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A Willmore type Problem for $S^2 \times S^2$

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Let $\phi: M^n \to S^{n+1}(1)$ be an immersion of a compact n-manifold into the Euclidean unit $(n+1)$-sphere, and h the second fundamental form of ϕ. By $\overset{\circ}{h}$ we denote the traceless part of h i.e.,

$$\overset{\circ}{h} = h - \frac{1}{n}(\mathrm{tr}_g\, h)g,$$

where g is the metric on M induced by ϕ. Define $w(\phi)$ by

$$w(\phi) = \int_{M^n} |\overset{\circ}{h}|^n \, dv_g \quad .$$

It is easy to see that $w(\phi)$ is a conformal invariant of the immersion ϕ, that is, $w(\phi) = w(a \circ \phi)$ for any Möbius transformation $a \in \mathrm{Conf}(S^{n+1})$. The famous Willmore conjecture is stated as follows: For any immersion $\phi: T^2 \to S^3(1)$ of 2-dimensional torus into the unit 3-sphere, the following will hold: (i) $w(\phi) \geqq w(\phi_{Clifford})$, where $\phi_{Clifford}: T^2 \to S^3(1)$ is the Clifford embedding $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2}) \subset S^3(1)$: (ii) $w(\phi) = w(\phi_{Clifford})$ if and only if ϕ is conformal to $\phi_{Clifford}$, that is, there exists a Möbius transformation $a \in \mathrm{Conf}(S^3)$ such that $\phi = a \circ \phi_{Clifford}$.

We consider the same problem for $S^2 \times S^2$. In order to state our result, we introduce a conformal invariant $\nu(g)$ of a Riemannian metric g of a compact n-manifold M, which is defined by

$$\nu(g) = \int_M |W_g|^{\frac{n}{2}} dv_g ,$$

where W_g is the Weyl conformal curvature tensor of g. It is easy to see that $\nu(g)$ depends on the conformal class of the metric g.

Theorem A. *Let $\phi: S^2 \times S^2 \to S^5(1)$ be an immersion, and g is the induced metric on $S^2 \times S^2$. Assume that*

$$(*) \qquad \nu(g) \geqq \frac{256}{3}\pi^2 .$$

Then, (i) $w(\phi) \geqq w(\phi_{Clifford})$;

(ii) $w(\phi) = w(\phi_{Clifford})$ *if and only if ϕ is conformal to the Clifford embedding* $\phi_{Clifford}: S^2(1/\sqrt{2}) \times S^2(1/\sqrt{2}) \subset S^5(1)$.

The author thinks the assumption (*) may be unnecessary. We have the following result which partially supports this conjecture. We denote by $\mathcal{M}(M)$ the space of all smooth Riemannian metrics of M.

Theorem B ([4], [5]). *The functional $\nu: \mathcal{M}(S^2 \times S^2) \to \mathbb{R}$ has the following properties:* (0) $\nu(g_0) = 256\pi^2/3$ *for the standard Einstein metric* g_0; (i) *if g is a Kähler metric for some complex structure, then* $\nu(g) \geqq \nu(g_0)$; (ii) g_0 *is a "strictly" stable critical point of the functional* ν; (iii) $\nu(g) > 64\pi^2$, *if the scalar curvature of g is nonnegative.*

From (i), we can see, for example, that $\nu(g) \geqq \nu(g_0)$ for any product metric $g = g_1 + g_2$, $g_i \in \mathcal{M}(S^2)$. In (ii), "strictly" means that

$(d/dt)^2\big|_{t=0}\nu(g(t)) \geqq 0$ for any variation $g(t)$ with $g(0) = g_0$, and the equality holds only when $(d/dt)\big|_{t=0}g(t) = fg_0 + \mathcal{L}_X g_0$ for some function f and some vector field X.

§1. Total conformal curvature.

In this section, we shall give a brief review on the conformal invariant $\nu(g)$ (cf. [4], [5]).

