On the outer pressure problem

of the one-dimensional polytropic ideal gas

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ABSTRACT

In this paper the existence of the global solution of the outer pressure problem of the one-dimensional polytropic ideal gas is proved (Theorem 1). We shall also show that under some suitable assumptions the solution converges to a stationary state (Theorem 2).

1. Introduction.

We consider the one-dimensional motion of the polytropic ideal gas with inward pressed and adiabatic ends. Assuming that the reference configuration is the unit interval [0,1], the motion is described by the following three equations in the Lagrangian coordinate corresponding to the law of mass, momentum and energy:

$$u_t = v_x, \tag{1.1}$$

$$v_t + R\left(\frac{\theta}{u}\right)_x = \mu\left(\frac{v_x}{u}\right)_x, \tag{1.2}$$

$$c_V\theta_t = -R\frac{\theta v_x}{u} + \mu\frac{v_x^2}{u} + \kappa\left(\frac{\theta_x}{u}\right)_x, \tag{1.3}$$

for $(x,t) \in [0,1] \times \mathbf{R}_+$ with the initial condition:

$$(u,v,\theta)(x,0) = (u_0,v_0,\theta_0)(x), \tag{1.4}$$

and the boundary conditions:

$$\left(-R\frac{\theta}{u} + \mu\frac{v_x}{u}\right)(0,t) = \left(-R\frac{\theta}{u} + \mu\frac{v_x}{u}\right)(1,t) = -P(t) < 0, \tag{1.5}$$

$$\theta_x(0,t) = \theta_x(1,t) = 0, \tag{1.6}$$

where the unknown functions (u,v,θ) are the specific volume, the velocity and the absolute temperature respectively, the given function $P(t)$ is the outer pressure, and the positive constants R, μ, c_V and κ are the gas constant, the coefficient of viscosity, the heat capacity at constant volume and the coefficient of heat conduction respectively. The suffix t or x denotes the partial differentiation with respect to the variable t or x.

For physical reasons, we assume

$$u_0 > 0, \ \ \theta_0 > 0. \tag{1.7}$$

Assume v is the velocity of the smooth solution. Integrating (1.2) over $[0,1] \times [0,t]$ by use of (1.5), we have

$$\int_0^1 v \, dx = \int_0^1 v_0 dx.$$

Since (1.1) - (1.3) are invariant with respect to adding any constant to v, without loss of generality, we may assume

$$\int_0^1 v_0 dx = 0. \tag{1.8}$$

Hence we have

$$\int_0^1 v \, dx = 0. \tag{1.9}$$

The local existence and the uniqueness of the solution of the outer pressure problem is established by Tani [9], [10] and Secchi-Valli [8] in the three-dimensional case. And in the one-dimensional case results on temporally global existence problem for the polytropic ideal gas with other boundary conditions are obtained by Kazhikhov [2], [3], Kazhikhov-Shelukhin [4] and Nagasawa [6] etc..

In this paper the global existence of solution for the problem (1.1) - (1.6) is established (Theorem 1), and the asymptotic behavior is studied under some suitable assumptions (Theorem 2). Roughly speaking, if $P(t)$ has a positive limit $\bar{P} > 0$ as $t \to +\infty$, then the solution (u,v,θ) converges to a stationary state $(\bar{u},0,\bar{\theta})$ in $W^{1,2}(0,1)$ as $t \to +\infty$, and the inner pressure $R\frac{\theta}{u}$ and the outer pressure P balance each other, *i.e.*,

$$R\frac{\bar{\theta}}{\bar{u}} = \lim_{t \to +\infty} R\frac{\theta}{u} = \lim_{t \to +\infty} P(t) = \bar{P}. \tag{1.10}$$

The convergence of the non-stationary solution to a stationary state was established in [3] in the case that the boundary condition (1.5) is replaced by

$$v(0,t) = v(1,t) = 0.$$

In this case, constants $\bar{u}$, $\bar{\theta}$ are determined by the initial value and the parameters of equations and $(\bar{u},0,\bar{\theta})$ is a stationary solution of the problem. In our case, however except the case of $P'(t) \equiv 0$, $\bar{u}$, $\bar{\theta}$ is not determined by them (see (2.3) and (2.4)) and $(\bar{u},0,\bar{\theta})$ is not a stationary solution.

Unfortunately for technical reasons, we must assume that $P(t)$ is positive-valued and $\bar{P} > 0$. From the physical point of view, $P(t)$ is a non-negative-valued function. The author showed in [6] that in the case of $P(t) \equiv 0$, the specific volume u is satisfied

$$u(x,t) \geq C(\log(1 + t))^k$$

for some constants $C > 0$, $k \geq 1$ which are determined by the initial value and the parameters of equations. Therefore in this case any global solution (u,v,θ) does not converge to a stationary state as $t \to +\infty$. (The global existence in this case was proved in [2] and [4].)

2. Main results.

For function spaces $H^{n+\alpha}$, $H_T^{\frac{n}{2}+\alpha}$ and $B_T^{\frac{n}{2}+\alpha}$ we should refer to [6, Eq. (2.2) - (2.6)]. And other spaces $W^{1,2}(0,1)$ etc. are commonly used ones.

Throughout this paper we assume that the compatibility conditions are valid if they are not written down explicitly.

We will established the following theorems:

Theorem 1. *If the initial and boundary data satisfying* (1.7) *and* (1.8) *belong to* $u_0 \in H^{1+\alpha}$, $v_0 \in H^{2+\alpha}$, $\theta_0 \in H^{2+\alpha}$ *for some* $\alpha \in (0,1)$, *and* $P(t) \in C^1$, *then there exists a temporally global and unique solution for the problem* (1.1) - (1.6) *such that this solution* (u,v,θ) *belongs to* $B_T^{\frac{1}{2}+\alpha} \times H_T^{\frac{2}{2}+\alpha} \times H_T^{\frac{2}{2}+\alpha}$ *and has generalized derivatives* u_{tt}, v_{xt}, $\theta_{xt} \in L^2((0,1) \times (0,T))$ *for any* $T \in (0,+\infty)$. *Moreover* $u > 0$ *and* $\theta > 0$.

Theorem 2. *Assume that the hypotheses of Theorem 1 hold, and that* $P(t)$ *satisfies*

$$m_P \equiv \inf_{t \in [0,+\infty)} P(t) > 0, \tag{2.1}$$

$$\mathrm{TV}(P) \equiv \int_0^{+\infty} |P'(t)| dt < +\infty. \tag{2.2}$$

Then the limits

$$\bar{P} = \lim_{t\to+\infty} P(t)\ (>0) \quad\text{and}\quad \lim_{t\to+\infty} \int_0^t P'(\tau) \int_0^1 u(x,\tau)dxd\tau$$

exist, and the solution (u,v,θ) *converges to a stationary state* $(\bar{u},0,\bar{\theta})$ *in* $W^{1,2}(0,1) \cap C[0,1]$ *as* $t \to +\infty$, *where* $\bar{u}$, $\bar{\theta}$ *are positive constants given by*

$$\bar{u} = \frac{R}{\bar{P}(c_V + R)}\left\{\int_0^1\left(\frac{1}{2}v_0^2 + c_V\theta_0 + P(0)u_0\right)dx + \int_0^{+\infty} P'(\tau)\int_0^1 u(x,\tau)dxd\tau\right\}, \tag{2.3}$$

$$\bar{\theta} = \frac{1}{c_V + R}\left\{\int_0^1\left(\frac{1}{2}v_0^2 + c_V\theta_0 + P(0)u_0\right)dx + \int_0^{+\infty} P'(\tau)\int_0^1 u(x,\tau)dxd\tau\right\}. \tag{2.4}$$

Thus $\bar{u}$ *and* $\bar{\theta}$ *satisfy* (1.10).

