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Global asymptotics of the outer pressure problem of free type

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ABSTRACT

Global asymptotics on the outer pressure problem of free type is discussed. The solution of the problem exists globally (Theorem 1), and some global asymptotics are investigated (Theorem 2). Under some additional assumptions, the solution converges to a stationary state (Theorem 3) with exponential rate (Theorem 4).

Key Words: outer pressure problem, one-dimensional polytropic ideal gas, global solution, asymptotic behavior of solution, convergence of solution

1. Introduction and results.

We consider the outer pressure problem of free type for the one-dimensional motion of the polytropic ideal gas. Find the functions (u,v,θ) satisfying the system of equations

$$u_t = v_x, \tag{1.1}$$

$$v_t = \left(-R\frac{\theta}{u} + \mu\frac{v_x}{u}\right)_x, \tag{1.2}$$

$$c_V\theta_t = \left(-R\frac{\theta}{u} + \mu\frac{v_x}{u}\right)v_x + \kappa\left(\frac{\theta_x}{u}\right)_x, \tag{1.3}$$

for $(x,t) \in [0,1] \times \mathbf{R}_+$, the initial condition

$$(u,v,\theta)(x,0) = (u_0,v_0,\theta_0)(x), \quad u_0 > 0, \quad \theta_0 > 0, \tag{1.4}$$

and the boundary conditions

$$v(0,t) = 0, \tag{1.5}$$

$$\left(-R\frac{\theta}{u} + \mu\frac{v_x}{u}\right)(1,t) = -P(l(t)), \tag{1.6}$$

$$\theta_x(0,t) = \theta_x(1,t) = 0, \tag{1.7}$$

where

$$l(t) = \int_0^1 u_0(x)dx + \int_0^t v(1,\tau)d\tau. \tag{1.8}$$

$P(\zeta)$ is the outer pressure. Here the specific volume u, the velocity v and the absolute temperature θ are unknown functions; the outer pressure $P(\zeta)$ is a given function; the gas constant R, the coefficient of viscosity μ, the heat capacity at constant volume c_V and the coefficient of heat conduction κ are positive constants. The subindex t or x indicates the partial differentiation with respect to the variable t or x.

In [2] we discussed the outer pressure problem of different type where the outer pressure P was a given function of t. In this paper, however, P is determined together with the solution. We assume $P(\zeta)$ is a not necessarily positive function in the class of $C^1(0,L)$ $(L \in (l(0),+\infty])$, satisfying

(P1) $$\overline{\lim_{\zeta \uparrow L}}\, \Psi(\zeta) > E_0,$$

and either

(P2) $$\overline{\lim_{\zeta \downarrow 0}}\, \Psi(\zeta) > E_0,$$

or

(P2)′ $$\overline{\lim_{\zeta \downarrow 0}}\, P(\zeta) < +\infty,$$

where

$$\Psi(\zeta) = \int_{l(0)}^{\zeta} P(\eta)d\eta, \tag{1.9}$$

$$E_0 = \int_0^1 \left(\frac{1}{2}v_0^2 + c_V\theta_0\right)(x)dx. \tag{1.10}$$

Physically, $P(\zeta)$ is a function like

$$P(\zeta) = \text{const.} > 0, \tag{1.11}$$

or

$$P(\zeta) = \frac{\text{const.}}{L-\zeta} > 0, \tag{1.12}$$

where in latter, L is a positive constant. In these cases we can easily check the assumption (P1) and (P2)′.

Firstly, in Section 2, under the above assumption, we shortly discuss the existence of the solution of (1.1) - (1.7) globally in time. For function spaces $H^{n+\alpha}$, $H_T^{n+\alpha}$ and $B_T^{n+\alpha}$, we should refer to [1, Eq. (2.2) - (2.6)].

Theorem 1. *Assume that the initial data which are compatible with* (1.5) - (1.7) *belong to* $u_0 \in H^{1+\alpha}$, $v_0 \in H^{2+\alpha}$, $\theta_0 \in H^{2+\alpha}$ *for some* $\alpha \in (0,1)$ *with* $u_0 > 0$, $\theta_0 > 0$ *and that the function* $P(\zeta)$ *satisfies* (P1) *and either* (P2) *or* (P2)′. *Then there exists a temporally global and unique solution* (u,v,θ) *for the problem* (1.1) - (1.7) *such that it belongs to* $B_T^{1+\alpha} \times H_T^{2+\alpha} \times H_T^{2+\alpha}$ *with* $u > 0$, $\theta > 0$ *and has generalized derivatives* u_{tt}, v_{xt}, $\theta_{xt} \in L^2((0,1)\times(0,T))$ *for any* $T > 0$.

The sequel of studies is an asymptotic analysis of the solution. In Section 3, we can establish some asymptotic properties of the solution.

Theorem 2. *Assume that the hypotheses of Theorem 1 hold. Then there exists positive constant* C *which depends on* R, μ, c_V, κ, $P(\zeta)$ *and initial data but not on* t *such that*

$$C^{-1} \le u, \theta \le C,$$

$$\int_0^t \left[\max_{x\in[0,1]} v^2(x,\tau) + \int_0^1 \left\{ u_x^2 + v_x^2 + \theta_x^2 + v_{xx}^2 + \theta_{xx}^2 + \left(R\frac{\theta}{u} - P(l(\tau)) \right)^2 \right\} dx \right] d\tau \le C,$$

$$\lim_{t\to+\infty} \int_0^1 (v^2 + u_x^2 + v_x^2 + \theta_x^2) dx = 0,$$

$$\lim_{t\to+\infty} (P(l(t))u(x,t) - R\theta(x,t)) = 0 \quad \text{uniformly in } x \in [0,1]$$

hold.

For the proof of this theorem we use similar techniques to those in [2], plus some new ones. To avoid a routine repetition, we shall only point out differences between our case and [2]. For details we should refer to [2].

Theorem 2 implies that v converges to zero in $W^{1,2}(0,1)$ as $t \to +\infty$, but we cannot clarify if u and θ converge to some constants or not. [2, Theorem 2] shows that, roughly speaking, if $\lim_{t\to+\infty} P(t)$ exists, then (u,v,θ) converges to a stationary state. In our case, the convergence of $P(l(t))$ is almost equivalent to the convergence of u (see (2.1)). Under the additional assumption, however, we can show the convergence of u and θ. Let Z be a set

$$Z = \{\zeta \in (0,L) \,;\, c_V K \zeta^{-\frac{R}{c_V}} + \Psi(\zeta) = E_0\}$$

where

$$K = \int_0^1 (u_0(x))^{\frac{R}{c_V}} \theta_0(x) dx + \int_0^{+\infty}\int_0^1 \left(\frac{\mu}{c_V} u^{\frac{R}{c_V}-1} v_x^2 - \frac{\kappa R}{c_V^2} u^{\frac{R}{c_V}-2} u_x \theta_x \right) dx d\tau. \tag{1.13}$$

KSTS/RR-86/008
July 27, 1986

The right-hand side of the above equation is meaningful because of Theorem 2. The meaning of K will be clarified in Lemma 3.6. Assuming that $P(\zeta)$ satisfies

(P3) Z does not contain any intervals,

we get the following theorem.

