phi^n_m(L^p(G))$. Moreover, we shall give some applications to the classical harmonic analysis on D.

§2. Notation. We shall use the same notations in [K]. For the readers who read first we shall brief them.

2.1. L^p functions on SU(1,1). Let G be SU(1,1), the group of 2x2 complex matrices g satisfying $\det g=1$ and ${}^t\bar{g}\begin{pmatrix}1&0\\0&-1\end{pmatrix}g=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, and let

$$K=\{\ k_\theta=\begin{pmatrix}e^{i\theta/2}&0\\0&e^{-i\theta/2}\end{pmatrix}\ ;\ 0\leq\theta\leq4\pi\ \},\qquad A=\{\ a_t=\begin{pmatrix}\mathrm{ch}\,t/2&\mathrm{sh}\,t/2\\\mathrm{sh}\,t/2&\mathrm{ch}\,t/2\end{pmatrix}\ ;\ t\in\mathbf{R}\ \}. \tag{2.1}$$

For a complex valued function f on G we put

$$\|f\|_p=(\int_G|f(g)|^p dg)^{1/p}\qquad(0<p<\infty), \tag{2.2}$$

where dg is the Haar measure on G normalized by the following integral formula for the Cartan decomposition $G=KCL(A^+)K$ ($A^+=\{\ a_t\ ;\ t>0\ \}$):

$$\int_G f(g)dg=\frac{1}{8\pi}\int_0^{4\pi}\int_0^{\infty}\int_0^{4\pi}f(k_\theta a_t k_{\theta'})\mathrm{sh}t\,d\theta\,dt\,d\theta', \tag{2.3}$$

where $d\theta$, $d\theta'$ and dt denote the Euclidean measures on $[0,4\pi)$ and $\mathbf{R}$ respectively. Then $L^p(G)$ $(0<p<\infty)$ denotes the space of all complex valued functions on G with finite $\|\cdot\|_p$-"norm". For $1\leq p<\infty$, these spaces are Banach spaces with the norm, especially, $L^2(G)$ is a Hilbert space with obvious inner product denoted by (,).

2.2. Bergman and Hardy spaces on D. Let $D=\{\ z\in\mathbf{C}\ ;\ |z|<1\}$ be the open unit disk in $\mathbf{C}$. For a complex valued function F on D we put

$$\|F\|_{p,r}=(\frac{1}{\pi}\int_D|F(z)|^p(1-|z|^2)^{2r}dz)^{1/p}\ (0<p<\infty,\ r\in\mathbf{R})$$
$$\|F\|_{H^p}=\lim_{r\to-\frac{1}{2}}(2r+1)\|F\|_{p,r}\qquad(0<p<\infty), \tag{2.4}$$

where dz is the Euclidean measure on D. Then the weighted Bergman space $A_{p,r}(D)$ and the Hardy space $H^p(D)$ are defined as the spaces of all holomorphic functions on D with finite $\|\cdot\|_{p,r}$ and $\|\cdot\|_{H^p}$-"norms" respectively. For $1\leq p<\infty$, these spaces are Banach spaces with the norms, especially, if p=2, they are Hilbert spaces with obvious inner products. Moreover, it is easy to see that when $r \leq -\frac{1}{2}$, $A_{p,r}(D)=\{0\}$, and each $\|F\|_{H^p}$ is also given by

$$\|F\|_{H^p} = \sup_{0<r<1} \left(\frac{1}{\pi}\int_0^{2\pi} |F(re^{i\theta})|^p d\theta\right)^{1/p}. \tag{2.5}$$

For $r > -\frac{1}{2}$ we put

$$e_{r+1}^{\ell}(z)=B(\ell+1,2r+1)z^{\ell} \quad (\ell \in \mathbf{N}). \tag{2.6}$$

Then $\{ e_{r+1}^{\ell} \,;\, \ell \in \mathbf{N} \}$ is a complete orthonormal basis of $A_{p,r}(D)$.

2.3. Norm-preserving operators. The group G=SU(1,1) acts transitively on D by the linear transformation $g\cdot z=(\alpha z+\beta)/(\bar{\beta}z+\bar{\alpha})$ ($z \in D$ and $g=\begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix} \in G$) and the isotropy subgroup at z=0 is equal to K, so we have the identification D=G/K. Here we put $J(g,z)=\bar{\beta}z+\bar{\alpha}$, and define the operator $T_{p,r}(g)$ ($0<p<\infty$, $r \in \mathbf{R}$ and $g \in G$) as follows.

$$T_{p,r}(g)F(z)=J(g^{-1},z)^{-4(1+r)/p}F(g^{-1}\cdot z) \quad (z \in D). \tag{2.7}$$

Then it is easy to see that each operator $T_{p,r}(g)$ preserves $\|\cdot\|_{p,r}$ and $\|\cdot\|_{H^p}$-"norms", and moreover, $(T_{p,r},A_{p,r}(D))$ and $(T_{p,r},H^p(D))$ are irreducible representations of G with respect to the topologies induced by the "norms".