Remark. The constant state $(\bar{u},0,\bar{\theta})$ is a stationary solution of (1.1) - (1.6) with $P(t) = \bar{P}$.

3. Global existence (a priori estimates).

The proof of Theorem 1 is based on the local existence theorem and on the a priori estimates. However the problem is the one-dimensional case of [9], so the local existence theorem can be established in the similar method. For details, we should refer to [9].

Therefore we investigate the a priori estimates of the solution (u,v,θ) such as

$$\|u(\cdot,t)\|^{(1+\alpha)} + \|v(\cdot,t)\|^{(2+\alpha)} + \|\theta(\cdot,t)\|^{(2+\alpha)} \le C(t), \tag{3.1}$$

$$\min\left\{\min_{x\in[0,1]} u(x,t),\ \min_{x\in[0,1]} \theta(x,t)\right\} \ge C^{-1}(t). \tag{3.2}$$

From now on, C, $C(\cdot)$, $C(\cdot,\cdot)$ etc. denote positive constants depending on (their argument(s) and) possibly the initial value and parameters of equations. For convenience we sometimes denote different constants by the same symbol C even in the same sentence. When different constants should be distinguished, we use the numbered symbol C_i.

Here we introduce the useful abbreviations as follows:

$$\begin{cases} m_u(t) = \min\limits_{x\in[0,1]} u(x,t),\ M_u(t) = \max\limits_{x\in[0,1]} u(x,t), \\ m_\theta(t) = \min\limits_{x\in[0,1]} \theta(x,t),\ M_\theta(t) = \max\limits_{x\in[0,1]} \theta(x,t). \end{cases}$$

In proving (3.1) and (3.2), we need several lemmas concerning the estimates of the solution and its derivatives.

In order to get the estimate (3.2), we proceed the argument under the assumption that $u > 0, \theta > 0$.

Remark. $\theta > 0$ is a consequence of $u > 0$. Indeed we apply the maximum principle [7] to (1.3). In consequence of (1.7), we conclude that

$$\begin{cases} \theta(x,t) \geq 0 & \text{on } [0,1] \times [0,+\infty), \\ \theta(x,t) > 0 & \text{on } (0,1) \times [0,+\infty). \end{cases}$$

If $\theta(0,t) = 0$ for some $t \in [0,+\infty)$, from the above inequalities and (1.6), $\theta_t(0,t) = 0$, $\theta_x(0,t) = 0$ and $\theta_{xx}(0,t) \geq 0$ hold. Therefore from (1.3),

$$\left(\mu \frac{v_x^2}{u} + \kappa \frac{\theta_{xx}}{u}\right)(0,t) = 0.$$

Hence it must be that

$$v_x(0,t) = \theta_{xx}(0,t) = 0.$$

But it is impossible because of (1.5). In the similar way, we get $\theta(1,t) > 0$.

Lemma 3.1. *We have*

$$\int_0^1 (v^2 + \theta + u)dx \leq C(t). \tag{3.3}$$

Proof. Integrating (1.1) over [0,1], we obtain

$$\left(\int_0^1 u\, dx\right)_t = v(1,t) - v(0,t).$$

Multiplying both sides of (1.2) by v, adding them to (1.3), and integrating, we have

$$\begin{aligned} \left\{\int_0^1 \left(\frac{1}{2}v^2 + c_V\theta\right)dx\right\}_t &= \left[- R\frac{\theta v}{u} + \mu\frac{v_x v}{u} + \kappa\frac{\theta_x}{u}\right]_{x=0}^{x=1} \\ &= - P(t)(v(1,t) - v(0,t)) \\ &= - P(t)\left(\int_0^1 u\, dx\right)_t. \end{aligned}$$

Therefore we have

$$\begin{aligned} &\int_0^1 \left(\frac{1}{2}v^2 + c_V\theta + P(t)u\right)dx \\ &\quad = \int_0^1 \left(\frac{1}{2}v_0^2 + c_V\theta_0 + P(0)u_0\right)dx + \int_0^t P'(\tau)\int_0^1 u(x,\tau)dxd\tau \\ &\quad \leq C + \int_0^t |(\log P(\tau))'|\cdot P(\tau)\int_0^1 u(x,\tau)dxd\tau. \end{aligned} \tag{3.4}$$

Now we apply Gronwall's lemma, and then we get

$$\int_0^1 \left(\frac{1}{2}v^2 + c_V\theta + P(t)u\right)dx \le C(t).$$

Since $P(t) > 0$, we have

$$\int_0^1 u\, dx \le C(t)P^{-1}(t) \le \tilde{C}(t). \quad \square$$

Lemma 3.2. *We have*

$$U(t) + \int_0^t V(\tau)d\tau \le C(t), \tag{3.5}$$

where

$$U(t) = \int_0^1 \left\{\frac{1}{2}v^2 + R(u - \log u - 1) + c_V(\theta - \log\theta - 1)\right\}dx, \tag{3.6}$$

$$V(t) = \int_0^1 \left(\mu\frac{v_x^2}{u\theta} + \kappa\frac{\theta_x^2}{u\theta^2}\right)dx. \tag{3.7}$$

Proof. From (1.1) - (1.3), (1.5) and (1.6), it is easy to show the following identity:

$$U'(t) + V(t) = (R - P(t))\left(\int_0^1 u\, dx\right)_t.$$

Therefore, by Lemma 3.1, we have

$$\begin{aligned} U(t) + \int_0^t V(\tau)d\tau &= U(0) + R\int_0^1 (u - u_0)dx \\ &- P(t)\int_0^1 u\, dx + P(0)\int_0^1 u_0 dx + \int_0^t P'(\tau)\int_0^1 u(x,\tau)dxd\tau \le C(t). \quad \square \end{aligned}$$

Lemma 3.3. *We have*

$$u(x,t) = \frac{1}{B(x,t)Y(t)}\left(u_0(x) + \int_0^t \frac{R}{\mu}\theta(x,\tau)B(x,\tau)Y(\tau)d\tau\right) \tag{3.8}$$

where

$$B(x,t) = \exp\left\{\frac{1}{\mu}\int_0^x (v_0(\xi) - v(\xi,t))d\xi\right\}, \tag{3.9}$$

$$Y(t) = \exp\left\{\frac{1}{\mu}\int_0^t P(\tau)d\tau\right\}. \tag{3.10}$$

Proof. Integrating (1.2) over $[0,x] \times [0,t]$, and taking exponent, we get

$$\frac{1}{u(x,t)}\exp\left(\frac{R}{\mu}\int_0^t \frac{\theta(x,\tau)}{u(x,\tau)}d\tau\right) = \frac{1}{u_0(x)}B(x,t)Y(t).$$