Theorem 3. *Assume that the hypotheses of Theorem 1 and* (P3) *hold. Then the solution* (u,v,θ) *converges to a stationary state* $(u_\infty,0,\theta_\infty)$ *in* $W^{1,2}(0,1)$ *as* $t\to+\infty$, *where* u_∞ *and* θ_∞ *are the roots of*

$$c_V\theta_\infty + \Psi(u_\infty) = E_0\,, \tag{1.14}$$

$$P(u_\infty)u_\infty = R\theta_\infty\,. \tag{1.15}$$

Moreover

$$K = u_\infty^{\frac{R}{c_V}}\theta_\infty = \exp\left\{\int_0^1 \log\left((u_0(x))^{\frac{R}{c_V}}\theta_0(x)\right)dx + \int_0^{+\infty}\int_0^1\left(\frac{\mu}{c_V}\frac{v_x^2}{u\theta} + \frac{\kappa}{c_V}\frac{\theta_x^2}{u\theta^2}\right)dxd\tau\right\} \tag{1.16}$$

holds.

Though for checking (P3), we must know the explicit value of K, it seems difficult to know it from initial data and P etc., because it may be equal to solve the problem explicitly, so we want to have a criterion not containing K.

Corollary of Theorem 3. *Let* Z' *be a set*

$$Z' = \{\zeta \in (0,L)\;;P(\zeta)\zeta = \frac{R}{c_V}(E_0 - \Psi(\zeta))\}.$$

Assume that the hypotheses of Theorem 1 and

(P3)′ Z' does not contain any intervals,

hold. Then (u,v,θ) *converges to a stationary state* $(u_\infty,0,\theta_\infty)$ *in* $W^{1,2}(0,1)$ *as* $t\to+\infty$.

In the case when $P(\zeta)$ is a function like (1.11) or (1.12), it is easy to check the assumption (P3)′.

In Section 4 we assume still more hypotheses on $P(\zeta)$. Let S, $\underline{s}$ and $\overline{s}_i$ $(i = 1, 2)$ be

$$S = \{\,(s_1,s_2) \in (0,L)\times(0,+\infty)\mid c_Vs_2 + \Psi(s_1) = E_0,\, P(s_1)s_1 = Rs_2\,\},$$

$$\underline{s} = \inf_{(s_1,s_2)\in S}\min\{s_1, s_2\},$$

$$\bar{s}_i = \sup_{(s_1,s_2)\in S} s_i,$$

Remark that S depends on R, c_V, E_0 and $P(\zeta)$ but not on μ, so $\underline{s}$ and $\bar{s}_i$ do not depend on μ either. We assume on $P(\zeta)$ that

$$(\text{P4}) \qquad \begin{cases} \text{(i)} \quad P(u_\infty) > \left|\dfrac{R}{c_V}P(u_\infty) + P'(u_\infty)u_\infty\right|, \\ \text{(ii)} \quad \underline{s} > 0,\ \bar{s}_1 < L,\ \bar{s}_2 < +\infty, \\ \text{(iii)} \quad \mu \text{ is sufficiently large.} \end{cases}$$

Then we can establish the following result concerning the rate of the convergence of the solution.

Theorem 4. *Assume the hypotheses of Theorem 3 and* (P4) *hold. Then there exist positive constants* λ, C *depending on* R, μ, c_V, κ, $P(\zeta)$ *and initial data such that*

$$\int_0^1 \{(u - u_\infty)^2 + v^2 + (\theta - \theta_\infty)^2 + u_x^2 + v_x^2 + \theta_x^2\}dx \le C\exp(-\lambda t).$$

One can readily check that in the case when $P(\zeta)$ is (1.12) and R is smaller than c_V, (P4) is fulfilled for suitable choices of initial data and μ. In thermodynamics,

$$\gamma = \frac{R}{c_V} + 1$$

is called the adiabatic exponent and its value belongs to (1,2) ($\gamma = \frac{5}{3}$ for He, Ar, etc., $\frac{7}{5}$ for O_2, N_2, etc., $\frac{4}{3}$ for CO_2, CH_4, etc.). Thus the assumption $R < c_V$ is meaningful.

Remark. Theorem 4 also holds under the assumption

$$(\text{P4})' \qquad P'(\zeta) = 0 \quad \text{on some open interval containing } u_\infty,$$

instead of (P4). Obviously in the case when $P(\zeta)$ is (1.11), (P4)′ is fulfilled for any initial data.

2. Comments on existence of solution.

To establish an existence theorem, we need a priori estimates of the solution. We assume $u > 0$, $\theta > 0$, $\int_0^1 u\,dx < L$, and introduce notations:

$$\begin{cases} m_u(t) = \min\limits_{x\in[0,1]} u(x,t),\ M_u(t) = \max\limits_{x\in[0,1]} u(x,t), \\ m_\theta(t) = \min\limits_{x\in[0,1]} \theta(x,t),\ M_\theta(t) = \max\limits_{x\in[0,1]} \theta(x,t). \end{cases}$$

Lemma 2.1. *There exist constants* $C \in (1,+\infty)$ *and* $C' \in (0,L)$ *which depend on parameters* R, μ, c_V, κ, $P(\zeta)$ *and initial data but not on* t *such that the estimates*

$$C^{-1} \le l(t) = \int_0^1 u(x,t)dx \le C', \tag{2.1}$$

$$C^{-1} \le \int_0^1 \theta(x,t)dx \le C, \tag{2.2}$$

$$\int_0^1 v^2(x,t)dx \le C, \tag{2.3}$$

$$-C \le P(l(t)), \ \Psi(l(t)) \le C. \tag{2.4}$$

hold.

Proof. Integrating (1.1) over $[0,1] \times [0,t]$ by virtue of the boundary conditions, we have the equality in (2.1).

From adding (1.3) to the equation obtained by multiplying (1.2) by v, and the integrating over $[0,1] \times [0,t]$, it follows

$$\int_0^1 \left(\frac{1}{2}v^2 + c_V\theta\right)(x,t)dx + \Psi(l(t)) = \int_0^1 \left(\frac{1}{2}v_0^2 + c_V\theta_0\right)(x)dx \equiv E_0 > 0 \tag{2.5}$$

by integration by parts, hence we get the boundedness of $\Psi(l(t))$ from above. Therefore by the assumption (P1), we have the boundedness of $l(t)$ $\left(=\int_0^1 u\, dx\right)$ from above. Dividing both sides of (1.3) by θ and integrating over $[0,1] \times [0,t]$, we have

$$\begin{aligned}\int_0^1(c_V\log\theta_0 + R\log u_0)dx &\le \int_0^1(c_V\log\theta + R\log u)dx \\ &\le c_V\log\left\{\int_0^1 \theta\, dx\right\} + R\log\left\{\int_0^1 u\, dx\right\}.\end{aligned} \tag{2.6}$$

Consequently, we have the boundedness of $\int_0^1 \theta\, dx$ from below, *i.e.*,

$$\int_0^1 \theta\, dx \ge \widetilde{C}\left(\int_0^1 u\, dx\right)^{-\frac{R}{c_V}} \ge \widetilde{C}' > 0.$$

Other estimates can be shown as follows. Firstly, we assume (P2). In this case the boundedness of $l(t)$ $\left(= \int_0^1 u\, dx\right)$ from below follows from the boundedness of $\Psi(l(t))$ from above. Therefore (2.4) follows. The boundedness of $\int_0^1(v^2 + \theta)dx$ from above is a consequence of (2.4) and (2.5).

Next we assume (P2)'. The boundedness of $P(l(t))$ from above follows from that of $l(t)$ and (P2)'. Therefore we have

$$\inf_{\zeta < l(0)} \Psi(\zeta) > -\infty.$$

From this and (2.5), $\int_0^1(v^2 + \theta)dx$ is bounded from above. From (2.6) we get

$$l(t) = \int_0^1 u\, dx \ge \widetilde{C}\left(\int_0^1 \theta\, dx\right)^{-\frac{c_V}{R}} \ge \widetilde{C}' > 0.$$

Thus (2.1) is proved, and yields (2.4). □

From now on, the symbol C denotes a generic constant with the same dependence as in Lemma 2.1.