§3. <u>Discrete series of</u> G. If $0<p<\infty$ and $r \in \mathbf{R}$ satisfy the relation: $4(1+r)/p=2n$, we put $T_n=T_{p,r}$. Then $(T_n,A_{2,n-1}(D))$ ($n \in \frac{1}{2}Z$ and $n \geq 1$) are nothing but the holomorphic discrete series representation of G. Let $f_{\ell m}^n(g)$ ($g \in G$ and $\ell, m \in \mathbf{N}$) denote the normalized matrix coefficients of T_n defiend by

$$f_{\ell m}^n(g)=[T_n(g)e_n^m,e_n^{\ell}]_{n-1}/\|[T_n(\cdot)e_n^m,e_n^{\ell}]_{n-1}\|_2, \tag{3.1}$$

where $[\ ,\ ]_r$ means the inner product of $A_{2,r}(D)$. Actually, by using hypergeometric function we can give the explicit form of $f^n_{\ell m}$: for example, if $\ell \geqq m$,

$$f^n_{\ell m}(g)=\frac{1}{2}\left(\frac{2n-1}{\pi}\right)^{1/2}\left(\frac{\Gamma(\ell+1)\Gamma(\ell+2n)}{\Gamma(m+1)\Gamma(m+2n)}\right)^{1/2}\frac{1}{\Gamma(\ell-m+1)}(1-r^2)^n \qquad (3.2)$$
$$\times F(-m;2n+\ell,\ell-m+1;r^2)r^{\ell-m}e^{-i\ell\theta}e^{-im\theta'}e^{-in(\theta+\theta')},$$

where $g=k_\theta a_t k_{\theta'} \in KCL(A^+)K$ and $r=\mathrm{th}\, t/2$ (cf. [Sa]). Some basic properties of $f^n_{\ell m}$, which will be used in the following arguments, are summalized as follows.

'emma 3.1. *Let $n \in \frac{1}{2}Z$, $n \geqq 1$ and ℓ, $m \in \boldsymbol{N}$.*

(1) $\| f^n_{\ell m}\|_2=1$.

(2) $f^n_{\ell m}(k_\theta g k_{\theta'})=e^{-i(\ell+n)\theta}e^{-i(m+n)\theta'}f(g) \quad (k_\theta, k_{\theta'} \in K, g \in G)$.

(3) $T_n(g)e^m_n=2(\frac{\pi}{2n-1})^{1/2}\sum_{\ell=0}^{\infty} f^n_{\ell m}(g)e^\ell_n$.

(4) $\frac{1}{4\pi}\int_0^{4\pi} e^{i(u+n)\theta}f^n_{\ell m}(xk_\theta y)d\theta=2(\frac{\pi}{2n-1})^{1/2}f^n_{\ell u}(x)f^n_{um}(y) \quad (u \in \boldsymbol{N}, x, y \in G)$.

(5) $f^n_{\ell m}*f^w_{uv}=\delta_{nw}\delta_{mu}2(\frac{\pi}{2n-1})f^n_{\ell v} \quad (w \in \frac{1}{2}Z, w \geqq 1 \text{ and } u, v \in \boldsymbol{N})$.

(6) $f^n_{\ell m}$ is the eigenfunction of the Laplace-Beltrami operator of G with the eigenvalue $4n(n-1)$.

(7) Let $1 \leqq p \leqq \infty$ and $(n,p)\neq(1,1)$. Then $\| f^n_{\ell m}\|_p < \infty$.

By the Plancherel formula for $L^2(G)$, each L^2 function f on G has the following decomposition:

$$f= f_p + {}^\circ f, \quad {}^\circ f= \sum_{\substack{n\in\frac{1}{2}Z, |n|\geqq 1\\ \ell, m\in\boldsymbol{N}}} \hat{f}(n;\ell,m)f^n_{\ell m}, \qquad (3.3)$$

where f_p consists of wave packets and $f^n_{\ell m}=\mathrm{conj}(f^{-n}_{\ell m})$ for $n \leqq -1$. Clearly this decomposition for $L^2(G) \cap L^p(G)$ $(1\leqq p\leqq 2)$ can be extended to $L^p(G)$. Then the next lemma is an easy consequence from Lemma 3.1 (5), (6), (7) and (3.3).