To calculate the exponent part of the left-hand side, we multiply by $\frac{R}{\mu}\theta$, and integrate in t. Thus, we have

$$\exp\left(\frac{R}{\mu}\int_0^t \frac{\theta}{u}\, d\tau\right) = 1 + \frac{1}{u_0}\int_0^t \frac{R}{\mu}\theta\cdot B\cdot Y\, d\tau. \quad \square$$

Lemma 3.4. *We have*

$$M_\theta(t) \le C(t)(1 + M_u(t)V(t)), \tag{3.11}$$

$$m_\theta(t) \ge C^{-1}\left(1 + \int_0^t \frac{d\tau}{m_u(\tau)}\right)^{-1}. \tag{3.12}$$

Proof. For any $t \ge 0$, there exists $y(t) \in [0,1]$ such that

$$\theta(y(t),t) = \int_0^1 \theta\, dx.$$

Therefore we have

$$\theta^{\frac{1}{2}}(x,t) = \theta^{\frac{1}{2}}(y(t),t) + \int_{y(t)}^x \frac{\theta_x(\xi,t)}{2\theta^{\frac{1}{2}}(\xi,t)} d\xi$$

$$\le \left(\int_0^1 \theta\, dx\right)^{\frac{1}{2}} \left\{1 + \frac{1}{2}\left(M_u(t)\int_0^1 \frac{\theta_x^2}{u\theta^2} dx\right)^{\frac{1}{2}}\right\}.$$

Using Lemma 3.1, we obtain (3.11).

Setting $\theta^{-1} = \omega$, we rewrite (1.3) as follows:

$$c_V\omega_t = \kappa\left(\frac{\omega_x}{u}\right)_x - \left\{2\kappa\frac{\theta}{u}\omega_x^2 + \frac{\mu}{u}\left(\omega v_x - \frac{R}{2\mu}\right)^2\right\} + \frac{R^2}{4\mu u}$$

$$\le \kappa\left(\frac{\omega_x}{u}\right)_x + \frac{R^2}{4\mu u}.$$

We multiply both sides by $2r\omega^{2r-1}$ $(r > \frac{1}{2})$, and integrate over [0,1]. Since $\omega_x = 0$ at $x = 0$ and 1, we get the following estimate through the integration by parts and Hölder's inequality,

$$2c_V r\left(\int_0^1 \omega^{2r}dx\right)^{1-\frac{1}{2r}}\left\{\left(\int_0^1 \omega^{2r}dx\right)^{\frac{1}{2r}}\right\}_t$$

$$\le -\kappa\int_0^1 2r(2r-1)\omega^{2r-2}\frac{\omega_x^2}{u}dx + 2c_V r\left(\int_0^1 \omega^{2r}dx\right)^{1-\frac{1}{2r}} \cdot \frac{R^2}{4\mu c_V}\left\{\int_0^1 \left(\frac{1}{u}\right)^{2r} dx\right\}^{\frac{1}{2r}}.$$

Majorizing the first term of the right-hand side by zero, dividing both sides by $2c_V r\left(\int_0^1 \omega^{2r}dx\right)^{1-\frac{1}{2r}}$, and integrating over [0,t], we have

$$\left(\int_0^1 \omega^{2r}dx\right)^{\frac{1}{2r}} \le C\left(1 + \int_0^t \frac{d\tau}{m_u(\tau)}\right).$$

Putting $r \to \infty$, we get (3.12). □

Lemma 3.5. *We have*

$$M_u(t) + \int_0^t M_\theta(\tau)d\tau \le C(t), \tag{3.13}$$

$$m_u(t),\ m_\theta(t) \ge C^{-1}(t). \tag{3.14}$$

Proof. From the assumption on $P(t)$, and Lemma 3.1, we have

$$0 < C^{-1}(t) \le B(x,t),\ Y(t) \le C(t).$$

Therefore, using (3.8), (3.11) and (3.12), we have (3.14) and the estimate

$$M_u(t) \le C(t)\Big(1 + \int_0^t M_u(\tau)V(\tau)d\tau\Big).$$

Taking into account of (3.5), we get

$$M_u(t) \le C(t)$$

by Gronwall's lemma. Using (3.11) and (3.5), we have

$$\int_0^t M_\theta(\tau)d\tau \le C(t). \quad \square$$

Lemma 3.6. *We have*

$$\int_0^1 (v^4 + \theta^2)dx + \int_0^t\int_0^1 (\theta_x^2 + v^2v_x^2)dxd\tau \le C(t). \tag{3.15}$$

Proof. Setting $w = \frac{1}{2}v^2 + c_V\theta$, we rewrite (1.2) and (1.3) in the following form:

$$w_t = \Big(-R\frac{\theta v}{u} + \mu\frac{vv_x}{u} + \kappa\frac{\theta_x}{u}\Big)_x.$$

Multiplying both sides by w, and integrating over $[0,1]$, we have

$$\begin{aligned}
&\Big(\int_0^1 \frac{1}{2}w^2dx\Big)_t + \kappa c_V\int_0^1 \frac{\theta_x^2}{u}dx \\
&= \int_0^1\Big\{\Big(R\frac{\theta v}{u} - \mu\frac{vv_x}{u}\Big)(vv_x + c_V\theta_x) - \kappa\frac{\theta_x vv_x}{u}\Big\}dx \\
&\quad - P(t)\Big\{v(1,t)\Big(\frac{1}{2}v^2(1,t) + c_V\theta(1,t)\Big) - v(0,t)\Big(\frac{1}{2}v^2(0,t) + c_V\theta(0,t)\Big)\Big\} \\
&\le \frac{\kappa c_V}{2}\int_0^1\frac{\theta_x^2}{u}dx + C\int_0^1\Big(\frac{\theta^2v^2}{u} + \frac{v^2v_x^2}{u}\Big)dx \\
&\quad + CP(t)(|v^3(1,t)| + |v^3(0,t)| + |v(1,t)\theta(1,t)| + |v(0,t)\theta(0,t)|).
\end{aligned}$$

Taking into account of (1.9), $|v^3|$ can be majorized as follows:

$$|v^3| \le 3\int_0^1 v^2|v_x|dx \le \Big(\epsilon\int_0^1 v^2v_x^2dx + C(\epsilon)\int_0^1 v^2dx\Big) \tag{3.16}$$

for any $\epsilon > 0$. From (3.13), (3.3) and (3.7), we have

$$|v| \le \int_0^1 |v_x| dx \le \left(\int_0^1 \frac{u\theta}{\mu} dx\right)^{\frac{1}{2}} \left(\int_0^1 \mu \frac{v_x^2}{u\theta} dx\right)^{\frac{1}{2}} \le C(t) V^{\frac{1}{2}}(t), \tag{3.17}$$

$$\theta \le \int_0^1 \theta\, dx + \int_0^1 |\theta_x| dx \le C(t) + \left(\int_0^1 \theta_x^2 dx\right)^{\frac{1}{2}}.$$

Therefore we have

$$|v\theta| \le C(t) V^{\frac{1}{2}}(t) + \epsilon \int_0^1 \theta_x^2 dx + C(\epsilon, t) V(t)$$

for any $\epsilon > 0$. Taking ϵ sufficiently small, we have

$$\left(\int_0^1 w^2 dx\right)_t + \int_0^1 \theta_x^2 dx \le C(t) \left\{1 + V(t) + \int_0^1 \left(\frac{\theta^2 v^2}{u} + \frac{v^2 v_x^2}{u}\right) dx\right\}. \tag{3.18}$$