Lemma 2.2. *We have*

$$U(t) + \int_0^t V(\tau)d\tau + \Psi(l(t)) \le C, \tag{2.7}$$

where

$$U(t) = \int_0^1 \left\{ \frac{1}{2}v^2 + R(u - \log u - 1) + c_V(\theta - \log\theta - 1) \right\} dx, \tag{2.8}$$

$$V(t) = \int_0^1 \left(\mu \frac{v_x^2}{u\theta} + \kappa \frac{\theta_x^2}{u\theta^2} \right) dx. \tag{2.9}$$

Proof. From (1.1) - (1.3), (1.5) - (1.7), we have

$$U'(t) + V(t) + \frac{d}{dt}\Psi(l(t)) = Rv(1,t).$$

Integrating this identity over $[0,t]$, we have the assertion by use of (2.1). □

Using above lemmas, a similar procedure to [2] gives us

Lemma 2.3. *We have an expression*

$$u(x,t) = \frac{1}{B(x,t)Y(t)} \left\{ u_0(x) + \int_0^t \frac{R}{\mu} \theta(x,\tau)B(x,\tau)Y(\tau)d\tau \right\}, \tag{2.10}$$

where

$$B(x,t) = \exp\left\{ \frac{1}{\mu} \int_x^1 (v(\xi,t) - v_0(\xi))d\xi \right\}, \tag{2.11}$$

$$Y(t) = \exp\left\{ \frac{1}{\mu} \int_0^t P(l(\tau))d\tau \right\}, \tag{2.12}$$

and estimates

$$C^{-1} \le B(x,t) \le C, \tag{2.13}$$

$$C^{-1} - Ce^{-C^{-1}t} \le \frac{1}{Y(t)} \int_0^t Y(\tau)d\tau \le C, \tag{2.14}$$

$$C^{-1}t - C \le \int_0^t P(l(\tau))d\tau \le C(t+1), \tag{2.15}$$

$$M_u(t) \le C, \tag{2.16}$$

$$M_\theta(t) \le C(1 + V(t)). \tag{2.17}$$

$$m_u(t) \ge C^{-1}e^{-Ct}, \tag{2.18}$$

$$m_\theta(t) \ge C^{-1}\left(1 + \int_0^t \frac{d\tau}{m_u(\tau)}\right)^{-1}, \tag{2.19}$$

hold.

Proof. The expression (2.10) can be got in a similar way to [2, Lemma 3.3]. (2.13) is a consequence of (2.3). To prove (2.14), we integrate both sides of (2.10) with respect to x. With help of (2.1), (2.2) and (2.13), we have

$$\frac{C^{-1}}{Y(t)}\left(1+\int_0^t Y(\tau)d\tau\right) \le 1 \le \frac{C}{Y(t)}\left(1+\int_0^t Y(\tau)d\tau\right). \tag{2.20}$$

Recalling the definition (2.12) of $Y(t)$, we have the right inequality of (2.14).

Multiplying (2.20) by $Y(t)$, applying Gronwall's lemma, we have

$$C^{-1}e^{C^{-1}t} \le Y(t) \le Ce^{C\,t}. \tag{2.21}$$

Taking logarithm, we obtain (2.15). From (2.20) and (2.21) we have the left inequality of (2.14).

(2.16) - (2.19) are obtained in a similar manner to [2, Lemmas 3.4 - 3.5]. □

Above lemmas play essential roles of establishing a priori estimates of the solution, which are in need of extending a local solution, by a similar manner to [2, § 2], while the local existence theorem has been already established by Tani even more general case [3]. Thus we have Theorem 1.

3. Global asymptotics.

It is important for investigating the asymptotic behavior of the solution to establish an estimate

$$\inf_{t\ge 0} m_u(t) > 0.$$

This can be derived from a similar procedure to [2, Lemma 4.2], if $P(l(t))$ is non-negative for large t. However it seems difficult to prove non-negativity of $P(l(t))$ directly. Therefore we need a closer analysis. We begin with proving the following lemma of Massera-Schäffer type. It is a slightly general version of [4, Lemma 4.6], so the proof is quite similar to its.

Lemma 3.1. *Let $\lambda(t)$ (≥ 0) and $\omega(t)$ be continuous functions satisfying that there exist positive constants C_i $(i = 1, \cdots, 4)$ such that*

$$C_1e^{C_2(t-\tau)} \le \exp\left\{\int_\tau^t \omega(s)ds\right\} \le C_3e^{C_4(t-\tau)} \quad \text{for } 0 \le \tau \le t. \tag{3.1}$$

We denote $\Lambda(t)$ by

$$\Lambda(t) = \int_t^{t+1}\lambda(\tau)d\tau.$$

Then for any $T \ge 0$ there exists a positive constant $C(T)$ depending on T, ω and λ, such that

$$\frac{1 - e^{-C_4[t-T]}}{C_3(e - e^{-(C_4+1)})}\inf_{\tau\geq T}\Lambda(\tau) \leq \int_0^t \exp\left\{-\int_\tau^t \omega(s)ds\right\}\lambda(\tau)d\tau$$
$$\leq \frac{1}{C_1(1 - e^{-C_2})}\sup_{\tau\geq T}\Lambda(\tau) + C(T)\exp\left\{-\int_0^t \omega(\tau)d\tau\right\}, \tag{3.2}$$

holds for $t \geq T$. *Here* $[\cdot]$ *is a Gaussian symbol. Especially,*

$$\frac{1}{C_3(e - e^{-(C_4+1)})}\varliminf_{t\to+\infty}\Lambda(t) \leq \varliminf_{t\to+\infty}\int_0^t \exp\left\{-\int_\tau^t \omega(s)ds\right\}\lambda(\tau)d\tau$$
$$\leq \varlimsup_{t\to+\infty}\int_0^t \exp\left\{-\int_\tau^t \omega(s)ds\right\}\lambda(\tau)d\tau \leq \frac{1}{C_1(1 - e^{-C_2})}\varlimsup_{t\to+\infty}\Lambda(t) \tag{3.3}$$

holds.

Proof. Only the right inequality of (3.2) will be shown; other estimates are derived similarly.

$$\int_0^t \exp\left\{-\int_\tau^t \omega(s)ds\right\}\lambda(\tau)d\tau$$
$$\leq \exp\left\{-\int_0^t \omega(s)ds\right\}\int_0^{T+1}\exp\left\{\int_0^\tau \omega(s)ds\right\}\lambda(\tau)d\tau + \sum_{j=0}^{\max\{0,[t-T]-1\}}\int_{t-j-1}^{t-j}\exp\left\{-\int_\tau^t \omega(s)ds\right\}\lambda(\tau)d\tau$$
$$\leq C(T)\exp\left\{-\int_0^t \omega(s)ds\right\} + C_1^{-1}\left(\sup_{t\geq T}\Lambda(t)\right)\sum_{j=0}^{\max\{0,[t-T]-1\}} e^{-C_2 j}.$$

Here we use (3.1). Since $\sum e^{-C_2 j}$ can be majorized by $(1 - e^{-C_2})^{-1}$, the desired estimate follows.

