Research Report

KSTS/RR-86/001
24 Jan. 1986

A remark on the central limit theorem for Weyl automorphism

by

Yutaka Kato
Tatsuo Kawata

Yutaka Kato

Department of Industrial Engineering
Hosei University

3-7-2, Kajinocho, Koganei
Tokyo, 184 Japan

Tatsuo Kawata

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku
Yokohama, 223 Japan

Dept. of Math., Fac. of Sci. & Tech., Keio Univ.
Hiyoshi 3-14-1, Kohoku-ku, Yokohama, 223 Japan

KSTS/RR-86/001
January 24, 1986

A REMARK ON THE CENTRAL LIMIT THEOREM FOR WEYL AUTOMORPHISM

Yutaka Kato [1] and Tatsuo Kawata

§1. INTRODUCTION Maruyama [3] has investigated the class of functions satisfying the central limit theorem for the Gaussian flow with the continuous singular spectral measure ν such that

$$R(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \, \nu(d\lambda) = O(|t|^{-\frac{1}{2}+\varepsilon}) \quad , |t| \longrightarrow \infty$$

for $\varepsilon > 0$. It seems to us that this result is the first one of the central limit theorem for a dynamical system with zero entropy . We have studied the central limit theorem for an ergodic automorphism on the one dimensional torus , for which the rate of convergence of Birkhoff's individual ergodic theorem is estimated uniformly for all intervals on the one dimensional torus , and as an application of the above fundamental result , we have proved that the class of functions satisfying the central limit theorem for Weyl automorphism is an uncountable dense set of the space of square-integrable functions , (see , Kato [2]) .

The purpose of this note is to prove directly the central limit theorem for Weyl automorphism by the aid of Salem-Zygmund's central limit theorem for lacunary trigonometric series . This idea has been

---

[1] Department of Industrial Engineering , Hosei University , 3-7-2 , Kajinocho , Koganei , Tokyo 184 , Japan .

pointed out in [2] . After completed [2] , we have recently known that Burton and Denker [1] have constructed a function satisfying the central limit theorem for Weyl automorphism by the similar idea.

In the forthcoming paper , we shall study the multi-dimensional version of Salem-Zygmund's central limit theorem . Our method of showing this will be to prove first the central limit theorem for multi-dimensional Weyl automorphism which will be made just in the same way as in [2] and then , contrary to our line of the method of this paper , the result is applied to show a type of multi-dimensional Salem-Zygmund's central limit theorem for lacunary trigonometric series .

§2. THE RESULT AND IT'S PROOF Let μ be Lebesgue measure on $\Omega = [0, 1)$ and T be Weyl automorphism on Ω , i.e. $T\omega = \{\omega + \xi\}$ for $\omega \in \Omega$, where ξ is an irrational number and $\{y\}$ denotes a fractional part of y . Let

$$Z_+ = \{1, 2, \ldots\} ,$$

$$Z_k = \left\{ n \in Z_+ ; \{n\xi\} \in \left(\frac{e^{-k-1}}{2} , \frac{e^{-k}}{2} \right] \right\} ,$$

for $k \in Z_+$ and put

$$(1) \ldots \qquad f_\gamma(\omega) = \sum_{k=1}^{\infty} 2 \left(\frac{1}{k} \right)^{\frac{1+\gamma}{2}} \cos 2\pi m_k \omega ,$$

where $m_k \in Z_k$ and $\gamma > 1$. The series in (1) obviously converges absolutely . Let m_k satisfy the Hadamard's gap condition :

(2) ... $$\frac{m_{k+1}}{m_k} \geq q > 1 .$$

PROPOSITION 1 . Under the condition (2) , we have

(3) ... $$v_{f_\gamma}(n) = \mathrm{Var}\left(\sum_{k=0}^{n-1} f_\gamma (T^k\omega) \right) \asymp \frac{n^2}{(\log n)^\gamma}$$

and

(4) ... $$\frac{1}{\sqrt{v_{f_\gamma}(n)}} \sum_{k=0}^{n-1} f_\gamma (T^k\omega) \xrightarrow{D} N(0 , 1) ,$$

as $n \longrightarrow \infty$, where $a_n \asymp b_n$ denotes that $0 < \liminf_{n\to\infty} a_n/b_n \leq \limsup_{n\to\infty} a_n/b_n < \infty$ and $X_n \xrightarrow{D} X$ means that X_n converges in distribution to X .

REMARK . (1) This proposition has been proved in [2] under a condition on ξ .

(2) Recently , Burton and Denker [1] have constructed a function $g(\omega)$ such that

$$v_g(n) = \mathrm{Var}\left(\sum_{k=0}^{n-1} g(T^k\omega) \right) \asymp n^\delta , \quad 0 < \delta < 2$$

and

$$\frac{1}{\sqrt{v_g(n)}} \sum_{k=0}^{n-1} g(T^k\omega) \xrightarrow{D} N(0 , 1) ,$$

as $n \longrightarrow \infty$, by the aid of Salem-Zygmund's central limit theorem for lacunary trigonometric series .