Here we use (3.13). Multiplying (1.2) by v^3, and integrating over [0,1], we have

$$\begin{aligned} \left(\int_0^1 \frac{1}{4} v^4 dx\right)_t &= -P(t)\{v^3(1,t) - v^3(0,t)\} + 3\int_0^1 \left(R\frac{\theta}{u} - \mu \frac{v_x}{u}\right) v^2 v_x dx \\ &\le C(t) \left(V(t) + \int_0^1 \frac{\theta^2 v^2}{u} dx\right) - 2\mu \int_0^1 \frac{v_x^2 v^2}{u} dx. \end{aligned} \tag{3.19}$$

Here we use (3.16) with ϵ sufficiently small and (3.17). From this estimate (3.18) and (3.17), we get

$$\begin{aligned} \int_0^1 (v^4 + \theta^2) dx + \int_0^t \int_0^1 (\theta_x^2 + v_x^2 v^2) dx d\tau &\le C(t) \left(1 + \int_0^t \max_{x \in [0,1]} v^2(x,\tau) \int_0^1 \theta^2 dx\right) \\ &\le C(t) \left(1 + \int_0^t V(\tau) \int_0^1 \theta^2 dx d\tau\right). \end{aligned}$$

Applying Gronwall's lemma, we get (3.15). □

Lemma 3.7. *We have*

$$\int_0^1 u_x^2 dx \le C(t). \tag{3.20}$$

Proof. Differentiating (3.8) with respect to x, we get an expression of u_x. From it, we have

$$\int_0^1 u_x^2 dx \le C(t) \left[\int_0^1 v^2 dx + 1 + \int_0^t \left\{\int_0^1 \theta_x^2 dx + M_\theta \int_0^1 \theta(1 + v^2) dx\right\} d\tau\right] \le C(t). \quad \square$$

Lemma 3.8. *We have*

$$\int_0^t \int_0^1 v_x^2 dx d\tau \le C(t). \tag{3.21}$$

Proof. Multiplying (1.2) by v, and integrating over [0,1], we have

$$\begin{aligned} \left(\int_0^1 \frac{1}{2} v^2 dx\right)_t + P(t) \left(\int_0^1 u\, dx\right)_t + \mu \int_0^1 \frac{v_x^2}{u} dx \\ = R \int_0^1 \frac{\theta v_x}{u} dx \le \frac{\mu}{2} \int_0^1 \frac{v_x^2}{u} dx + C \int_0^1 \frac{\theta^2}{u} dx. \end{aligned}$$

Therefore, we have

$$\int_0^1 \frac{1}{2}v^2dx + P(t)\int_0^1 u\,dx + \frac{\mu}{2}\int_0^t\int_0^1 \frac{v_x^2}{u}dx$$
$$\le \int_0^1 \frac{1}{2}v_0^2dx + P(0)\int_0^1 u_0dx + \int_0^t|P'(\tau)|\int_0^1 u\,dxd\tau + C\int_0^t\int_0^1 \frac{\theta^2}{u}dxd\tau$$
$$\le C(t),$$

from (3.3) and (3.13) - (3.15). □

Lemma 3.9. *We have*

$$\int_0^1 v_x^2dx + \int_0^t\int_0^1 v_{xx}^2dxd\tau \le C(t)\left[1 + \left\{\int_0^t v_t^2(0,\tau)d\tau\right\}^{\frac{1}{2}} + \left\{\int_0^t v_t^2(1,\tau)d\tau\right\}^{\frac{1}{2}}\right]. \tag{3.22}$$

Proof. Multiplying (1.2) by v_{xx}, and integrating over [0,1], we have

$$\left(\int_0^1 \frac{1}{2}v_x^2dx\right)_t + \int_0^1 \mu\frac{v_{xx}^2}{u}dx$$
$$= \int_0^1\left(R\frac{\theta_xv_{xx}}{u} - R\frac{u_x\theta v_{xx}}{u^2} + \mu\frac{v_xu_xv_{xx}}{u^2}\right)dx + v_x(1,t)v_t(1,t) - v_x(0,t)v_t(0,t)$$
$$\le \int_0^1 \frac{\mu}{2}\frac{v_{xx}^2}{u}dx + C(t)\left[\int_0^1 \theta_x^2dx + \left(M_\theta^2(t) + \max_{x\in[0,1]} v_x^2\right)\int_0^1 u_x^2dx\right]$$
$$+ v_x(1,t)v_t(1,t) - v_x(0,t)v_t(0,t). \tag{3.23}$$

From (3.15), we have

$$\int_0^t M_\theta^2(\tau)d\tau \le \int_0^t\left(\int_0^1 \theta^2dx + \int_0^1 2\theta|\theta_x|dx\right)d\tau \le C\int_0^t\int_0^1(\theta^2 + \theta_x^2)dxd\tau \le C(t). \tag{3.24}$$

Integrating (3.23) over [0,t], we get

$$\int_0^1 v_x^2dx + \int_0^t\int_0^1 v_{xx}^2dx \le C(t)\left[1 + \int_0^t \max_{x\in[0,1]} v_x^2(x,\tau)d\tau\right.$$
$$\left. + \left\{\int_0^t v_t^2(1,\tau)d\tau\int_0^t v_x^2(1,\tau)d\tau\right\}^{\frac{1}{2}} + \left\{\int_0^t v_t^2(0,\tau)d\tau\int_0^t v_x^2(0,\tau)d\tau\right\}^{\frac{1}{2}}\right], \tag{3.25}$$

from (3.20) and (3.24). We must evaluate the right-hand side of (3.25). From (3.21), we have

$$\int_0^t \max_{x\in[0,1]} v_x^2(x,\tau)d\tau \le \int_0^t\left\{\int_0^1 v_x^2dx + \int_0^1 2|v_xv_{xx}|dx\right\}d\tau$$
$$\le \epsilon\int_0^t\int_0^1 v_{xx}^2dxd\tau + C(\epsilon)\int_0^t\int_0^1 v_x^2dxd\tau$$
$$\le \epsilon\int_0^t\int_0^1 v_{xx}^2dxd\tau + C(\epsilon,t). \tag{3.26}$$

From (1.5), we have

$$\int_0^t v_x^2(1,\tau)d\tau \le C\int_0^t(P^2(\tau)M_u^2(\tau) + M_\theta^2(\tau))d\tau \le C(t). \tag{3.27}$$

In the same way, we have

$$\int_0^t v_x^2(0,\tau)d\tau \le C(t). \tag{3.28}$$

From (3.25), (3.26) with sufficiently small ϵ, and (3.27) - (3.28), we get the assertion. □

Lemma 3.10. *For any* $\epsilon > 0$, *we have*

$$\int_0^1 \theta_x^2 dx + \int_0^t\int_0^1 \theta_{xx}^2 dxd\tau \le \epsilon\int_0^t(v_t^2(1,\tau) + v_t^2(0,\tau))d\tau + C(\epsilon,t). \tag{3.29}$$