□

Lemma 3.2. $u(x,t)$ *and* $\frac{1}{B(x,t)Y(t)}\int_0^t \frac{R}{\mu}B(x,\tau)Y(\tau)\int_0^1 \theta(\xi,\tau)d\xi d\tau$ *are asymptotically equivalent with respect to sup-norm:*

$$\lim_{t\to+\infty}\sup_{x\in[0,1]}\left|u(x,t) - \frac{1}{B(x,t)Y(t)}\int_0^t \frac{R}{\mu}B(x,\tau)Y(\tau)\int_0^1 \theta(\xi,\tau)d\xi d\tau\right| = 0. \tag{3.4}$$

Proof. We have

$$\theta^{\frac{1}{2}}(x,t) \leq \left(\int_0^1 \theta(\xi,t)d\xi\right)^{\frac{1}{2}} + \int_0^1 \frac{|\theta_x(\xi,t)|}{2\theta^{\frac{1}{2}}(\xi,t)}d\xi$$
$$\leq \left(\int_0^1 \theta(\xi,t)d\xi\right)^{\frac{1}{2}}\left(1 + CV^{\frac{1}{2}}(t)\right).$$

Here we use (2.16). Therefore by (2.2), we have

$$\theta(x,t) \leq \int_0^1 \theta(\xi,t)d\xi + C\cdot\epsilon + C(\epsilon)V(t)$$

for any $\epsilon > 0$. Consequently, from (2.10), (2.21) and (2.12) we have

$$u(x,t) - \frac{1}{B(x,t)Y(t)}\int_0^t \frac{R}{\mu}B(x,\tau)Y(\tau)\int_0^1 \theta(\xi,\tau)d\xi d\tau$$
$$\le \frac{C}{Y(t)}\left\{1 + \int_0^t(\epsilon + C(\epsilon)V(\tau))Y(\tau)d\tau\right\}$$
$$\le C\left[e^{-C^{-1}t} + \int_0^t \exp\left\{-\int_\tau^t \frac{P(l(s))}{\mu}ds\right\}(\epsilon + C(\epsilon)V(\tau))d\tau\right].$$

A similar argument to one from which (2.21) is derived shows that $\frac{P(l(t))}{\mu}$ satisfies the condition of $\omega(t)$ in Lemma 3.1, while from (2.7) we have

$$\lim_{t\to+\infty}\int_t^{t+1}(\epsilon + C(\epsilon)V(\tau))d\tau = \epsilon.$$

Consequently by means of (3.3), we have

$$u(x,t) - \frac{1}{B(x,t)Y(t)}\int_0^t \frac{R}{\mu}B(x,\tau)Y(\tau)\int_0^1 \theta(\xi,\tau)d\xi d\tau \le C\cdot\epsilon$$

for t large uniformly with respect to $x \in [0,1]$.

On the other hand from

$$\theta^{\frac{1}{2}}(x,t) + C\left(\int_0^1 \theta(\xi,t)d\xi\right)^{\frac{1}{2}} V^{\frac{1}{2}}(t) \ge \left(\int_0^1 \theta(\xi,t)d\xi\right)^{\frac{1}{2}},$$

we get

$$\theta(x,t) \ge \int_0^1 \theta(\xi,t)d\xi - C\cdot\epsilon - C(\epsilon)V(t).$$

Thus in a similar way, we have

$$u(x,t) - \frac{1}{B(x,t)Y(t)}\int_0^t \frac{R}{\mu}B(x,\tau)Y(\tau)\int_0^1 \theta(\xi,\tau)d\xi d\tau \ge -C\cdot\epsilon$$

for t large uniformly with respect to $x \in [0,1]$.

Since $\epsilon > 0$ is arbitrary, we obtain (3.4). □

Lemma 3.3. *We have*

$$\inf_{t\ge 0} m_u(t) > 0. \tag{3.5}$$

Proof. From Lemma 3.2, there exists $T_1 \ge 0$ such that

$$u(x,t) \ge \frac{1}{2B(x,t)Y(t)}\int_0^t \frac{R}{\mu}B(x,\tau)Y(\tau)\int_0^1 \theta(\xi,\tau)d\xi d\tau \tag{3.6}$$

for $t \ge T_1$. By virtue of (2.2), (2.13) and (2.14), if necessary we take T_1 larger, we find that the right-hand side of (3.6) is positive uniformly with respect to $t \in [T_1,+\infty)$. This fact and (2.18) yield the assertion. □

By the help of Lemma 3.3, although the details do not carry out, the same procedure as [2, Lemmas 4.3 - 4.7] with trivial modifications leads the followings.

Lemma 3.4. *We have*

$$\int_0^1 (v^4 + \theta^2 + u_x^2)dx + \int_0^t \left[\max_{x \in [0,1]} v^2(x,\tau) + \int_0^1 \left\{ v_x^2 + \theta u_x^2 + \theta_x^2 + \left(R\frac{\theta}{u} - P(l(\tau)) \right)^2 \right\} dx \right] d\tau \le C, \tag{3.7}$$

$$\lim_{t \to +\infty} \int_0^1 v^2(x,t)dx = 0. \tag{3.8}$$

Due to this lemma, we have

Lemma 3.5. $u(x,t)$, $\int_0^1 u(x,t)dx$, $\Theta(x,t)$ *and* $\int_0^1 \Theta(x,t)dx$ *are asymptotically equivalent with respect to sup-norm each other. Here*

$$\Theta(x,t) = \frac{1}{Y(t)} \int_0^t \frac{R}{\mu} Y(\tau)\theta(x,\tau)d\tau. \tag{3.9}$$

Proof. Application of Lemma 3.1 gives us

$$\lim_{t \to +\infty} \sup_{x \in [0,1]} \left| \frac{1}{B(x,t)Y(t)} \int_0^t \frac{R}{\mu} B(x,\tau)Y(\tau) \int_0^1 \theta(\xi,\tau)d\xi d\tau - \frac{1}{Y(t)} \int_0^t \frac{R}{\mu} Y(\tau) \int_0^1 \theta(\xi,\tau)d\xi d\tau \right| = 0$$

with help of (2.11) and (3.8). Thus from Lemma 3.2, the asymptotic equivalence between $u(x,t)$ and $\int_0^1 \Theta(x,t)dx$ follows.

To establish the equivalence between $\int_0^1 u(x,t)dx$ and $\int_0^1 \Theta(x,t)dx$, we integrate (1.2) over $[x,1]$:

$$\left(\int_x^1 v(\xi,t)d\xi \right)_t = \left(R\frac{\theta}{u} - \mu\frac{u_t}{u} \right)(x,t) - P(l(t)).$$

Here we use (1.1). Multiplying both sides by $u(x,t)$ and integrating over $[0,1]$, we have by means of (1.5) and (2.1)

$$\frac{d}{dt}\left\{ l(t) + \frac{1}{\mu}\int_0^1 u(x,t)\int_x^1 v(\xi,t)d\xi dx \right\} + \frac{P(l(t))}{\mu}\left\{ l(t) + \frac{1}{\mu}\int_0^1 u(x,t)\int_x^1 v(\xi,t)d\xi dx \right\}$$
$$= \frac{1}{\mu}\left\{ \int_0^1 (v^2 + R\theta)(x,t)dx + \frac{P(l(t))}{\mu}\int_0^1 u(x,t)\int_x^1 v(\xi,t)d\xi dx \right\}. \tag{3.10}$$

Therefore we get an expression

$$l(t) + \frac{1}{\mu}\int_0^1 u(x,t)\int_x^1 v(\xi,t)d\xi dx$$
$$= \frac{1}{Y(t)}\left[\int_0^t \frac{Y(\tau)}{\mu}\left\{ \int_0^1 (v^2 + R\theta)(x,\tau)dx + \frac{P(l(\tau))}{\mu}\int_0^1 u(x,\tau)\int_x^1 v(\xi,\tau)d\xi dx \right\} d\tau + l(0) + \frac{1}{\mu}\int_0^1 u_0(x)\int_x^1 v_0(\xi)d\xi dx \right]. \tag{3.11}$$

By virtue of (2.1), (2.21) and (3.8), the same method as in the proof of Lemma 3.2 yields the desired equivalence.

