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Hamiltonian version of the Seifert Conjecture

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ABSTRACT

A Hamiltonian H is called *classical* if it is of the form

" *kinetic energy* " + " *potential* " .

It is known that on *every* compact energy surface of classical Hamiltonian, there is a periodic solution on it [Hay 3][G-Z].

In this note, we give a plausibility of the following Conjecture: *If the energy surface is homeomorphic to* S^{2n-1} *, then there are* n *periodic solutions on it.*

This is considered as a Hamiltonian version of the Seifert Conjecture: *any flow on* S^3 *has a closed orbit.*

We also present an open problem concerning a characterization of manifolds.

§1 Introduction

In considering Hamiltonian systems, there are three levels of the formulation. Let n be the degree of freedom.

The first is the usual one.

level 1 Hamiltonian $H = H(q, p)$ is a C^∞ function from $\mathbf{R}^{2n}$ to $\mathbf{R}$, and the equation is

$$\dot{q} = H_p, \qquad \dot{p} = -H_q \quad . \tag{1.1}$$

$\mathbf{R}^{2n}$ in level 1 is considered as the cotangent bundle $T^*\mathbf{R}^n$. In general, let N be an n -dimensional configuration manifold such as S^2 , describing a spherical pendulum, $SO(3)$, a rigid body with one point fixed , and so on. The case $N = \mathbf{R}^n$, describing particles in a free space, gives level 1. Mathematically, N is an arbitrary C^∞ manifold.

In order to give Lagrangian $T - U$, where T means kinematic energy and U is a potential, a Riemannian metric and a real valued function on N is endowed.

Roughly speaking, " *Classical mechanics* " is " *real valued function on Riemannian manifold* " .

In this case, we have the symplectic 2-form

$$\Omega = dp \wedge dq \tag{1.2}$$

and the equation corresponding (1.1) is given by the vector field ξ_H defined by

$$dH = \xi_H \lrcorner\, \Omega \equiv \Omega(\xi_H, \cdot) \ . \tag{1.3}$$

This can be done because

$$\Omega : \textit{non-degenerate, that is, } \Omega(\xi, \cdot) = 0 \ \textit{ iff } \ \xi = 0 \ . \tag{1.4}$$

And in this case $dp \wedge dq = d(pdq)$, hence

$$\Omega : \textit{closed form.} \tag{1.5}$$

Definition 1 Let N be an n - dimensional configuration manifold and $H = T^*N \longrightarrow \mathbf{R}$. H is called *classical* if $H = H(z)$ is of the form

$$H(z) = \frac{1}{2} \mid z \mid^2 + U(\pi^*(z)), \tag{1.6}$$

where $| \ |$ is the norm on the cotangent bundle deduced from the Riemannian metric on N ,

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(1.7) $\pi^* : T^*N \longrightarrow N$

is the usual projection and

(1.8) $U : N \longrightarrow \mathbf{R}$; C^∞ *function.*

level 2 A Hamiltonian $H : T^*N \longrightarrow \mathbf{R}$ is classical and equation of motion is given by the vector field ξ_H on T^*N defined by (1.3) .

The most general formulation for Hamiltonian systems is the following one.

level 3 Let P be a 2n-dimensional manifold with symplectic 2-form Ω , that is, closed non-degenerated 2-form.

Hamiltonian H is a real valued function on P and the Hamiltonian vector field ξ_H on P is defined by (1.3). We call this (P, Ω) a *symplectic manifold.*

Let (P, Ω) be a symplectic manifold and $e \in \mathbf{R}$ a regular value of a Hamiltonian $H : P \longrightarrow \mathbf{R}$. Then

(1.9) $\Sigma = H^{-1}(e)$

is a hypersurface (a submanifold of P of codimension 1), which is an invariant set of the flow generated by ξ_H (because $\xi_H \cdot H = dH \cdot \xi_H = \Omega(\xi_H, \xi_H) = 0$) .