Proof. Multiplying (1.3) by θ_{xx}, and integrating over [0,1], we have

$$\begin{aligned}
&\left\{\int_0^1 \frac{c_V}{2}\theta_x^2 dx\right\}_t + \int_0^1 \kappa\frac{\theta_{xx}^2}{u}dx \\
&= \int_0^1\left(R\frac{\theta v_x\theta_{xx}}{u} - \mu\frac{v_x^2\theta_{xx}}{u} + \kappa\frac{u_x\theta_x\theta_{xx}}{u^2}\right)dx \\
&\le \int_0^1 \frac{\kappa}{2}\,\frac{\theta_{xx}^2}{u}dx + C(t)\left\{\max_{x\in[0,1]} v_x^2(x,t)\int_0^1(\theta^2 + v_x^2)dx + \max_{x\in[0,1]} \theta_x^2\int_0^1 u_x^2 dx\right\}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\int_0^1 \theta_x^2 + \int_0^t\int_0^1\theta_{xx}^2 dxd\tau \\
&\le C(t)\left[1 + \int_0^t\left\{\max_{x\in[0,1]} v_x^2(x,\tau)\left(1 + \int_0^1 v_x^2 dx\right) + \max_{x\in[0,1]} \theta_x^2(x,\tau)\right\}d\tau\right].
\end{aligned} \tag{3.30}$$

Here we use (3.15) and (3.20). From (3.22) and (3.26), we have

$$\int_0^t \max_{x\in[0,1]} v_x^2 d\tau \le \epsilon_1\int_0^t(v_t^2(1,\tau) + v_t^2(0,\tau))d\tau + C(\epsilon_1,t), \tag{3.31}$$

and

$$\begin{aligned}
&\int_0^t \max_{x\in[0,1]} v_x^2(x,\tau)\cdot\int_0^1 v_x^2(x,\tau)dxd\tau \\
&\le \int_0^t \max_{x\in[0,1]} v_x^2(x,\tau)d\tau\cdot \max_{\tau\in[0,t]}\int_0^1 v_x^2(x,\tau)dx \\
&\le \left\{\epsilon_2\int_0^t\int_0^1 v_{xx}^2 dxd\tau + C(\epsilon_2,t)\right\} \\
&\quad\times C(t)\left[1 + \left\{\int_0^t v_t^2(1,\tau)d\tau\right\}^{\frac{1}{2}} + \left\{\int_0^t v_t^2(0,\tau)d\tau\right\}^{\frac{1}{2}}\right] \\
&\le \epsilon_2 C(t)\left[1 + \left\{\int_0^t v_t^2(1,\tau)d\tau\right\}^{\frac{1}{2}} + \left\{\int_0^t v_t^2(0,\tau)d\tau\right\}^{\frac{1}{2}}\right]^2 \\
&\quad + C(\epsilon_2,t)\left[1 + \left\{\int_0^t v_t^2(1,\tau)d\tau\right\}^{\frac{1}{2}} + \left\{\int_0^t v_t^2(0,\tau)d\tau\right\}^{\frac{1}{2}}\right]
\end{aligned}$$

$$\leq \epsilon_3 \int_0^t (v_t^2(1,\tau) + v_t^2(0,\tau))d\tau + C(\epsilon_3,t), \tag{3.32}$$

where

$$\epsilon_3 = \epsilon_2 C(t) + \text{any positive number.}$$

In the same way, from (1.6) and (3.15), we have

$$\begin{aligned}\int_0^t \max_{x\in[0,1]} \theta_x^2(x,\tau)d\tau &\leq \int_0^t\int_0^1 2|\theta_x\theta_{xx}|dxd\tau \\ &\leq \epsilon_4\int_0^t\int_0^1 \theta_{xx}^2 dxd\tau + C(\epsilon_4)\int_0^t\int_0^1 \theta_x^2 dxd\tau \\ &\leq \epsilon_4\int_0^t\int_0^1 \theta_{xx}^2 dxd\tau + C(\epsilon_4,t).\end{aligned} \tag{3.33}$$

Setting ϵ_j's sufficiently small, we get (3.29) from (3.30) - (3.33). □

Lemma 3.11. *For any* $\epsilon > 0$, *we have*

$$\int_0^t\int_0^1 \theta_t^2 dxd\tau \leq \epsilon\int_0^t (v_t^2(1,\tau) + v_t^2(0,\tau))d\tau + C(\epsilon,t). \tag{3.34}$$

Proof. From (1.3) and (3.14), we have

$$\begin{aligned}\int_0^t\int_0^1 \theta_t^2 dxd\tau \leq C(t)\int_0^t\Bigg\{&\Big(M_\theta^2(t) + \max_{x\in[0,1]} v_x^2(x,\tau)\Big)\int_0^1 v_x^2 dx \\ &+ \max_{x\in[0,1]} \theta_x^2(x,\tau)\int_0^1 u_x^2 dx + \int_0^1 \theta_{xx}^2 dx\Bigg\}d\tau.\end{aligned}$$

It yields (3.34) by use of (3.24), (3.32), (3.20), (3.33) and (3.29). □

Lemma 3.12. *The function* $v(x,t)$ *possesses a generalized derivative* $v_{xt} \in L^2((0,1)\times(0,t))$, *and we have*

$$\int_0^1 v_t^2 dx + \int_0^t\int_0^1 v_{xt}^2 dxd\tau + \int_0^t (v_t^2(1,\tau) + v_t^2(0,\tau))d\tau \leq C(t). \tag{3.35}$$

Proof. For a positive number Δt and a function $f(x,t)$,

$$\Delta f \overset{\text{def}}{=} f(x,t+\Delta t) - f(x,t).$$

From (1.2), we get

$$\left(\frac{\Delta v}{\Delta t}\right)_t = \left\{\frac{\Delta\left(-R\frac{\theta}{u} + \mu\frac{v_x}{u}\right)}{\Delta t}\right\}_x.$$

Multiplying both sides by $\frac{\Delta v}{\Delta t}$, and integrating them over [0,1], we get by the integration by parts with the boundary condition (1.5),

$$\left\{\frac{1}{2}\int_0^1\left(\frac{\Delta v}{\Delta t}\right)^2dx\right\}_t = \left[-\frac{\Delta P}{\Delta t}\frac{\Delta v}{\Delta t}\right]_{x=0}^{x=1}$$

$$+\int_0^1\left\{R\frac{1}{u(x,t+\Delta t)}\frac{\Delta\theta}{\Delta t}\frac{\Delta v_x}{\Delta t} + R\theta(x,\tau)\frac{\Delta\left(\frac{1}{u}\right)}{\Delta t}\frac{\Delta v_x}{\Delta t}\right.$$

$$\left.-\mu\frac{1}{u(x,t+\Delta t)}\left(\frac{\Delta v_x}{\Delta t}\right)^2 - \mu\frac{\Delta\left(\frac{1}{u}\right)}{\Delta t}v_x(x,\tau)\frac{\Delta v_x}{\Delta t}\right\}dx.$$

We evaluate the first term of the right-hand side as follows:

$$\left|\frac{\Delta P}{\Delta t}\right|\left|\frac{\Delta v}{\Delta t}\right| \le \max_{\tau\in[t,t+\Delta t]}|P'(\tau)|\left(\left(\frac{\Delta v}{\Delta t}\right)^2+1\right),$$

$$\left(\frac{\Delta v}{\Delta t}\right)^2 \le \int_0^1\left\{\left(\frac{\Delta v}{\Delta t}\right)^2 + 2\left|\frac{\Delta v_x}{\Delta t}\right|\left|\frac{\Delta v}{\Delta t}\right|\right\}dx$$

$$\le \epsilon\int_0^1\left(\frac{\Delta v_x}{\Delta t}\right)^2dx + C(\epsilon)\int_0^1\left(\frac{\Delta v}{\Delta t}\right)^2dx.$$