The equivalence between $\Theta(x,t)$ and $\int_0^1 \Theta(x,t)dx$ is also derived from (3.7) and the application of Lemma 3.1. □

Lemma 3.6. $(l(t))^{\frac{R}{c_V}}\int_0^1 \theta(x,t)dx$ *has a limit as* $t \to +\infty$ *:*

$$\lim_{t\to+\infty}(l(t))^{\frac{R}{c_V}}\int_0^1 \theta(x,t)dx = K,$$

and K *is a positive constant given by*

$$K = \int_0^1 (u_0(x))^{\frac{R}{c_V}}\theta_0(x)dx + \int_0^{+\infty}\int_0^1 \left(\frac{\mu}{c_V}u^{\frac{R}{c_V}-1}v_x^2 - \frac{\kappa R}{c_V^2}u^{\frac{R}{c_V}-2}u_x\theta_x\right)dxd\tau. \tag{3.12}$$

Proof. Multiplying both sides of (1.3) by $c_V^{-1}u^{\frac{R}{c_V}}$, and integrating over $[0,1]\times[t',t]$, we have

$$\left|\int_0^1 u^{\frac{R}{c_V}}\theta(x,t)dx - \int_0^1 u^{\frac{R}{c_V}}\theta(x,t')dx\right| \le C\int_{t'}^{t}\int_0^1\left(v_x^2 + \theta u_x^2 + \theta_x^2 + \frac{\theta_x^2}{\theta^2}\right)dxd\tau \to 0 \quad \text{as } t', t \to +\infty.$$

Here we use (1.1), (1.7), (2.16), (3.5), (3.7), and (2.7). Thus, with the help of Lemma 3.5, (2.1) and (2.2), we find that $(l(t))^{\frac{R}{c_V}}\int_0^1 \theta(x,t)dx$ has a limit. And the above calculation implies (3.12).
□

Lemma 3.7. $P(l(t))u(x,t)$ *and* $\int_0^1 R\theta(x,t)dx$ *are asymptotically equivalent with respect to sup-norm, especially* $P(l(t))$ *is strictly positive for large* t.

Proof. By Lemma 3.5 and (2.1), it is sufficient to show the asymptotic equivalence between $P(l(t))l(t)$ and $\int_0^1 R\theta(x,t)dx$.

Put $\bar{l}$ and $\underline{l}$ such that

$$\bar{l} = \overline{\lim_{t\to+\infty}}\, l(t),$$
$$\underline{l} = \underline{\lim}_{t\to+\infty}\, l(t).$$

Remark that from (2.1),

$$0 < \underline{l} \le \bar{l} < L$$

is valid.

Firstly we assume $\underline{l} = \overline{l}$, which is denoted by l_∞ in this paragraph. Of course, in this case

$$\lim_{t \to +\infty} P(l(t)) = P(l_\infty)$$

exists, and (2.15) implies $P(l_\infty)$ must be positive. Moreover, taking $t \to +\infty$ in (2.5), we find that $\int_0^1 \theta(x,t)dx$ has a limit, say θ_∞ in this paragraph. Therefore by L'Hospital's rule we have

$$\lim_{t \to +\infty} \int_0^1 \Theta(x,t)dx = \lim_{t \to +\infty} \frac{\frac{R}{\mu} Y(t) \int_0^1 \theta(x,t)dx}{\frac{P(l(t))}{\mu} Y(t)} = \frac{R\theta_\infty}{P(l_\infty)}.$$

From this equality, Lemma 3.5 and (2.1), the assertion of this lemma is valid.

Next we assume $\underline{l} \neq \overline{l}$. From (2.5),

$$c_V K(l(t))^{-\frac{R}{c_V}} + \Psi(l(t)) = E_0 - \int_0^1 \frac{1}{2} v^2(x,t)dx + c_V \left\{ K(l(t))^{-\frac{R}{c_V}} - \int_0^1 \theta(x,t)dx \right\}$$

is valid, so taking account of (3.8), Lemma 3.6 and (2.1), $l(t)$ is close to some root of

$$c_V K \zeta^{-\frac{R}{c_V}} + \Psi(\zeta) = E_0$$

for t sufficiently large. Since $l(t)$ is continuous, the assumption $\underline{l} \neq \overline{l}$ implies

$$c_V K \zeta^{-\frac{R}{c_V}} + \Psi(\zeta) \equiv E_0 \quad \text{on } [\underline{l}, \overline{l}],$$

which yields

$$P(\zeta)\zeta = RK\zeta^{-\frac{R}{c_V}} \quad \text{on } [\underline{l}, \overline{l}].$$

Therefore we have

$$\lim_{t \to +\infty} \left| P(l(t))l(t) - \int_0^1 R\theta(x,t)dx \right| = \lim_{t \to +\infty} \left| RK(l(t))^{-\frac{R}{c_V}} - \int_0^1 R\theta(x,t)dx \right| = 0. \quad \square$$

From above lemmas, we can establish the following lemma, which is proved in a similar manner to [2, Lemma 4.9] with only trifling modifications.

Lemma 3.8. *We have*

$$\int_0^1 (u_x^2 + v_x^2 + \theta_x^2)dx + \int_0^t \int_0^1 (v_{xx}^2 + \theta_{xx}^2)dxd\tau \leq C, \tag{3.13}$$

$$\lim_{t \to +\infty} \int_0^1 (u_x^2 + v_x^2 + \theta_x^2)dx = 0. \tag{3.14}$$

Theorem 2 can be easily proven from Lemmas 3.2 - 3.8.

Next we shall prove Theorem 3 and its corollary. They are quite easy.

Proof of Theorem 3. By Theorem 2, it is sufficient to show that there exist positive constants u_∞ and θ_∞ such that

$$\lim_{t\to+\infty}\int_0^1 u(x,t)dx \left(= \lim_{t\to+\infty} l(t)\right) = u_\infty ,$$
$$\lim_{t\to+\infty}\int_0^1 \theta(x,t)dx = \theta_\infty .$$

As stated in the proof of Lemma 3.7, if $\underline{l} \neq \overline{l}$, then

$$[\underline{l}, \overline{l}] \subset Z.$$

Consequently if (P3) is fulfilled, then $\underline{l} = \overline{l}$ holds, which is u_∞.