Lemma 3.2. *Let us suppose that f belongs to* $L^p(G)$ $(1 \leq p \leq 2)$. *Then for* $n \in \frac{1}{2}Z$, $n \geqq 1$ *and* ℓ, $m \in N$ *we have*

(1) $f_p * f^n_{\ell m} = 0.$

(2) $f * f^n_{\ell m} = c_n \sum_{u=0}^{\infty} \hat{f}(n;u,m) f^n_{um}$, *where* $c_n = 2\left(\frac{\pi}{2n-1}\right)^{\frac{1}{2}}$.

Therefore, if we put $P^n_m(f) = c_n^{-1} f * f^n_{mm}$, P^n_m is the projection operator which maps f in $L^p(G)$ to the discrete part of f being of the form $\sum_{u=0}^{\infty} \hat{f}(n;u,m) f^n_{um}$. One of the important properties of P^n_m is the following

Proposition 3.3. *Let* $n \in \frac{1}{2}Z$, $n \geqq 1$, $m \in N$, $1 \leq p \leq 2$ *and suppose that* $(n,p) \neq (1,1)$. *Then* $P^n_m(L^p(G))$ *is contained in* $L^p(G)$.

Proof. We shall use Lemma 3.1 (7). Let f be in $L^p(G)$. If $n > 1$ and $1 \leq p \leq 2$, then $\| P^n_m(f) \|_p = c_n^{-1} \| f * f^n_{mm} \|_p \leq c_n^{-1} \| f^n_{mm} \|_1 \| f \|_p < \infty$, and if $n=1$ and $1 < p \leq 2$, $\| P^n_m(f) \|_p \leq C_p \| f \|_p < \infty$ by the Kunze-Stein phenomenon on G (see [C] and [CS]). Q.E.D.

4. Φ^n_m-transform. Let $n \in \frac{1}{2}Z$, $n \geqq 1$, $m \in N$ and $1 \leq p \leq 2$. For an f in $L^p(G)$ we shall define $\Phi^n_m(f)$ as follows.

$$\Phi^n_m(f)(z) = c_n^{-1} T_n(\bar{f}) e^m_n(z) \quad (z \in D), \tag{4.1}$$

where $T_n(f)$ is the operator defined by $T_n(f) = \int_G f(g) T_n(g) dg$. By using Lemmas 3.1 and 3.2 we easily see the following

Lemma 4.1. $\Phi^n_m(L^p(G)) = \Phi^n_m(P^n_m(L^p(G)))$ *and* $\Phi^n_m(\Sigma a_\ell f^n_{\ell m}) = \Sigma \bar{a}_\ell e^\ell_n$ *for a finite sum* Σ. *In particular,* Φ^n_m *is injective on* $P^n_m(L^p(G))$.

Our aim is to give a characterization of $\Phi^n_m(L^p(G))$ as a space of holomorphic functions on D. First we shall prove the independence of m for $\Phi^n_m(L^p(G))$.

Theorem 4.2. *Let $n \in \frac{1}{2}Z$, $n \geqq 1$, $m \in \boldsymbol{N}$ and $1 \leqq p \leqq 2$. Then $\Phi^n_m(L^p(G))=\Phi^n_0(L^p(G))$.*

Proof. Obviously, it is enough to prove that the correspondence of $f=\sum_{\ell=0}^{\infty} a_\ell f^n_{\ell m}$ to $\tilde{f}=\sum_{\ell=0}^{\infty} a_\ell f^n_{\ell 0}$ gives a bijection between $P^n_m(L^p(G))_\infty$ and $P^n_0(L^p(G))$.
The case of $(n,p)\neq(1,1)$. Let us suppose that $f=\sum_{\ell=0}^{\infty} a_\ell f^n_{\ell m}$ belongs to $P^n_m(L^p(G))$. Then by Proposition 3.3 f also belongs to $L^p(G)$, and it follows from the same argument in the proof of Proposition 3.3 that $\|\tilde{f}\|_p = d\| f * f^n_{m0}\|_p \leqq c_{p,n}\| f\|_p < \infty$. Therefore, we see that $\tilde{f}$ belongs to $P^n_0(L^p(G))$. Clearly, this argument is reversible by noting that $f=c\tilde{f}*f^n_{0m}$. Thus the desired result follows from Lemma 4.1.