Therefore, we have

$$\left\{\int_0^1\left(\frac{\Delta v}{\Delta t}\right)^2dx\right\}_t + \int_0^1\left(\frac{\Delta v_x}{\Delta t}\right)^2dx + \left(\frac{\Delta v}{\Delta t}\right)^2(1,t) + \left(\frac{\Delta v}{\Delta t}\right)^2(0,t)$$

$$\le C(t)\left[1+\int_0^1\left\{\left(\frac{\Delta\theta}{\Delta t}\right)^2 + \left(\frac{\Delta\left(\frac{1}{u}\right)}{\Delta t}\right)^2(\theta^2+v_x^2) + \left(\frac{\Delta v}{\Delta t}\right)^2\right\}dx\right].$$

Integrating over [0,t], we get

$$\int_0^1\left(\frac{\Delta v}{\Delta t}\right)^2dx + \int_0^t\int_0^1\left(\frac{\Delta v_x}{\Delta t}\right)^2dxd\tau + \int_0^1\left\{\left(\frac{\Delta v}{\Delta t}\right)^2(1,\tau) + \left(\frac{\Delta v}{\Delta t}\right)^2(0,\tau)\right\}d\tau$$

$$\le C(t)\left[\int_0^1\left(\frac{\Delta v}{\Delta t}\right)^2dx\bigg|_{t=0} + 1\right.$$

$$\left.+\int_0^t\int_0^1\left\{\left(\frac{\Delta\theta}{\Delta t}\right)^2 + \left(\frac{\Delta v}{\Delta t}\right)^2 + \left(\frac{\Delta\left(\frac{1}{u}\right)}{\Delta t}\right)^2(\theta^2+v_x^2)\right\}dxd\tau\right].$$

By the application of Gronwall's lemma, we have

$$\int_0^t\left\{\left(\frac{\Delta v}{\Delta t}\right)^2(1,\tau) + \left(\frac{\Delta v}{\Delta t}\right)^2(0,\tau)\right\}d\tau + \int_0^1\left(\frac{\Delta v}{\Delta t}\right)^2dx + \int_0^t\int_0^1\left(\frac{\Delta v_x}{\Delta t}\right)^2dxd\tau$$

$$\leq C(t)\left[\int_0^1\left(\frac{\Delta v}{\Delta t}\right)^2 dx\bigg|_{t=0} + 1 + \int_0^t\int_0^1\left\{\left(\frac{\Delta\theta}{\Delta t}\right)^2 + \left(\frac{\Delta\left(\frac{1}{u}\right)}{\Delta t}\right)^2(\theta^2 + v_x^2)\right\}dxd\tau\right]. \tag{3.36}$$

The right-hand side is bounded when $\Delta t \to 0$:

$$\overline{\lim_{\Delta t\to 0}}\,(\text{the right-hand side})$$

$$\leq C(t)\left[\int_0^1\left\{-R\left(\frac{\theta_0}{u_0}\right)' + \mu\left(\frac{v_0'}{u_0}\right)'\right\}dx + 1 + \int_0^t\left\{\int_0^1 \theta_t^2 dx + \max_{x\in[0,1]} v_x^2\int_0^1(\theta^2 + v_x^2)dx\right\}d\tau\right]$$

$$\leq C(\epsilon,t) + \epsilon\int_0^t(v_t^2(0,\tau) + v_t^2(1,\tau))d\tau,$$

for any $\epsilon > 0$. Therefore the left-hand side of (3.36) is also meaningful when $\Delta t \to 0$, and we obtain the assertion ([5, Chap.II, Lemma 4.11]). □

From the previous lemma and (1.1), we recognize that a generalized derivative u_{tt} exists in $L^2((0,1) \times (0,t))$ for any $t > 0$.

Lemma 3.13. *We have*

$$\int_0^1(v_x^2 + \theta_x^2)dx + \int_0^t\int_0^1(v_{xx}^2 + \theta_{xx}^2 + \theta_t^2)dxd\tau \leq C(t). \tag{3.37}$$

Proof. The assertion follows from Lemmas 3.9 - 3.12. □

Hölder estimate (3.1) can be obtained from these lemmas with Sobolev's imbedding theorem, Nirenberg's interpolation theorem and Schauder estimates of the parabolic equations (see [1]).

The existence of a generalized derivative $\theta_{xt} \in L^2((0,1) \times (0,t))$ can be obtained in the similar manner to Lemma 3.12 by use of (1.3).

4. Asymptotic behavior.

In this section, we shall investigate the asymptotic behavior of the solution under the additional assumptions (2.1) and (2.2). It is easy from these to see that there exist positive numbers $\overline{P}$ and M_P such that

$$\lim_{t\to+\infty} P(t) = \overline{P}, \tag{4.1}$$

$$\sup_{t\in[0,+\infty)} P(t) = M_P. \tag{4.2}$$

Under these assumptions, looking back the proofs of Lemmas 3.1 - 3.4, we obtain

Lemma 4.1. *We have*

$$\int_0^1 (v^2 + \theta + u)dx + U(t) + \int_0^t V(\tau)d\tau \leq C, \tag{4.3}$$
$$C^{-1} \leq B(x,t) \leq C, \tag{4.4}$$
$$M_\theta(t) \leq C(1 + M_u(t)V(t)). \tag{4.5}$$

The constant C *does not depend on* t.

In consequence of this and next lemmas, constants $C(t)$ in (3.17) and (3.19) do not depend on t. Therefore when we quote these in this and next sections, we should interpret them with the replacement of $C(t)$ with C. We can get some more estimates not depending on t with some revision of the proofs.

Lemma 4.2. *We have*

$$M_u(t) \leq C, \tag{4.6}$$
$$m_u(t) \geq k. \tag{4.7}$$

Here k *is a positive constant not depending on* t *(see Remark after this lemma).*

Proof. From (3.8) and (4.5), we have

$$Y(t)M_u(t) \leq C\Big(1 + \int_0^t Y(\tau)d\tau + \int_0^t M_u(\tau)V(\tau)Y(\tau)d\tau\Big).$$

Applying Gronwall's lemma, we get

$$M_u(t) \leq CY^{-1}(t)\Big(1 + \int_0^t Y(\tau)d\tau\Big)\exp\Big(C\int_0^t V(\tau)d\tau\Big) \leq CY^{-1}(t)\Big(1 + \int_0^t Y(\tau)d\tau\Big). \tag{4.8}$$

Here we use (4.3). From the definition (3.10) of $Y(t)$ and the assumption (2.1) on $P(t)$,

$$\lim_{t\to+\infty} Y^{-1}(t) = 0, \tag{4.9}$$

and

$$\begin{aligned}\int_0^t Y^{-1}(t)Y(\tau)d\tau &= \int_0^t \exp\Big\{-\frac{1}{\mu}\int_\tau^t P(s)ds\Big\}d\tau \\ &\leq \int_0^t \exp\Big\{-\frac{m_P}{\mu}(t-\tau)\Big\}d\tau \leq \frac{\mu}{m_P}.\end{aligned} \tag{4.10}$$

From (4.8) - (4.10), we get (4.6).