We remark that if Σ is also an energy surface of another Hamiltonian $\tilde{H}$, then the flow on Σ obtained from $\tilde{H}$ is equal to the one from H up to time change. In particular the number of periodic orbits on Σ coinside. So we can define

Definition 2 Let Σ be a hypersurface of a symplectic manifold (P, Ω) . Then we define $\mu(\Sigma)$ as the number of periodic orbits of the flow on Σ given by *any* Hamiltonian $H : P \longrightarrow \mathbf{R}$ having Σ as a regular level surface, that is, $\Sigma = H^{-1}(e)$ for a regular value e of H .

From now on, we assume that Σ is a *compact* hypersurface of P (we take $P = T^*N$ in level 2 and $P = \mathbf{R}^{2n}$ in level 1). There is a non-compact Σ without periodic orbit. Consider, for example, a particle in a free space without external force.

Our concern is to give an estimate to $\mu(\Sigma)$ when we are given (P, Ω) and Σ .

In 1978, P. Rabiowitz [Rab] obtained the excellent

Theorem 1 (level 1) $\mu(\Sigma) \geq 1$ if Σ *is star-shaped.*

And we have [Hay 3][G-Z]

Theorem 2 (level 2) $\mu(\Sigma) \geq 1$ if Σ *is compact.*

A rough proof of Theorem 2 is found in [Hay 1].

Remark that , in level 2, e is a regular value of H iff e is a regular value of U . So

(1.10) $W = \pi^*(\Sigma) = \{ q \in N ; U(q) \leq e \}$,

(1.11) $\partial W = \{ q \in N ; U(q) = e \}$,

(1.12) $\mathring{W} = \{ q \in N ; U(q) < e \}$.

H. Seifert [Sei 1] obtained this result for the case $W \approx D^n$ and stated, in the footnote, that " one may obtain $\mu(\Sigma) \geq n$ for such a case using the method of Lusternik-Schnirelmann. "

Remark that $W \approx D^n$ implies $\Sigma \approx S^{2n-1}$.

In this note, we pose

Conjecture : (level 2) $\mu(\Sigma) \geq n$ if $\Sigma \approx S^{2n-1}$.

For the Hamiltonian describing the harmonic oscillators with the angular frequencies $\omega_1, \ldots, \omega_n > 0$, $\mu(\Sigma) = n$ if $\omega_1, \ldots, \omega_n$ are independent over **Q** and $\mu(\Sigma) = \infty$ otherwise (that is, when a resonance occures), where Σ is any regular energy surface of the Hamiltonian.

For such a case, Σ is an ellipsoid, hence $\Sigma \approx S^{2n-1}$. Therefore the estimate of the conjecture cannot be improved.

In the following sections, we shall give a plausibility of this conjecture.

We also present an open problem :

determine manifolds obtained as a compact energy surface of a classical Hamiltonian.

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As for the level 3, A. Weinstein [Wei] defined the concept of contact type, which is invariant under canonical transformations, and posed a conjecture:

In level 3, $\mu(\Sigma) \geq 1$ *if* Σ *is of contact type and* $H^1(\Sigma; \mathbf{R}) = 0$

For more details, see [Wei].

Optimists may pose :

In level 3, $\mu(\Sigma) \geq n$ *if* Σ *is compact.*

But there are no circumstancial evidences, so this is doubtful.

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§ 2 Seifert Conjecture

In 1950, H. Seifert [Sei 2] found a closed orbit of any flow on S^3 "*near*" the special flow which is obtained from the Hamiltonian system describing two harmonic oscillators with the same angular frequency.

Since then the following has been called

Seifert Conjecture : any flow on S^3 has a closed orbit.

Recently, a $C^{3-\epsilon}$ counter example for this conjecture is found [Har].

Our interest is concentrated on odd dimensional spheres, because a tangent vector field of the even dimensional sphere has always a singular point and a Hamiltonian energy surface is always odd dimensional.