Before the proof of the case of $(n,p)=(1,1)$ we shall prove a lemma, which will play an important role in §5.

Lemma 4.3. *For each $f^1_{\ell m}$ there exists an L^1 function $[f^1_{\ell m}]$ such that $f^1_{\ell m}=[f^1_{\ell m}]*f^1_{mm}$.*

Proof. For a fixed $\alpha > 0$ we put

$$[f^1_{\ell m}](g)=C^\alpha_{\ell m}|f^1_{00}(g)|^\alpha f^1_{\ell m}(g) \quad (g \in G),$$

where $C^\alpha_{\ell m}$ is the constant determined by

$$C^\alpha_{\ell m}c_1 \int_G |f^1_{00}(g)|^\alpha |f^1_{\ell m}(g)|^2 dg=1. \tag{4.2}$$

Then it is easy to see that $[f^1_{\ell m}]$ belongs to $L^1(G)$ and moreover,

$$[f^1_{\ell m}]*f^1_{mm}(x)$$

$$= \int_G [f^1_{\ell m}](xy^{-1})f^1_{mm}(y)dy$$

$$= \int_G [f^1_{\ell m}](y^{-1}) \int_K e^{i(n+\ell)\theta} f^1_{mm}(yk_\theta x)dkdy$$

$$= c_1 \int_G [f^1_{\ell m}](y^{-1})\overline{f^1_{\ell m}}(y)dy f^1_{\ell m}(x)$$

$$= c_1 C^\alpha_{\ell m} \int_G |f^1_{00}(y)|^\alpha |f^1_{\ell m}(y)|^2 dy f^1_{\ell m}(x)$$

$$= f^1_{\ell m}(x).$$

Therefore, $[f^1_{\ell m}]$ satisfies the desired properties. Q.E.D.

The case of $(n,p)=(1,1)$. Let us suppose that $f=\sum_{\ell=0}^{\infty} a_\ell f^1_{\ell m}$ belongs to $P^1_m(L^1(G))$. This means that there exists an L^1 function $[f]$ on G such that $f=[f]*f^1_{mm}$. Therefore, $\tilde{f}=c_1^{-1}f*f^1_{m0}=[f]*f^1_{m0}$. Then by Lemma 4.3 we can choose an L^1 function $[f^1_{m0}]$ on G such that $f^1_{m0}=[f^1_{m0}]*f^1_{00}$, and thus, $\tilde{f}= [f]*[f^1_{m0}]*f^1_{00}$. Here we note that $[f]*[f^1_{m0}]$ belongs to $L^1(G)$. Therefore, $\tilde{f}$ belongs to $P^1_0(L^1(G))$. Clearly, this argument is reversible as before, so we can obtain the desired result.

This completes the proof of the theorem. Q.E.D.

§5. <u>Characterization of</u> $\Phi^n_0(L^p(G))$. Let $n \in \frac{1}{2}Z$, $n \geq 1$ and $1\leq p\leq 2$. By Theorem 4.2 our problem is reduced to the case of m=0. We shall give a characterization of $\Phi^n_0(L^p(G))$.

First we shall consider the case of $(n,p)\neq(1,1)$. Let us suppose that $f=\sum_{\ell=0}^{\infty} a_\ell f^n_{\ell 0}$ belongs to $P^n_0(L^p(G))$ and we put $F(z)=\Phi^n_0(f)(z)=\sum_{\ell=0}^{\infty}\bar{a}_\ell e^\ell_n$. Then we see that

$$\begin{aligned} f(g) &= \frac{1}{2}\pi^{-\frac{1}{2}}(1-r^2)^n \sum_{\ell=0}^{\infty} a_\ell B(\ell+1,2n-1)^{-\frac{1}{2}} r^\ell e^{-i\ell\theta} e^{-in(\theta+\theta')} \\ &= \frac{1}{2}\pi^{-\frac{1}{2}}(1-r^2)^n \overline{F}(re^{i\theta})e^{-in(\theta+\theta')}, \end{aligned} \tag{5.1}$$

where $g=k_\theta a_t k_{\theta'} \in KCL(A^+)K$ and $r=\mathrm{th}\, t/2$. Since f belongs to $L^p(G)$ by Proposition 3.3, this relation deduces that $\|f\|_p=c\|F\|_{p,\frac{1}{2}np-1} < \infty$. This means that when $(n,p)\neq(1,1)$, f belongs to $P_0^n(L^p(G))$ if and only if $F=\Phi_0^n(f)$ belongs to $A_{p,\frac{1}{2}np-1}(D)$, and Φ_0^n preserves the norm up to a constant multiplication.