From the conformal invariance of $\nu(g)$, we have

$$\text{(a)} \qquad \nu(g) = [\sup\{ \frac{\int |W_{\tilde g}|^2 dv_{\tilde g}}{(\int dv_{\tilde g})^{\frac{n-4}{n}}};\ \tilde g \text{ is conformal to } g\}]^{\frac{n}{4}};$$

$$\text{(b)} \qquad \nu(g) = \inf\{\mathrm{Vol}(M,\tilde g);\ \tilde g \text{ is conformal to } g, \text{ and } |W_{\tilde g}(x)| \leqq 1 \text{ for all } x \in M\}.$$

The expression (a) bears some resemblance to the Yamabe constant $\mu(g)$, which is defined by

$$\mu(g) = \inf\{ \frac{\int R_{\tilde g} dv_{\tilde g}}{(\int dv_{\tilde g})^{\frac{n-2}{n}}};\ \tilde g \text{ is conformal to } g\}.$$

For the Yamabe constant μ, it is known that $\inf\{\mu(M,g);\ g \in \mathcal{M}(M)\} = -\infty$, if $n = \dim M \geqq 3$. Correspondingly, we have

(c) $\sup\{\nu(g); g \in \mathcal{M}(M)\} = +\infty$, <u>if</u> $\dim M \geqq 4$.

This property motivates us to the following definition.

<u>Definition 1.1.</u> $\nu(M) := \inf\{\nu(g);\ g \in \mathcal{M}(M)\}$.

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From the expression (b) of $\nu(g)$, we can see

(d) $\nu(M) \leqq c_n \mathrm{Min\ Vol}(M)$ <u>for some constant</u> c_n <u>depending on</u> $n = \dim M$, <u>where</u> $\mathrm{Min\ Vol}(M) = \inf\{\mathrm{Vol}(M,g);\ |\text{sect. curv. of } g| \leqq 1\}$ ([3]).

If M admits a free S^1 action, then Min Vol(M) = 0 ([3]), hence $\nu(M) = 0$. For example, $\nu(S^2 \times S^3) = 0$. Since $S^2 \times S^3$ has no conformal flat metrics, $\nu(M) = 0$ does not always imply that M has a flat conformal structure.

As for lower bounds of $\nu(M)$, we have

(e) <u>For any Pontrjagin number</u> p, <u>there is a constant</u> c_p <u>such that</u> $|p(M)| \leqq c_p \nu(M)$.

In particular, if dim M = 4, we have

(f) $48\pi^2 |\mathrm{sgn}(M)| \leqq \nu(M)$. <u>If</u> M <u>admits a half conformally flat metric, then the equality holds</u>.

It is known that a connected sum of conformally flat manifolds admits a flat conformal structure ([6]). The following is related to this fact.

(g) $\nu(M_1 \# M_2) \leqq \nu(M_1) + \nu(M_2)$.

From (f) and (g), we can see, for example, that $\nu(kCP^2) = 48k\pi^2$. There exist M_1 and M_2 for which the strict inequality of (g) holds. For example, it is the case when $M_1 = CP^2 \# CP^2$ and $M_2 = -CP^2$.

In dimension 4, the Gauss Bonnet formula gives some information on $\nu(g)$.

(h) *Suppose that* dim M = 4 *and* $g \in \mathfrak{M}(M)$ *is a Kähler metric for some complex structure of* M. *Then*,

$$\nu(g) \geqq -16\pi^2 \mathrm{sgn}(M) + \frac{64}{3}\pi^2 \chi(M),$$

with equality implying that g *is a Kähler Einstein metric*.

From this, we see (0) and (i) of Theorem B.

(i) *If* dim M = 4, *then* $\nu(g) \geqq 32\pi^2\chi(M) - \mu(g)^2/6$,

with equality implying that g *is conformal to an Einstein metric*.

Since $S^2 \times S^2$ admits no conformally flat metrics, we have $\mu(g) < 8\sqrt{6}\pi$ for any $g \in \mathfrak{M}(S^2 \times S^2)$ (cf. [1]), which shows (iii) of Theorem B.

To show (ii) of Theorem B, we compute variational formulas for $\nu : \mathfrak{M}(M) \to \mathbb{R}$. When dim M = 4, these formulas take relatively simple form. For details, see [4].

§2. Proof of Theorem A.

In this section, we prove the following result, which is a generalization of Theorem A.