Although [Hay 4] gives a higher dimensional version of [Sei 2], it is known that there is a C^∞ flow without closed orbit on S^{2n-1}, $n \geq 3$ [Wil].

Considering this fact with Theorem 1, one may feel it is easier to find a periodic orbit on Hamiltonian energy surface than general flow on a manifold.

The reason of it may be the fact that Hamiltonian system is derived by the Principle of Least Action :

(2.1) $\delta \int pdq - Hdt = 0$

or for the classical Hamiltonian $H = T + U$,

(2.2) $\delta \int (T - U)\, dt = 0$.

Recall that it is not so easy to find a singular point of a tangent vector field of a manifold, but any function on a compact manifold always holds two critical points.

We already have Theorem 2, so our interest is the number of periodic solutions (at least in level 2) on Σ .

A plausibility of the Conjecture proposed in §1 consists of following facts.

Let N be an n - dimensional configuration manifold,

$$H = T + U : T^*N \longrightarrow \mathbf{R}$$

a classical Hamiltonian, and $\Sigma \equiv H^{-1}(e) \subset T^*N$ a regular energy surface. And put $W \equiv \pi^*(\Sigma) = \{ q \in N; U(q) \leq e \}$.

Fact 1 *A periodic orbit on Σ is given by an orthogonal geodesic chord of W (precisely speaking, a manifold diffeomorphic to W) w.r.t. the Jacobi metric.*

Here an *orthogonal geodesic chord* of a Riemannian manifold with boundary is a geodesic connecting two points of the boundary orthogonally. We called

$$ds^2 = (e - U) T$$

the Jacobi metric. This fact is proved as Gauss' Theorem (see for example [Sei 1]).

Fact 2 $\Sigma \approx S^{2n-1} \Leftrightarrow W$ *: compact contractible* .

Fact 3 *Let $W = W^n$ be a Riemannian manifold with (geodesically)* convex *boundary. If W is compact contractible, then there are at least* n *orthogonal geodesic chords.*

A proof of Fact 2 is given in §3.

Fact 3 is the main result of [Hay 6].

There is a compact contractible Riemannian manifold without orthogonal geodesic chord. So the assumption "*convex boundary*" is necessary.

In [Hay 4], there is given an example with n periodic orbits on $\Sigma \approx S^{2n-1}$. [Hay5] and resp. [BLMR] give $\mu(\Sigma) \geq n$ in level 1, where Σ is convex or resp. star-shaped "near" an ellipsoid. These are also considered as circumstantial evidences of the Conjecture.

§3. Proof of Fact 2.

(=>) Let $\Sigma \approx S^{2n-1}$.

We regard N as the subset of T^*N , considering it as the 0 section, in particular $\partial W \subset \Sigma$. The restriction $\pi : \Sigma \setminus \partial W \longrightarrow \mathring{W}$ of $\pi^* : T^*N \longrightarrow N$ to $\Sigma \setminus \partial W$ gives a fibration

(3.1) $$S^{n-1} \xrightarrow{i} \Sigma \setminus \partial W \xrightarrow{\pi} \mathring{W} .$$

Since W is an n - manifold with boundary,

(3.2) $\pi_k(W) = 0$ for $k = 1, 2, \ldots, n-1$

implies W is contractible by the Theorem of J.H.C. Whitehead and the Theorem of Hurewitz.

First we claim

(3.3) ∂W *is connected.*

In fact, if not so, there is an arc ω connecting different components of ∂W . Let B be one of them. Then π^{*-1} (Im ω) intersects with B transversally. Considering the intersection form in the (2n-1)-sphere, this can't be possible, proving (3.3). □

Also we have

(3.4) $\pi_k (\Sigma \setminus \partial W) \cong 0$ for $k = 1, 2, \ldots, n-2$.

To prove this, consider the codimension and use the general position lemma.