Next we shall consider the case of $(n,p)=(1,1)$. For a holomorphic function $F(z)=\sum_{\ell=0}^{\infty}a_\ell z^\ell$ on D the fractional derivative (resp. integral) of F of order $\alpha \geqq 0$ is defined as follows.

$$F^{[\alpha]}(z)=\sum_{\ell=0}^{\infty} a_\ell \frac{\Gamma(\ell+1+\alpha)}{\Gamma(\ell+1)} z^\ell$$
$$\text{(resp. } F_{[\alpha]}(z)=\sum_{\ell=0}^{\infty} a_\ell \frac{\Gamma(\ell+1)}{\Gamma(\ell+1+\alpha)} z^\ell). \tag{5.2}$$

By using this derivative we shall define

$$\| F\|_{H_0^1}= \lim_{\alpha\to 0} \alpha\| F^{[\alpha]}\|_{1,\frac{1}{2}\alpha-\frac{1}{2}}. \tag{5.3}$$

Then the space $H_0^1(D)$ is defined as the space of all holomorphic functions on D with finite $\|\cdot\|_{H_0^1}$-norm.

Lemma 5.1. *For each $g \in G$ $T_1(g)e_1^0(z)$ belongs to $H_0^1(D)$ and the norm is uniformly bounded on $g \in G$.*

Proof. From the definition (2.7) it is easy to see that

$$z^{-1}(zT_1(g)e_1^0(z))^{[\alpha]}= \frac{\Gamma(\alpha+2)(1-r^2)e^{-i(\theta+\theta')}}{(1-re^{i\theta}z)^{\alpha+2}} ,$$

where $g=k_\theta a_t k_{\theta_1'} \in KCL(A^+)K$ and $r=\mathrm{th}\, t/2$. Then we have $\| (T_1(g)e_1^0)^{[\alpha]}\|_{1,\frac{1}{2}\alpha-\frac{1}{2}} \leqq C\Gamma(\alpha+2)(1+\alpha^{-1})$, where C does not depend on $g \in G$. Thus the desired result is obtained. Q.E.D.

Now we shall give a characterization of $\Phi_0^1(L^1(G))$. First let us suppose that f belongs to $L^1(G)$. Since $\Phi_0^1(f)(z)=c_1^{-1}(T_1(\cdot)e_1^0(z),f)$, it follows from Lemma 5.1 that $\| \Phi_0^1(f)\|_{H_0^1} \leqq c_1^{-1}(\| T_1(\cdot)e_1^0\|_{H_0^1}, |f|) \leqq c_1^{-1}c\| f\|_1 < \infty$.

This means that $\Phi_0^1(f)$ belongs to $H_0^1(D)$ for all $f \in L^1(G)$.

Conversely let us suppose that $F(z)=\sum_{\ell=0}^{\infty}\bar{a}_\ell e_1^\ell(z)$ belongs to $H_0^1(D)$. Here we put $f(g)=\sum_{\ell=0}^{\infty}a_\ell f_{\ell 0}^1(g)$, and let $[f_{\ell 0}^1]$ ($\ell \in \mathbf{N}$) be the L^1 functions on G constructed in the proof of Lemma 4.3. Actually, for a fixed $\alpha > 0$ they are given by

$$[f_{\ell 0}^1](g)=C_{\ell 0}^{\alpha}|f_{00}^1(g)|^{\alpha}f_{\ell 0}^1(g) \quad (g \in G),$$

where

$$C_{\ell 0}^{\alpha}=\frac{2^{\alpha}\pi^{\frac{1}{2}(1+\alpha)}\Gamma(\ell+2+\alpha)}{\Gamma(\alpha+1)\Gamma(\ell+2)}.$$

Here we put $[f]=\sum_{\ell=0}^{\infty}a_\ell[f_{\ell 0}^1]$. Then it follows from Lemmas 4.1 and 4.3 that $c_n\cdot\Phi_0^1([f])=\Phi_0^1(f)=F$, and moreover,

$$\|[f]\|_1=\frac{2^{\alpha}\pi^{\frac{1}{2}(1+\alpha)}}{\Gamma(\alpha+1)}\left\| |f_{00}^1|^{\alpha}\sum_{\ell=0}^{\infty}a_\ell\frac{\Gamma(\ell+2+\alpha)}{\Gamma(\ell+2)}f_{\ell 0}^1\right\|_1$$
$$=\frac{2\pi}{\Gamma(\alpha+1)}\|z^{-1}(zF)^{[\alpha]}\|_{1,\frac{1}{2}\alpha-\frac{1}{2}}.$$

Obviously, this is finite, because F belongs to $H_0^1(D)$. Therefore, we see that F belongs to $\Phi_0^1(L^1(G))$.