Theorem 2.1. *Let* $\phi : S^p \times S^p \to S^{2p+1}(1)$ *be an immersion, and* $g \in \mathfrak{M}(S^p \times S^p)$ *the induced metric. Assume that* $p \geqq 2$ *and*

(2.1) $$\nu(g) \geqq \nu(g_0),$$

where g_0 *is the standard Einstein metric, i.e.*, $(S^p \times S^p, g_0) = S^p(1) \times S^p(1)$. *Then*, (i) $w(\phi) \geqq w(\phi_{\mathrm{Clifford}})$; *and*

(ii) $w(\phi) = w(\phi_{\text{Clifford}})$ if and only if ϕ is conformal to the Clifford embedding $\phi_{\text{Clifford}}: S^p(1/\sqrt{2}) \times S^p(1/\sqrt{2}) \subset S^{2p+1}(1)$.

Remark. If p is odd, the assumption (2.1) looks too restrictive, since we know from (d) in §1 that $\nu(S^p \times S^p) = 0$ for odd p.

Proof. From the Gauss equation, the curvature tensor of g is given by

$$R_{ijk\ell} = g_{ik}g_{j\ell} - g_{i\ell}g_{jk} + h_{ik}h_{j\ell} - h_{i\ell}h_{jk} ,$$

where h is the second fundamental form of ϕ. From this, a direct calculation shows

(2.2) $$|W_g|^2 = 2\frac{n-2}{n-1}|\mathring{h}|^4 - \frac{2n}{n-2}|(\mathring{h}\circ\mathring{h})^\circ|^2 \leqq 2\frac{n-2}{n-1}|\mathring{h}|^4,$$

where n = 2p, and

$$\mathring{h}_{ij} = h_{ij} - \frac{1}{n}(\mathrm{tr}_g h)g_{ij} , \qquad (\mathring{h}\circ\mathring{h})^\circ_{ij} = \mathring{h}_{ik}\mathring{h}^k{}_j - \frac{1}{n}|\mathring{h}|^2 g_{ij} .$$

Hence,

(2.3) $$w(\phi) \geqq \left(\frac{2p-1}{4(p-1)}\right)^{\frac{p}{2}}\nu(g).$$

It follows from (2.2) that the equality in (2.3) occurs only when $(\mathring{h}\circ\mathring{h})^\circ \equiv 0$. This equality condition is equivalent to that at each point, either ϕ is umbilic or ϕ has two distinct principal curvatures each of which is of multiplicity p. Thus the argument used by Cecil and Ryan [2] shows that $w(\phi) = ((2p-1)/(4(p-1)))^{p/2}\nu(g)$ if and only if ϕ is conformal to the embedding

$$\phi_\alpha : S^p\left(\sqrt{\frac{1}{\alpha+1}}\right) \times S^p\left(\sqrt{\frac{\alpha}{\alpha+1}}\right) \subset S^{2p+1}(1).$$

for some α.

In particular, for the Clifford embedding $\phi_{Clifford} = \phi_1$,

$$(2.4) \quad w(\phi_{Clifford}) = \left(\frac{2p-1}{4(p-1)}\right)^{\frac{p}{2}} \nu(g_0).$$

Hence, from (2.3), (2.4) and the assumption (2.1), we have

$$w(\phi) \geqq \left(\frac{2p-1}{4(p-1)}\right)^{\frac{p}{2}} \nu(g) \geqq \left(\frac{2p-1}{4(p-1)}\right)^{\frac{p}{2}} \nu(g_0) = w(\phi_{Clifford}),$$

which proves the assertion (i).

Now, suppose $w(\phi) = w(\phi_{Clifford})$. Then by the above argument, ϕ must be conformal to ϕ_α for some α. Moreover, then $\nu(g_0) = \nu(S^p(1/\sqrt{\alpha+1}) \times S^p(\sqrt{\alpha/(\alpha+1)}))$ should hold. On the other hand, we get, by direct calculation

$$\nu(S^p(\sqrt{\frac{1}{\alpha+1}}) \times S^p(\sqrt{\frac{\alpha}{\alpha+1}})) = \{1 + \frac{(\sqrt{\alpha}-1)^2}{2\sqrt{\alpha}}\}^p \nu(g_0).$$

Therefore, $\alpha = 1$, and ϕ is conformal to $\phi_{Clifford}$. □

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