Taking $t \to +\infty$ in (2.5), we find the existence of θ_∞, and (1.14) is valid. Here we use (3.8). (1.15) follows from Lemma 3.7. Lemma 3.6 yields the first equality of (1.16). The second equality of (1.16) is shown as follows. Dividing both sides of (1.3) by $c_V\theta$, and integrating over $[0,1] \times [0,t]$, we have

$$\int_0^1 \log\left((u(x,t))^{\frac{R}{c_V}}\theta(x,t)\right)dx = \int_0^1 \log\left((u_0(x))^{\frac{R}{c_V}}\theta_0(x)\right)dx + \frac{1}{c_V}\int_0^t\int_0^1 V(\tau)d\tau.$$

Taking into account of the uniform convergence of (u,θ) to (u_∞,θ_∞), the second equality follows. □

Proof of Corollary of Theorem 3. If (P3) is not fulfilled, Z contains some interval $[a,b]$. On this interval

$$c_V K + \zeta^{\frac{R}{c_V}}\Psi(\zeta) = E_0\zeta^{\frac{R}{c_V}}$$

holds. Differentiating with respect to ζ, we have

$$P(\zeta)\zeta = \frac{R}{c_V}(E_0 - \Psi(\zeta)) \quad \text{on } [a,b].$$

Thus (P3)′ is not fulfilled. □

4. Rate of convergence.

In the remainder of this paper, we devote ourselves to prove Theorem 4. Since under the hypotheses of Theorem 3, the solution (u,v,θ) converges to $(u_\infty,0,\theta_\infty)$ in $W^{1,2}(0,1)$ as $t \to +\infty$, we may assume that

$$\int_0^1 \{(u - u_\infty)^2 + v^2 + (\theta - \theta_\infty)^2 + u_x^2 + v_x^2 + \theta_x^2\} dx < \delta < 1, \tag{4.1}$$

for arbitrary small δ. Thus, we have

$$C^{-1}(\delta)\int_0^1 \{(u - u_\infty)^2 + v^2 + (\theta - \theta_\infty)^2\} dx \le \tilde{U}(t) \le C(\delta)\int_0^1 \{(u - u_\infty)^2 + v^2 + (\theta - \theta_\infty)^2\} dx, \tag{4.2}$$

$$|u - u_\infty| + |l - u_\infty| + |\theta - \theta_\infty| \le c(\delta) < 1, \tag{4.3}$$

where

$$\tilde{U}(t) = \int_0^1 \left\{ \frac{1}{2\theta_\infty} v^2 + R\left(\frac{u}{u_\infty} - \log\frac{u}{u_\infty} - 1\right) + c_V\left(\frac{\theta}{\theta_\infty} - \log\frac{\theta}{\theta_\infty} - 1\right)\right\} dx, \tag{4.4}$$

and $C(\delta)$, $c(\delta)$ are positive constants depending on δ and satisfying $c(\delta) \to 0$ as $\delta \to 0$.

In what follows, the symbol C means $C(\delta)$, and $c(\delta)$ is a generic constant satisfying above properties.

Lemma 4.1. *We have*

$$\tilde{U}'(t) + (1 - \epsilon_1)V(t) \le \frac{(1 + c(\delta))u_\infty}{4\epsilon_1 \mu \theta_\infty}(P'(u_\infty))^2 (l(t) - u_\infty)^2 \tag{4.5}$$

for any $\epsilon_1 \in (0,1)$. $P'(u_\infty)$ *means* $\left.\frac{dP(\zeta)}{d\zeta}\right|_{\zeta = u_\infty}$.

Proof. From (1.1) - (1.3), (1.5) - (1.7) and (1.15), we have

$$\tilde{U}'(t) + V(t) = \frac{1}{\theta_\infty}(P(u_\infty) - P(l(t)))\int_0^1 v_x dx$$

$$\le \epsilon_1 \int_0^1 \mu \frac{v_x^2}{u\theta} dx + \frac{1}{4\epsilon_1 \mu}\left(\int_0^1 u\theta\, dx\right) \cdot \frac{1}{\theta_\infty^2}(P'(l(t) + \gamma(t)(u_\infty - l(t)))^2 (l(t) - u_\infty)^2 \tag{4.6}$$

for some $\gamma(t) \in (0,1)$. Here we use the mean value theorem. Recalling (4.3), we get (4.5). □

Lemma 4.2. *We have*

$$(l(t) - u_\infty)^2 \le (1 + \epsilon_2)G(t) + (1 + \epsilon_3)(1 + c(\delta))\frac{u_\infty^3 \theta_\infty}{4\mu^3}V(t) + \frac{C(\epsilon_2, \epsilon_3)}{Y^2(t)} \tag{4.7}$$

for any $\epsilon_2, \epsilon_3 > 0$. *Here*

$$G(t) = \frac{1 + c(\delta)}{Y^2(t)}\left[\int_0^t \frac{Y(\tau)}{\mu}\left\{\left(\left|1 - \frac{R}{2c_V}\right|\frac{c(\delta)\theta_\infty^{1/2}}{\mu} + \frac{P(u_\infty)u_\infty}{2\mu^{3/2}}\right) u_\infty^{1/2}\theta_\infty^{1/2} V^{1/2}(\tau) + \left|\frac{R}{c_V}P(u_\infty) + P'(u_\infty)u_\infty\right| |l(\tau) - u_\infty|\right\} d\tau\right]^2. \tag{4.8}$$

Proof. We multiply (1.2) by v, add the result to (1.3), and integrate over $[0,1] \times [t, +\infty)$.

Integration by parts and the fact that $\int_0^1 v^2(x,t)dx \to 0$ as $t \to +\infty$ yield

$$\int_0^1 \theta(x,t)dx - \theta_\infty = -\frac{1}{2c_V}\int_0^1 v^2(x,t)dx + \frac{1}{c_V}(\Psi(u_\infty) - \Psi(l(t))). \tag{4.9}$$

Moreover it follows from the definition (2.12) of $Y(t)$ that

$$\frac{1}{Y(t)}\int_0^1 \frac{Y(\tau)}{\mu}P(l(\tau))d\tau = 1 - \frac{1}{Y(t)}. \tag{4.10}$$

By use of (3.9), (3.11), (1.14), (1.15), (4.9), (4.10) and the mean value theorem, we have

$$\begin{aligned}
l(t) - u_\infty &= l(t) - \int_0^1 \Theta(x,t)dx + \int_0^1 \Theta(x,t)dx - u_\infty \\
&= \frac{1}{Y(t)}\left[\int_0^t \frac{Y(\tau)}{\mu}\left\{\int_0^1 v^2(x,\tau)dx + \frac{P(l(\tau))}{\mu}\int_0^1 u(x,\tau)\int_x^1 v(\xi,\tau)d\xi dx\right\}d\tau\right. \\
&\quad \left. + l(0) + \frac{1}{\mu}\int_0^1 u_0(x)\int_x^1 v_0(\xi)d\xi dx\right] - \frac{1}{\mu}\int_0^1 u(x,t)\int_x^1 v(\xi,t)d\xi dx \\
&\quad + \frac{1}{Y(t)}\left[\int_0^t \frac{RY(\tau)}{\mu}\left\{\int_0^1 \theta(x,\tau)dx - \theta_\infty + \frac{u_\infty}{R}(P(u_\infty) - P(l(\tau)))\right\}d\tau - u_\infty\right] \\
&= \frac{1}{Y(t)}\left[\int_0^t \frac{Y(\tau)}{\mu}\left\{\left(1 - \frac{R}{2c_V}\right)\int_0^1 v^2(x,\tau)dx + \frac{P(l(\tau))}{\mu}\int_0^1 u(x,\tau)\int_x^1 v(\xi,\tau)d\xi dx\right.\right. \\
&\quad \left. + \left(\frac{R}{c_V}P(l(\tau) + \gamma(\tau)(u_\infty - l(\tau))) + P'(l(\tau) + \gamma(\tau)(u_\infty - l(\tau)))u_\infty\right)(u_\infty - l(\tau))\right\}d\tau \\
&\quad \left. + l(0) + \frac{1}{\mu}\int_0^1 u_0(x)\int_x^1 v_0(\xi)d\xi dx - u_\infty\right] - \frac{1}{\mu}\int_0^1 u(x,t)\int_x^1 v(\xi,t)d\xi dx
\end{aligned}$$

for some $\gamma(t) \in (0,1)$.