Summalizing the results we just obtained, we have the following

Theorem 5.2. *Let $n \in \frac{1}{2}\mathbf{Z}$, $n \geq 1$ and $1 \leq p \leq 2$.*

(1) *If $(n,p)\neq(1,1)$, then $\Phi_0^n(L^p(G))=A_{p,\frac{1}{2}np-1}(D)$ and $\Phi_0^n : P_0^n(L^p(G)) \to A_{p,\frac{1}{2}np-1}(D)$ is bijective and norm-preserving.*

(2) *If $(n,p)=(1,1)$, then $\Phi_0^1(L^1(G))=H_0^1(D)$ and $\Phi_0^1 : P_0^1(L^1(G)) \to H_0^1(D)$ is bijective.*

Remark 5.3. Noting the proof in the case of $(n,p)=(1,1)$, we see that if $\|F\|_{1,\frac{1}{2}\alpha-\frac{1}{2}} < \infty$ for an $\alpha > 0$, then $\|F\|_{1,\frac{1}{2}\alpha-\frac{1}{2}} < \infty$ for all $\alpha > 0$ and moreover $\|F\|_{H_0^1} < \infty$. Therefore, it is easy to see that for each $\alpha > 0$ $H_0^1(D)$ is also defined as the space of all holomorphic functions F on D such that $F^{[\alpha]}$ belongs to $A_{1,\frac{1}{2}\alpha-\frac{1}{2}}(D)$.

<u>Corollary</u> 5.4. *$L^1(G) \cap P_0^1(L^1(G)) = \{0\}$.*

Proof. If $f \in L^1(G)$ belongs to $P_0^1(L^1(G))$, then $\|f\|_1 = 2\pi^{\frac{1}{2}}\|\Phi_0^1(f)\|_{1,-\frac{1}{2}} < \infty$. Obviously, this means that $\Phi_0^1(f) \equiv 0$ and thus $f \equiv 0$. Q.E.D.

§6. <u>The space</u> $H_0^1(D)$. In this section we shall state some basic properties of the space $H_0^1(D)$ introduced in the previous section.

<u>Proposition</u> 6.1.

(1) $H_0^1(D)$ is dense in $H^1(D)$.

(2) $H_0^1(D)$ is not contained in and does not contain $H^\infty(D)$. In particular $H_0^1(D) \subsetneqq H^1(D)$.

Proof. Let f be in $L^1(G)$. Then $\|\Phi_0^1(f)\|_{H^1} \leq (\|T_1(\cdot)e_1^0\|_{H^1}, |f|) \leq C\|f\|_1 < \infty$, and thus, $H_0^1(D) = \Phi_0^1(L^1(G)) \subset H^1(D)$ (see Theorem 5.2 (2)). Here we note that $\Phi_0^1(L^1(G))$ is G-invariant under the action of $T_1(g)$ $(g \in G)$. Therefore, since $(T_1, H^1(D))$ is irreducible, $H_0^1(D)$ must be dense in $H^1(D)$. This proves (1).

Let F be an extremal function in $H^\infty(D)$. Then $|F'(z)| = (1-r^2)^{-1}\|F\|_\infty$ (cf. [D], p.144, Ex.7). Therefore, F' does not belong to $A_{1,0}(D)$. This means that F does not belong to $H_0^1(D)$ (see Remark 5.3), and thus, $H_0^1(D)$ does not contain $H^\infty(D)$. Let $\mathfrak{D}$ be the space of all holomorphic functions F on D such that $\int_D |F'(z)|^2 dz < \infty$. Then by using Schwarz's inequality and Remark 5.3 we see that $\mathfrak{D}$ is contained in $H_0^1(D)$. However, it is well-known that $\mathfrak{D}$ is not contained in $H^\infty(D)$ (cf. [D], p.106, Ex.7). Thus $H_0^1(D)$ is not contained in $H^\infty(D)$. Moreover, since $H^\infty(D)$ is contained in $H^1(D)$, the last assertion is obvious. Q.E.D.