PROOF . The relation (3) is easily proved by Proposition 2 in [2] . From (1) , we have

$$f_\gamma(\omega) = \sum_n \hat{f}_n e^{2\pi i m_n \omega} ,$$

(5) ... $$\hat{f}_n = \hat{f}_{-n} = \left(\frac{1}{n}\right)^{\frac{1+\gamma}{2}} , \quad \gamma > 1 ,$$

$$m_{-n} = - m_n .$$

Let

$$F_p(\omega) = \frac{1}{\sqrt{v_{f_\gamma}(p)}} \sum_{k=0}^{p-1} f_\gamma(T^k\omega)$$

$$= \frac{1}{\sqrt{v_{f_\gamma}(p)}} \sum_{|n| \leq n_0(p)} \hat{f}_n e^{2\pi i m_n \omega} \cdot \frac{1 - e^{2\pi i m_n p \xi}}{1 - e^{2\pi i m_n \xi}}$$

$$+ \frac{p}{\sqrt{v_{f_\gamma}(p)}} \sum_{|n| > n_0(p)} \hat{f}_n e^{2\pi i m_n \omega}$$

$$+ \frac{1}{\sqrt{v_{f_\gamma}(p)}} \sum_{|n| > n_0(p)} \hat{f}_n e^{2\pi i m_n \omega} \cdot \left(\frac{1 - e^{2\pi i m_n p \xi}}{1 - e^{2\pi i m_n \xi}} - p \right)$$

$$= F_{1,p}(\omega) + F_{2,p}(\omega) + F_{3,p}(\omega) , \qquad (\text{ say })$$

where $n_0(p) = \min \left\{ n > 0 \; ; \; \{ m_n \xi \} \leq \frac{1}{2p(\log p)^{1/6}} \right\}$. Since

$$\left| \frac{\sin \pi px}{\sin \pi x} - p \right| = O(\ p(\ \log p\ )^{-\frac{1}{3}}\ )\ \ ,\ 0 < x \leq \frac{1}{2p(\ \log p\ )^{1/6}}$$

for sufficiently large p , it follows from the proof of Proposition 2 in [2] that

$$E \left| F_{3,p}(\omega) \right|^2 = O(\ \frac{p^2(\ \log p\ )^{-\frac{2}{3}}}{v_{f_\gamma}(p)} \sum_{|n| > n_0(p)} |\hat{f}_n|^2\ )$$

(6) ...

$$= O(\ (\ \log p\ )^{\gamma - \frac{2}{3}} \int_0^{\frac{1}{2}p^{-1}(\ \log p\ )^{-\frac{1}{6}}} dH_{f_\gamma}(x)\ )$$

$$= O(\ (\ \log p\ )^{-\frac{2}{3}}\ )$$

and

$$E \left| F_{1,p}(\omega) \right|^2 = \frac{1}{v_{f_\gamma}(p)} \sum_{|n| \leq n_0(p)} |\hat{f}_n|^2 \left| \frac{\sin \pi m_n p \xi}{\sin \pi m_n \xi} \right|^2$$

$$= O(\ \frac{1}{v_{f_\gamma}(p)} \int_{\frac{1}{2}p^{-1}(\log p)^{-\frac{1}{6}}}^{\frac{1}{2}} \left| \frac{\sin \pi px}{\sin \pi x} \right|^2 dH_{f_\gamma}(x)\ )$$

(7) ...

$$= O(\ \frac{(\log p)^{\gamma}}{p^2} \cdot \frac{(\ 2p(\log p)^{1/6}\ )^2}{(\ \log(\ 2p(\log p)^{1/6}\ )\ )^{1+\gamma}}\ )$$

$$= O(\ (\ \log p\ )^{-\frac{2}{3}}\ )\ \ .$$

From (6) and (7) , we have

$$F_{1,p}(\omega) \xrightarrow{P} 0 ,$$

$$F_{3,p}(\omega) \xrightarrow{P} 0 ,$$

and

$$E \left| F_{2,p}(\omega) \right|^2 \longrightarrow 1 ,$$

as $p \longrightarrow \infty$. Furthermore it follows from (5) that

$$\frac{\hat{f}_k}{\left(\sum_{n \geq k} | \hat{f}_n |^2 \right)^{\frac{1}{2}}} \longrightarrow 0 ,$$

as $k \longrightarrow \infty$. Therefore it follows from Salem-Zygmund's central limit theorem for lacunary trigonometric series (Theorem 5.29 in Zygmund [4]) that

$$F_{2,p}(\omega) \xrightarrow{D} N(0,1) ,$$

as $p \longrightarrow \infty$, so that we have

$$F_p(\omega) \xrightarrow{D} N(0,1) .$$

This completes the proof of Proposition 1 .

REFERENCES :

[1] Burton , R. & M. Denker : On the central limit theorem for dynamical systems , preprint .

[2] Kato , Y. : Central limit theorem for Weyl automorphism , preprint .

[3] Maruyama , G. : Nonlinear functionals of Gaussian stationary processes and their applications , Proc. Third Japan-USSR Sym. on Prob. Theory , Lecture Notes in Math. 550 , Springer , 1976 .

[4] Zygmund , A. : Trigonometric Series , Vol. II , Cambridge Univ. Press , 1959 .