Furthermore we have

(3.5) $$\pi_{n-1}(\Sigma \setminus \partial W) \cong \mathbf{Z} .$$

In fact, $\pi_{n-1}(\Sigma \setminus \partial W)$

$\cong H_{n-1}(\Sigma \setminus \partial W)$ (*by* (3.4) and *Hurewitz*)

$\cong \check{H}^{n-1}(\partial W)$ (*Alexander duality*)

$\cong H_0(\partial W)$ (*Poincaré duality*)

$\cong \mathbf{Z}$ (*by* (3.3))

Now we have

Lemma 3.1 $i_* : \pi_{n-1}(S^{n-1}) \cong \pi_{n-1}(\Sigma \setminus \partial W)$
where $i : S^{n-1}$ *(as a fiber)* $\subset \Sigma \setminus \partial W$.

(proof) First we claim that

(3.6) *there is a section* $s : \mathring{W} \longrightarrow \Sigma \setminus \partial W$ *, i.e.,* $\pi \circ s = i_d$.

In fact, considering a Morse function, there is a vector field with finitely many singular points in $\mathring{W}$. Connecting these singular points with points of ∂W by smooth curves and removing neighborhoods of the curves, one has a vector field without singular points on a manifold diffeomorphic to W . Pullback this vector field to the original W and consider the corresponding section of cotangent bundle of W . By rescaling the section, we have the desired section $s : \mathring{W} \longrightarrow \Sigma \setminus \partial W$, giving (3.6).

Now we restrict the fibration $\pi : \Sigma \setminus \partial W \longrightarrow \mathring{W}$ to $W \setminus C$, where C is a small open collar of ∂W in W . Thus we have the fibration

$$S^{n-1} \xrightarrow{i} \Sigma \setminus Q \xrightarrow{\pi'} W \setminus C$$

where π' is the restriction and $Q = \pi^{*-1}(C)$ is an open subset of Σ .

Since $\Sigma \setminus Q$ is compact, we can consider the intersection form

$$\circ : H_{n-1}(\Sigma \setminus Q) \otimes H_n(\Sigma \setminus Q, \partial Q) \longrightarrow \mathbf{Z} \ .$$

Let $s' : W \setminus C \longrightarrow \Sigma \setminus Q$ be the section given by (3.6). Then the image Im s' determines an element σ of $H_n(\Sigma \setminus Q, \partial Q)$. And the fibre S^{n-1} is regarded as an element of $H_{n-1}(\Sigma \setminus Q)$, write this $[S^{n-1}]$.

Since s' is a section, we have

$$[S^{n-1}] \circ \sigma = \pm 1 ,$$

in particular, $[S^{n-1}]$ is a generator of $H_{n-1}(\Sigma \setminus Q)$.

By (3.4) and Hurewitz, we can identify

$$\pi_{n-1}(\Sigma \setminus Q) = H_{n-1}(\Sigma \setminus Q),$$

giving the proof. □

The homotopy exact sequence of the fibration (3.1), and (3.4), (3.5) and Lemma 3.1 gives (3.2), proving (=>) . □

(<=) Let $W = \pi^*(\Sigma)$ be compact contractible manifold. Then cotangent bundle T^*W is trivial, and

$$\Sigma \approx \partial(W \times D^n) = \partial W \times D^n \cup W \times S^{n-1} .$$

We put $X_1 = \partial W \times D^n$, $X_2 = W \times S^{n-1}$ and $X = X_1 \cup X_2$.

For the case $n = 1$ and 2 , we have $W \approx D^n$, hence $\Sigma \approx S^{2n-1}$.

We assume that $n \geq 3$. Then $dim\ \Sigma \geq 5$.

Hence by the resolution of the Poincaré Conjecture, the followings give $\Sigma \approx S^{2n-1}$,

(3.7) X : *simply connected,*

(3.8) X : *homology sphere.*

To prove (3.7), use the Theorem by van Kampen.

Recalling that ∂W is a homology sphere, Mayer-Vietoris exact sequence yields (3.8). □

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