Proposition 6.2. *Let us suppose that F belongs to $H^1(D)$. Then for each $\alpha > 0$ $F_{[\alpha]}$ belongs to $H^1_0(D)$.*

Proof. We easily see that $\| (F_{[\alpha]})^{[\alpha]} \|_{1,\frac{1}{2}\alpha-\frac{1}{2}} = \| F \|_{1,\frac{1}{2}\alpha-\frac{1}{2}} < \infty$ for each $\alpha > 0$ and F in $H^1(D)$. Then the assertion is clear from Remark 5.3. Q.E.D.

For $\mu \in R$ and $a \in L^1(\partial D)$ we put

$$P_\mu(a)(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r^2)}{|1-re^{i(\theta-\phi)}|^{2(1+\mu)}} a(\phi) d\phi \qquad (z = re^{i\theta}),$$

and let $H^1_{(\mu)}(D)$ be the space of all holomorphic functions on D being of the form $P_\mu(a)$ for an a in $L^1(\partial D)$. Then $H^1(D) = H^1_{(0)}(D)$ by [D], Corollary 2 in p. 34, and moreover we have the following

Proposition 6.3. *For each $\mu < 0$ $H^1_{(\mu)}(D) \subset H^1_0(D) \subseteq H^1(D)$.*

Proof. Let us suppose that F belongs to $H^1_{(\mu)}(D)$, that is, $F = P_\mu(a)$ for an a in $L^1(\partial D)$. Without loss of generality we may assume that $-\frac{1}{2} < \mu < 0$. By using the integral formula for the fractional derivative (cf. [K], (9.1)), $F^{[\varepsilon]}$ can be written as

$$F^{[\varepsilon]}(z) = \frac{1}{\Gamma(1-\varepsilon)} \int_D F(\zeta) |\zeta|^{2\varepsilon} (1-|\zeta|^2)^{-\varepsilon} (1-\bar{\zeta}z)^{-2} d\zeta.$$

Now we shall take an $\varepsilon > 0$ such that $0 < \varepsilon < -2\mu$. Then by using Theorem A in [DRS] we see that

$$\begin{aligned}
&\| F^{[\varepsilon]} \|_{1,\frac{1}{2}\varepsilon-\frac{1}{2}} \\
&\leqq \frac{1}{\Gamma(1-\varepsilon)} \frac{1}{2\pi} \| a \|_1 \sup_{0 \leqq \phi < 2\pi} \int_D \frac{(1-|\zeta|^2)^{1-\varepsilon}}{|1-\zeta e^{i\phi}|^{2(1+\mu)}} \int_D \frac{(1-|z|^2)^{\varepsilon-1}}{|1-\bar{\zeta}z|^2} dz d\zeta \\
&\leqq C \| a \|_1 \int_0^1 (1-\eta)^{1-\varepsilon-2(1+\mu)} d\eta \int_0^1 (1-r)^{\varepsilon-1} dr \\
&< \infty,
\end{aligned}$$

where $|\zeta|=\eta$ and $|z|=r$. Then the desired result is obvious from Remark 5.3.

Q.E.D.

§7. Applications. We shall apply the characterization of $\Phi_m^n(L^p(G))$ to the classical harmonic analysis on D. The following theorem, which is obtained in [DRS], Theorem 5, is an easy consequence of Theorem 5.2 and Remark 5.3.

Theorem 7.1. *Let* $r > -\frac{1}{2}$ *and* $\beta \geqq 0$. *Then*

(1) *If F belongs to* $A_{1,r}(D)$, *then* $F^{[\beta]}$ *belongs to* $A_{1,r+\frac{1}{2}\beta}(D)$.

(2) *If F belongs to* $A_{1,r+\frac{1}{2}\beta}(D)$, *then* $F_{[\beta]}$ *belongs to* $A_{1,r}(D)$.

Moreover, if we replace $A_{1,-\frac{1}{2}}(D)=\{0\}$ *by the space* $H_0^1(D)$, *the assertions are also valid for the case of* $r=-\frac{1}{2}$.

To the detail of the proof see [K].