To get (4.7), dividing both sides of (1.3) by θ and integrating over $[0,1] \times [0,t]$,

$$\begin{aligned}\int_0^1 (c_V\log\theta_0 + R\log u_0)dx &\leq \int_0^1 (c_V\log\theta + R\log u)dx \\ &\leq c_V\log\Big\{\int_0^1 \theta\, dx\Big\} + R\log\Big\{\int_0^1 u\, dx\Big\}.\end{aligned}$$

Here we use Jensen's inequality. Thus,

$$\theta(y(t),t) = \int_0^1 \theta\, dx \geq C\left(\int_0^1 u\, dx\right)^{-\frac{R}{c_V}} \geq C > 0. \tag{4.11}$$

Here $y(t)$ is the same as in Lemma 3.4. Therefore, we have

$$\begin{aligned}
\theta^{\frac{1}{2}}(x,t) &= \theta^{\frac{1}{2}}(y(t),t) + \int_{y(t)}^{x} \frac{\theta_x(\xi,t)}{2\theta^{\frac{1}{2}}(\xi,t)} d\xi \\
&\geq \left\{\int_0^1 \theta\, dx\right\}^{\frac{1}{2}} \left\{1 - \frac{1}{2}\left(M_u(t)\int_0^1 \frac{\theta_x^2}{u\theta^2} dx\right)^{\frac{1}{2}}\right\} \\
&\geq C - \tilde{C} V^{\frac{1}{2}}(t),
\end{aligned}$$

i.e.,

$$\theta \geq C(1 - V(t)).$$

From (3.8), we have

$$m_u(t) \geq C Y^{-1}(t)\left(1 + \int_0^t Y(\tau)d\tau - \int_0^t Y(\tau)V(\tau)d\tau\right). \tag{4.12}$$

Since $V(t)$ $(\geq 0) \in L^1(0,+\infty)$, for any $\epsilon > 0$ there exists $T_1 = T_1(\epsilon)$ such that

$$\int_{T_1}^{+\infty} V(\tau)d\tau < \epsilon. \tag{4.13}$$

When $t \geq T_1$,

$$\begin{aligned}
Y^{-1}(t)\int_0^t Y(\tau)d\tau &\geq \int_0^t \exp\left\{-\frac{1}{\mu}\int_\tau^t P(s)ds\right\}d\tau \\
&\geq \int_0^t \exp\left\{-\frac{M_P}{\mu}(t-\tau)\right\}d\tau \\
&\geq \frac{\mu}{M_P}\left\{1 - \exp\left(-\frac{M_P}{\mu}T_1\right)\right\}.
\end{aligned} \tag{4.14}$$

From (2.1), $Y(t)$ is a monotone increasing function of t, so we have

$$\begin{aligned}
0 &\leq Y^{-1}(t)\int_0^t Y(\tau)V(\tau)d\tau \\
&= Y^{-1}(t)\int_0^{T_1} Y(\tau)V(\tau)d\tau + Y^{-1}(t)\int_{T_1}^t Y(\tau)V(\tau)d\tau \\
&\leq Y^{-1}(t)Y(T_1)\int_0^{+\infty} V(\tau)d\tau + \int_{T_1}^t V(\tau)d\tau \\
&\leq C\exp\left\{-\frac{1}{\mu}\int_{T_1}^t P(\tau)d\tau\right\} + \epsilon \\
&\leq C\exp\left\{-\frac{m_P}{\mu}(t - T_1)\right\} + \epsilon,
\end{aligned} \tag{4.15}$$

from (4.3) and (4.13). From (4.12), (4.14) and (4.15), if ϵ is sufficiently small and T_2 ($\geq T_1(\epsilon)$) is sufficiently large, then for $t \geq T_2$,

$$m_u(t) \geq \frac{\mu}{2M_P}.$$

Therefore, taking into account of (3.14), we have

$$m_u(t) \geq \min\left\{\frac{\mu}{2M_P},\, C^{-1}(T_2)\right\},$$

where $C^{-1}(t)$ is the same as in (3.14). □

Remark. It seems that the constant k in the previous lemma can not be expressed explicitly by the constants R, μ, c_V, κ and the initial data.

Lemma 4.3. *We have*

$$\int_0^1 (v^4 + \theta^2)dx + \int_0^t \left\{\int_0^1 (\theta_x^2 + v^2 v_x^2)dx + \int_0^1 v^2 dx \cdot \int_0^1 v_x^2 dx\right\} d\tau \leq C(k). \tag{4.16}$$

Proof. The proof is almost the same as of Lemma 3.6. w is the same as in Lemma 3.6. From (1.2), (1.3) and (1.6), we have

$$\begin{aligned}\left(w - c_V\int_0^1 \theta\, dx\right)_t &= vv_t + c_V\theta_t - \left(c_V\int_0^1 \theta\, dx\right)_t \\ &= \left(-R\frac{\theta v}{u} + \mu\frac{vv_x}{u} + \kappa\frac{\theta_x}{u}\right)_x - \int_0^1 \left(-R\frac{\theta v_x}{u} + \mu\frac{v_x^2}{u}\right)dx.\end{aligned}$$

Therefore, by use of (2.1) and (4.2) we have

$$\begin{aligned}&\left\{\frac{1}{2}\int_0^1 \left(w - c_V\int_0^1 \theta\, d\xi\right)^2 dx\right\}_t \\ &= \int_0^1 \left(-R\frac{\theta v}{u} + \mu\frac{vv_x}{u} + \kappa\frac{\theta_x}{u}\right)_x \left(w - c_V\int_0^1 \theta\, d\xi\right)dx \\ &\quad - \int_0^1 \left(w - c_V\int_0^1 \theta\, d\xi\right)dx \cdot \int_0^1 \left(-R\frac{\theta v_x}{u} + \mu\frac{v_x^2}{u}\right)dx \\ &= P(t)\left[-\frac{1}{2}v^3(1,t) + \frac{1}{2}v^3(0,t)\right. \\ &\quad \left. + c_V v(1,t)\left\{\int_0^1 \theta\, dx - \theta(1,t)\right\} - c_V v(0,t)\left\{\int_0^1 \theta\, dx - \theta(0,t)\right\}\right] \\ &\quad - \int_0^1 \left(-R\frac{\theta v}{u} + \mu\frac{vv_x}{u} + \kappa\frac{\theta_x}{u}\right)\left(vv_x + c_V\theta_x\right)dx \\ &\quad - \int_0^1 \frac{1}{2}v^2 dx \cdot \int_0^1 \left(-R\frac{\theta v_x}{u} + \mu\frac{v_x^2}{u}\right)dx\end{aligned}$$

$$\le C\Big[|v(1,t)|^3 + |v(0,t)|^3$$

$$+ \left|v(1,t)\left\{\int_0^1 \theta\, dx - \theta(1,t)\right\}\right| + \left|v(0,t)\left\{\int_0^1 \theta\, dx - \theta(0,t)\right\}\right|\Big]$$