On the other hand, by use of (1.5), (4.3) and (2.9), we have

$$\begin{aligned}
\int_0^1 v^2 dx &\le \int_0^1 v_x^2 dx \le \frac{1 + c(\delta)}{\mu}u_\infty\theta_\infty\int_0^1 \mu\frac{v_x^2}{u\theta}dx \\
&\le \frac{1 + c(\delta)}{\mu}u_\infty\theta_\infty V^{1/2}(\tau)V^{1/2}(\tau) \le \frac{1 + c(\delta)}{\mu}u_\infty\theta_\infty c(\delta)V^{1/2}(\tau),
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
&\left|\frac{P(l(\tau))}{\mu}\int_0^1 u(x,\tau)\int_x^1 v(\xi,\tau)d\xi dx\right| \\
&\qquad \le \frac{1 + c(\delta)}{2\mu^{3/2}}P(u_\infty)u_\infty^{3/2}\theta_\infty^{1/2}\left(\int_0^1 \mu\frac{v_x^2}{u\theta}dx\right)^{1/2} \le \frac{1 + c(\delta)}{2\mu^{3/2}}P(u_\infty)u_\infty^{3/2}\theta_\infty^{1/2}V^{1/2}(\tau), \\
&\left|\frac{R}{c_V}P(l(\tau) + \gamma(\tau)(u_\infty - l(\tau))) + P'(l(\tau) + \gamma(\tau)(u_\infty - l(\tau)))u_\infty\right| \\
&\qquad \le (1 + c(\delta))\left|\frac{R}{c_V}P(u_\infty) + P'(u_\infty)u_\infty\right|, \\
&\left|\frac{1}{\mu}\int_0^1 u(x,t)\int_x^1 v(\xi,t)d\xi dx\right| \\
&\qquad \le \frac{1 + c(\delta)}{2\mu^{3/2}}u_\infty^{3/2}\theta_\infty^{1/2}\left(\int_0^1 \mu\frac{v_x^2}{u\theta}dx\right)^{1/2} \le \frac{1 + c(\delta)}{2\mu^{3/2}}u_\infty^{3/2}\theta_\infty^{1/2}V^{1/2}(t).
\end{aligned}$$

Therefore we can get the assertion. □

From (4.3) and the fact $P(u_\infty) > 0$ (see Lemma 3.7), we may assume $P(l(t)) > 0$ for any $t \geq 0$. Therefore by means of the same procedure as one in [2, Lemma 5.3], we can establish the following lemma.

Lemma 4.3. *We have*

$$G'(t) + (1 - c(\delta))\frac{P(u_\infty)}{\mu}G(t) \leq \frac{1 + c(\delta)}{\mu P(u_\infty)}\left\{\left(\left|1 - \frac{R}{c_V}\right|\frac{c(\delta)\theta_\infty^{1/2}}{\mu} + \frac{P(u_\infty)u_\infty}{2\mu^{3/2}}\right)u_\infty^{1/2}\theta_\infty^{1/2}V^{1/2}(t) + \left|\frac{R}{c_V}P(u_\infty) + P'(u_\infty)u_\infty\right||l(t) - u_\infty|\right\}^2. \tag{4.12}$$

Lemma 4.4. *There exist some positive constants* λ_1, λ_2 *and* C *such that*

$$\lambda_1\tilde{U}'(t) + G'(t) + \lambda_2\{(l(t) - u_\infty)^2 + G(t) + V(t)\} \leq \frac{C}{Y^2(t)} \tag{4.13}$$

holds.

Proof. We fix $\epsilon_1 \in (0,1)$. From (4.5) and (4.12), we have

$$\begin{aligned}
&G'(t) + \epsilon_4(l(t) - u_\infty)^2 + \epsilon_5 G(t)\\
&\leq G'(t) + (\epsilon_4(1 + \epsilon_2) + \epsilon_5)G(t) + \epsilon_4(1 + \epsilon_3)(1 + c(\delta))\frac{u_\infty^3\theta_\infty}{4\mu^3}V(t) + \frac{\epsilon_4 C(\epsilon_2,\epsilon_3)}{Y^2(t)}\\
&\leq \frac{1 + c(\delta)}{\mu P(u_\infty)}\left\{\left(\left|1 - \frac{R}{2c_V}\right|\frac{c(\delta)\theta_\infty^{1/2}}{\mu} + \frac{P(u_\infty)u_\infty}{2\mu^{3/2}}\right)u_\infty^{1/2}\theta_\infty^{1/2}V^{1/2}(t)\right.\\
&\quad\left. + \left|\frac{R}{c_V}P(u_\infty) + P'(u_\infty)u_\infty\right||l(t) - u_\infty|\right\}^2\\
&\quad + \epsilon_4(1 + \epsilon_3)(1 + c(\delta))\frac{u_\infty^3\theta_\infty}{4\mu^3}V(t) + \frac{\epsilon_4 C(\epsilon_2,\epsilon_3)}{Y^2(t)},
\end{aligned} \tag{4.14}$$

if $\epsilon_2, \epsilon_4, \epsilon_5$ and $\delta > 0$ are sufficiently small such that

$$\epsilon_4(1 + \epsilon_2) + \epsilon_5 < (1 - c(\delta))\frac{P(u_\infty)}{\mu}. \tag{4.15}$$

Moreover (4.5) and (4.14), we have

$$\begin{aligned}
&\epsilon_6\tilde{U}'(t) + G'(t) + \epsilon_6(1 - \epsilon_1)V(t) + \epsilon_4(l(t) - u_\infty)^2 + \epsilon_5 G(t)\\
&\leq \frac{1 + c(\delta)}{\mu}\left\{\frac{\epsilon_6 u_\infty}{4\epsilon_1\theta_\infty}(P'(u_\infty))^2 + \frac{1 + \epsilon_7}{P(u_\infty)}\left(\frac{R}{c_V}P(u_\infty) + P'(u_\infty)u_\infty\right)^2\right\}(l(t) - u_\infty)^2\\
&\quad + \frac{(1 + c(\delta))u_\infty\theta_\infty}{\mu^3}\left\{\frac{\epsilon_4(1 + \epsilon_3)u_\infty^2}{4} + \frac{C(\epsilon_7)}{P(u_\infty)}\left(\left|1 - \frac{R}{2c_V}\right|c(\delta)\theta_\infty^{1/2} + \frac{P(u_\infty)u_\infty}{2\mu^{1/2}}\right)^2\right\}V(t)
\end{aligned}$$

$$+ \frac{\epsilon_4 C(\epsilon_2,\epsilon_3)}{Y^2(t)}. \tag{4.16}$$

We choose ϵ_2, ϵ_5 and δ sufficiently small, so we can choose ϵ_4 sufficiently close to $\frac{P(u_\infty)}{\mu}$ (see (4.15)). Therefore (P4)−(i) implies

$$\epsilon_4 > \frac{(1+c(\delta))}{\mu}\left\{\frac{\epsilon_6 u_\infty}{4\epsilon_1\theta_\infty}(P'(u_\infty))^2 + \frac{1+\epsilon_7}{P(u_\infty)}\left(\frac{R}{c_V}P(u_\infty) + P'(u_\infty)u_\infty\right)^2\right\} \tag{4.17}$$

for sufficiently small δ, ϵ_6 and $\epsilon_7 > 0$. Since $(u_\infty,\theta_\infty) \in S$, $\underline{s} \le u_\infty \le \bar{s}_1$, $\underline{s} \le \theta_\infty \le \bar{s}_2$ hold, where $\underline{s}$, $\bar{s}_i$ $(i = 1, 2)$ are positive constants not depending on μ (see (P4)−(ii)). Consequently if μ is sufficiently large (see (P4)−(iii)), then

$$\epsilon_6(1-\epsilon_1) > \frac{(1+c(\delta))u_\infty\theta_\infty}{\mu^3}\left\{\frac{\epsilon_4(1+\epsilon_3)u_\infty^2}{4} + \frac{C(\epsilon_7)}{P(u_\infty)}\left(\left|1 - \frac{R}{2c_V}\right|c(\delta)\theta_\infty^{1/2} + \frac{P(u_\infty)u_\infty^2}{2\mu^{1/2}}\right)^2\right\} \tag{4.18}$$

is valid. (4.16) - (4.18) yield the assertion. □

Lemma 4.5. *We have*

$$\left\{\int_0^1\left(\mu(\log u)_x - v\right)^2 dx\right\}_t + C^{-1}\int_0^1\left(\mu(\log u)_x - v\right)^2 dx \le CV(t), \tag{4.19}$$

$$\tilde{U}(t) \le C\left\{\int_0^1\left(\mu(\log u)_x - v\right)^2 dx + G(t) + V(t) + \frac{1}{Y^2(t)}\right\}. \tag{4.20}$$