Next we shall recall the explicit form of the matrix coefficients $f_{\ell m}^n$ (see (3.2)). Let $G_m(\alpha;\gamma;x)$ be the Jacobi polynomial with degree m. Then since $G_m(\ell-m+2n;\ell-m+1;1)=(-1)^m\Gamma(2n+m)\Gamma(\ell-m+1)/\Gamma(2n)\Gamma(\ell+1)$, it is easy to see that $G_m(\ell-m+2n;\ell-m+1;r)$ has the expansion being of the form

$$G_m(\ell-m+2n;\ell-m+1;r)$$
$$=(-1)^m \frac{\Gamma(2n+m)\Gamma(\ell-m+1)}{\Gamma(2n)\Gamma(\ell+1)}r^m$$
$$+\frac{\Gamma(\ell-m+1)}{\Gamma(\ell)}(1-r)(Q_{m,m-1}^n(\ell)(1-r)^{m-1}+Q_{m,m-2}^n(\ell)(1-r)^{m-2}+\ldots+Q_{m,0}^n),$$

where $Q_{m,k}^n$ is a polynomial of ℓ with degree k whose coefficients only depend on n, m and k. Therefore, since $F(-m;2n+\ell,\ell-m+1;r)=G_m(\ell-m+2n;\ell-m+1;r)$, $f_{\ell m}^n(g)$

($\ell \geqq m$, $g=k_\theta a_t k_{\theta'}$ and $r=\operatorname{th} t/2$) has the expansion:

$$f^n_{\ell m}(g)=C^n_m f^n_{\ell 0}(g)r^{-m}\ \{r^{2m}+(1-r^2)P^n_{m,m}(\ell) + \ldots +(1-r^2)P^n_{n,1}(\ell)\}\ e^{-m\theta'}, \tag{7.1}$$

where $C^n_m=(-1)^m(\Gamma(m+2n)/\Gamma(m+1)\Gamma(2n))^{\frac{1}{2}}$ and $P^n_{m,k}(\ell)=(-1)^m\frac{\Gamma(2n)}{\Gamma(2n+m)}\,\ell\, Q^n_{m,k}(\ell)$. By using these polynomials $P^n_{m,k}$, we shall define a differential operator $D^n_{m,k}$ as follows. For a holomorphic function $F(z)=\sum_{\ell=0}^{\infty}a_\ell z^\ell$

$$D^n_{m,k}F(z)=\sum_{\ell=0}^{\infty}P^n_{m,k}(\ell)a_\ell z^\ell. \tag{7.2}$$

Moreover, we put

$$D^n_m F(z)=C^n_m\ \{F_{(m)}|z|^m+(1-|z|^2)^m D^n_{m,m}F_{(m)}(z)|z|^{-m} \tag{7.3}$$
$$+(1-|z|^2)^{m-1}D^n_{m,m-1}F_{(m)}(z)|z|^{-m}+\ldots+(1-|z|^2)^{-1}D^n_{m,1}F_{(m)}(z)|z$$

where $F_{(m)}(z)=\sum_{\ell=m}^{\infty}a_\ell z^\ell$.

Now we shall recall Proposition 4.2 and its proof. Then we see that $F(z)=\sum_{\ell=0}^{\infty}a_\ell e^\ell_n(z)$ belongs to $\Phi^n_0(L^p(G))$ if and only if $\sum_{\ell=0}^{\infty}a_\ell f^n_{\ell m}$ belongs to $P^n_m(L^p(G))$, and thus by (7.1),(7.3), if and only if $\| D^n_m F\|_{p,\frac{1}{2}np-1} < \infty$ when $(n,p)\neq(1,1)$, and $\| D^n_m F^{[\alpha]}\|_{1,\frac{1}{2}\alpha-\frac{1}{2}} < \infty$ for an $\alpha > 0$ when $(n,p)=(1,1)$. Here let $A_{p,\frac{1}{2}np-1}(D)_m$ (resp. $H^1_0(D)_m$) be the space of all holomorphic functions F on D such that $\| D^n_m F\|_{p,\frac{1}{2}np-1} < \infty$ (resp. $\| D^n_m F^{[\alpha]}\|_{1,\frac{1}{2}\alpha-\frac{1}{2}} < \infty$ for an $\alpha > 0$). Then we have the following

Theorem 7.2. *Let $n \in \frac{1}{2}Z$, $n \geqq 1$ and $1\leqq p\leqq 2$.*

(1) If $(n,p)\neq(1,1)$, then $A_{p,\frac{1}{2}np-1}(D)_m=A_{p,\frac{1}{2}np-1}(D)$ for $m \in N$.

(2) If $(n,p)=(1,1)$, then $H^1_0(D)_m=H^1_0(D)$ for $m \in N$.

Corollary 7.3. *Let us suppose that* $(n,p)\neq(1,1)$. *Then if* F *belongs to* $A_{p,\frac{1}{2}np-1}(D)$, *then* F' *belongs to* $A_{p,\frac{1}{2}(n+1)p-1}(D)$.

Proof. This is an easy consequence from the case of m=1 in Theorem 7.2.

Q.E.D.

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