$$- \frac{\kappa c_V}{2}\int_0^1 \frac{\theta_x^2}{u}dx + C\int_0^1\left(\frac{\theta^2 v^2}{u} + \frac{v^2 v_x^2}{u}\right)dx$$

$$- \left(\frac{\mu}{2} - \epsilon\right)\int_0^1 v^2 dx\cdot\int_0^1 \frac{v_x^2}{u}dx + C(\epsilon)\int_0^1 v^2 dx\cdot\int_0^1 \theta^2 dx.$$

for any $\epsilon > 0$. Taking ϵ sufficiently small, and taking into account of (4.6) and (4.7), we obtain

$$\left\{\int_0^1\left(w - c_V\int_0^1 \theta d\xi\right)^2 dx\right\}_t + \int_0^1 \theta_x^2 dx + \int_0^1 v^2 dx\cdot\int_0^1 v_x^2 dx$$

$$\le C\left\{|v(1,t)|^3 + |v(0,t)|^3 + \left|v(1,t)\left\{\int_0^1 \theta - \theta(1,t)\right\}\right|\right.$$

$$\left. + \left|v(0,t)\left\{\int_0^1 \theta - \theta(0,t)\right\}\right| + \int_0^1 \frac{v^2 v_x^2}{u}dx + k\max_{x\in[0,1]} v^2\int_0^1 \theta^2 dx\right\}. \tag{4.17}$$

From (3.16) and (3.17), we have

$$|v|^3 \le \epsilon\int_0^1 v^2 v_x^2 dx + C(\epsilon)V(t), \tag{4.18}$$

and

$$\left|v\left(\int_0^1 \theta dx - \theta\right)\right| \le CV^{\frac{1}{2}}(t)\left|\int_{y(t)}^x \theta_x dx\right| \le \epsilon\int_0^1 \theta_x^2 dx + C(\epsilon)V(t), \tag{4.19}$$

where $y(t)$ is the same as in Lemma 3.4.

From (4.17) - (4.19) with ϵ sufficiently small, (3.17) and (3.19), we get

$$\left[\int_0^1\left\{\left(w - c_V\int_0^1 \theta\, d\xi\right)^2 + Cv^4\right\}dx\right]_t + \int_0^1(\theta_x^2 + v^2 v_x^2)dx + \int_0^1 v^2 dx\cdot\int_0^1 v_x^2 dx$$

$$\le CV(t)\left(1 + k\int_0^1 \theta^2 dx\right). \tag{4.20}$$

On the other hand, we have

$$\int_0^1\left(w - c_V\int_0^1 \theta\, d\xi\right)^2 dx = \int_0^1 w^2 dx - 2c_V\int_0^1 w\, dx\cdot\int_0^1 \theta\, dx + c_V^2\left(\int_0^1 \theta\, dx\right)^2. \tag{4.21}$$

From (4.3), we have

$$\int_0^1 w\, dx\cdot\int_0^1 \theta\, dx \le \left\{\int_0^1 w^2 dx\right\}^{\frac{1}{2}}\int_0^1 \theta\, dx \le \frac{1}{2}\int_0^1 w^2 dx + C. \tag{4.22}$$

Integrating (4.20) over $[0,t]$, and by use of (4.3), (4.21) and (4.22), we have

$$\int_0^1 w^2 dx + \int_0^t\left\{\int_0^1\left(\theta_x^2 + v^2 v_x^2\right)dx + \int_0^1 v^2 dx\cdot\int_0^1 v_x^2 dx\right\}d\tau$$

$$\leq C + C(k)\int_0^t V(\tau)\int_0^1 w^2 dxd\tau.$$

Applying Gronwall's lemma, we get the assertion. □

Lemma 4.4. *We have*

$$\int_0^1 u_x^2 dx + \int_0^t\int_0^1 \theta u_x^2 dxd\tau \leq C(k). \tag{4.23}$$

Proof. Multiplying (1.2) by $(\log u)_x$, and integrating over [0,1], we have

$$\begin{aligned}
&\frac{\mu}{2}\left[\int_0^1\left\{(\log u)_x\right\}^2 dx\right]_t + \int_0^1 R\frac{\theta}{u}\left(\frac{u_x}{u}\right)^2 dx \\
&= \left\{\int_0^1 v(\log u)_x dx\right\}_t - \int_0^1\left\{v\left(\frac{v_x}{u}\right)_x - R\frac{\theta_x u_x}{u^2}\right\}dx \\
&= \left\{\int_0^1 v(\log u)_x dx\right\}_t + \int_0^1\left(\frac{v_x^2}{u} + R\frac{\theta_x u_x}{u^2}\right)dx - \left[v\left(\frac{v_x}{u}\right)\right]_{x=0}^{x=1}.
\end{aligned} \tag{4.24}$$

Multiplying (1.2) by v, and integrating over [0,1], we have

$$\frac{1}{2}\left(\int_0^1 v^2 dx\right)_t = \left[\mu v\left(\frac{v_x}{u}\right)\right]_{x=0}^{x=1} - \int_0^1\left(\mu\frac{v_x^2}{u} - R\frac{\theta v u_x}{u^2} + R\frac{\theta_x v}{u}\right)dx. \tag{4.25}$$

From (4.24) and (4.25), we have

$$\begin{aligned}
&\frac{1}{2}\left[\int_0^1\left\{\left(\mu(\log u)_x\right)^2 + v^2\right\}dx\right]_t + \int_0^1 R\mu\frac{\theta}{u}\left(\frac{u_x}{u}\right)^2 dx \\
&= \mu\left\{\int_0^1 v(\log u)_x dx\right\}_t + \int_0^1 R\left(\mu\frac{\theta_x u_x}{u^2} + \frac{\theta v u_x}{u^2} - \frac{\theta_x v}{u}\right)dx.
\end{aligned}$$

Integrating over $[0,t]$ by use of Lemma 4.2, we have

$$\begin{aligned}
&\int_0^1(u_x^2 + v^2)dx + \int_0^t\int_0^1 \theta u_x^2 dxd\tau \\
&\leq C(k)\left\{1 + \int_0^1 |vu_x|dx + \int_0^t\int_0^1\left(|\theta_x u_x| + |\theta v u_x| + |\theta_x v|\right)dxd\tau\right\} \\
&\leq \frac{1}{2}\left(\int_0^1 u_x^2 dx + \int_0^t\int_0^1 \theta u_x^2 dxd\tau\right) \\
&\quad + C(k)\left\{1 + \int_0^1 v^2 dx + \int_0^t\int_0^1\left(\frac{\theta_x^2}{\theta} + \theta v^2 + \theta_x^2 + v^2\right)dxd\tau\right\}.
\end{aligned}$$

From (4.16), (4.3) and (3.17), we have

$$\begin{aligned}
&\int_0^t\int_0^1\left(\frac{\theta_x^2}{\theta} + \theta v^2 + \theta_x^2 + v^2\right)dxd\tau \\
&\leq C(k)\int_0^t\left\{\int_0^1\left(\frac{M_u\theta_x^2}{u\theta^2} + \theta_x^2\right)dx + \max_{x\in[0,1]} v^2\left(1 + \int_0^1 \theta\, dx\right)\right\}d\tau \\
&\leq C(k)\int_0^t\left(V(\tau) + \int_0^1\theta_x^2 dx\right)d\tau \leq C(k).
\end{aligned}$$