Proof. For (4.19) see [2, Lemma 5.2]. For (4.20) we begin with the estimate of v^2. From (1.5) and Theorem 3, it is clear that

$$v^2 \le \int_0^1 v_x^2 dx \le CV(t). \tag{4.21}$$

Therefore, we have from (4.7) and (4.21)

$$\begin{aligned}(u-u_\infty)^2 &\le 2\left(u - \int_0^1 u\,dx\right)^2 + 2\left(\int_0^1 u\,dx - u_\infty\right)^2 \\ &\le C\left(\int_0^1 u_x^2 dx + G(t) + V(t) + \frac{1}{Y^2(t)}\right) \\ &\le C\left\{\int_0^1\left(\mu(\log u)_x - v\right)^2 dx + G(t) + V(t) + \frac{1}{Y^2(t)}\right\}.\end{aligned} \tag{4.22}$$

Here we use $\int_0^1 u\,dx = l(t)$.

Next we evaluate $(\theta-\theta_\infty)^2$. By use of (4,9), (4.1), (4.21), (4.22), (4.7) and (2,9), we have

$$\begin{aligned}(\theta-\theta_\infty)^2 &\le 2\left(\theta - \int_0^1 \theta\,dx\right)^2 + 2\left(\int_0^1 \theta\,dx - \theta_\infty\right)^2 \\ &\le C\left(V(t) + (l(t) - u_\infty)^2\right)\end{aligned}$$

$$\leq C\left[G(t) + V(t) + \frac{1}{Y^2(t)}\right]. \tag{4.23}$$

(4.20) follows from (4.2) and (4.21) - (4.23). □

In a quite similar manner to one in [2, Lemmas 5.6 - 5.7], we can establish the following lemma.

Lemma 4.6. *We have*

$$\left[\int_0^1\left\{\left(v_x - \frac{1}{\mu}(R\theta - P(l(t))u)\right)^2 + P(l(t))u\left(\frac{u_x}{u}\right)^2 + \theta_x^2\right\}dx\right]_t$$
$$+ C^{-1}\int_0^1\left\{\left(v_x - \frac{1}{\mu}(R\theta - P(l(t))u)\right)^2 + P(l(t))u\left(\frac{u_x}{u}\right)^2 + \theta_x^2\right\}dx$$
$$\leq C\left\{\int_0^1\left(\mu(\log u)_x - v\right)^2dx + G(t) + V(t) + \frac{1}{Y^2(t)}\right\}. \tag{4.24}$$

Now we proceed to prove Theorem 4. From (4.13), (4.19), (4.20) and (4.24), we find that there exist positive constants λ_i $(i = 3, \cdots, 7)$ such that

$$\Bigg[\lambda_3\widetilde{U}(t) + \lambda_4G(t) + \lambda_5\int_0^1\left(\mu(\log u)_x - v\right)^2dx$$
$$+ \lambda_6\left\{\int_0^1\left\{\left(v_x - \frac{1}{\mu}(R\theta - P(l(t))u)\right)^2 + P(l(t))u\left(\frac{u_x}{u}\right)^2 + \theta_x^2\right\}dx\right\}\Bigg]_t$$
$$+ \lambda_7\Bigg[\lambda_3\widetilde{U}(t) + \lambda_4G(t) + \lambda_5\int_0^1\left(\mu(\log u)_x - v\right)^2dx$$
$$+ \lambda_6\left\{\int_0^1\left\{\left(v_x - \frac{1}{\mu}(R\theta - P(l(t))u)\right)^2 + P(l(t))u\left(\frac{u_x}{u}\right)^2 + \theta_x^2\right\}dx\right\}\Bigg]$$
$$\leq \frac{C}{Y^2(t)}. \tag{4.25}$$

It follows from the definition (2.12) of $Y(t)$ and the fact $P(l(t)) \to P(u_\infty) > 0$ as $t \to +\infty$, there exists a positive constant λ_8 such that

$$\frac{1}{Y^2(t)} \leq C\exp(-\lambda_8 t). \tag{4.26}$$

(4.25) and (4.26) assert that

$$\int_0^1\left\{(u - u_\infty)^2 + v^2 + (\theta - \theta_\infty)^2 + u_x^2 + \theta_x^2 + \left(v_x - \frac{1}{\mu}(R\theta - P(l(t))u)\right)^2\right\} \leq C\exp(-\lambda t)$$

for some $\lambda > 0$, which yields

$$\int_0^1 v_x^2 dx \le 2\int_0^1 \left(v_x - \frac{1}{\mu}(R\theta - P(l(t))u)\right)^2 dx + \frac{2}{\mu^2}\int_0^1 (R\theta - P(l(t))u)^2 dx$$
$$\le C\int_0^1 \left\{\left(v_x - \frac{1}{\mu}(R\theta - P(l(t))u)\right)^2 + (\theta - \theta_\infty)^2 + (u - u_\infty)^2 + u_x^2\right\} dx \le C\exp(-\lambda t).$$

Thus we complete the proof of Theorem 4.

Remark. From (4.13), (4.19) and (4.20), we have

$$\left\{\tilde\lambda_3 \tilde U(t) + \tilde\lambda_4 G(t) + \tilde\lambda_5 \int_0^1 \left(\mu(\log u)_x - v\right)^2 dx\right\}_t$$
$$+ \tilde\lambda_7 \left\{\tilde\lambda_3 \tilde U(t) + \tilde\lambda_4 G(t) + \tilde\lambda_5 \int_0^1 \left(\mu(\log u)_x - v\right)^2 dx\right\} \le \frac{C}{Y^2(t)},$$

which yields

$$\int_0^1 \{(u - u_\infty)^2 + v^2 + (\theta - \theta_\infty)^2 + u_x^2\} dx \le C\exp(-\tilde\lambda t).$$

From the above procedure we find $\tilde\lambda_7 > \lambda_7$, which implies $\tilde\lambda > \lambda$.

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References.

[1] Nagasawa, T., On the one-dimensional motion of the polytropic ideal gas non-fixed on the boundary, to appear in *J. Differential Equations*.

[2] Nagasawa, T., On the outer pressure problem of the one-dimensional polytropic ideal gas, to appear.

[3] Tani, A., Free boundary problems for the equations of motion of general fluids, *Lecture Note on Num. Appl. Anal.*, **6** (1983), 211-219.

[4] Yamamoto, M., "*Stability of ordinary differential equations*," Jikkyo-Shuppan, Tokyo, 1979 (in Japanese).