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K-theory for the C^*-algebras of discrete Heisenberg groups

by

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ABSTRACT

In the present paper we show that $K_j(C^*(G))\cong\mathbf{Z}^3$ $j=0,1$ and that $\tau_*(K_0(C^*(G)))=\mathbf{Z}$, where $C^*(G)$ is the C^*-algebra of the discrete Heisenberg group G and τ is the canonical trace on $C^*(G)$.

§1. Preliminaries.

By the discrete Heisenberg group we mean the group G defined as that of the following matrices;

$$G=\left\{\begin{bmatrix}1 & m & l\\ 0 & 1 & k\\ 0 & 0 & 1\end{bmatrix};k,l,m\in\mathbf{Z}\right\}.$$

We take two closed subgroups M and N of G as follows;

$$M=\left\{\begin{bmatrix}1 & m & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix};m\in\mathbf{Z}\right\}$$

and

$$N=\left\{\begin{bmatrix}1 & 0 & l\\ 0 & 1 & k\\ 0 & 0 & 1\end{bmatrix};k,l\in\mathbf{Z}\right\}.$$

It is clear that $M\cong\mathbf{Z}$,$N\cong\mathbf{Z}^2$,so that we may identify M with $\mathbf{Z}$ and N with $\mathbf{Z}^2$. An action of M on N is defined by

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$$m.z = mzm^{-1} = (k, l + mk)$$

for $m \in M$ and $z = (k, l) \in N$. Then G is isomorphic to the semidirect product $N \times_s M$ of N by M with the multiplication

$$(z, m)(z', m') = (z + m.z', m + m')$$

for (z, m) and $(z', m') \in N \times_s M$. Therefore we identify G with $\mathbf{Z}^2 \times_s \mathbf{Z}$ and write the element of G as (k, l, m) where $(k, l) \in \mathbf{Z}^2 = N$ and $m \in \mathbf{Z} = M$. Further by definition of crossed products and the Fourier transformation we see that $C^*(G)$ is isomorphic to the crossed product $C(T^2) \times_\alpha \mathbf{Z}$ where α is the automorphism on $C(T^2)$ defined by

$$\alpha(f)(s, t) = f(s + t, t)$$
$$f \in C(T^2), (s, t) \in T^2$$

and T^2 is the two dimensional torus.

Let τ be the finite faithful trace on $C^*(G)$ defined by $\tau(x) = x(e)$ where $x \in l^1(G)$ and e is the unit element of G , and let σ be the trace on $C(T^2) \times_\alpha \mathbf{Z}$ by $\sigma(y) = \int_{T^2} y(0, s, t)\, ds dt$ where $y \in l^1(\mathbf{Z}, C(T^2))$. Then we see easily that $\tau = \sigma$ on $l^1(G)$. In what follows, we compute

$$K_j(C(T^2) \times_\alpha \mathbf{Z})\ (j = 0, 1) \text{ and } \sigma_*(K_0(C(T^2) \times_\alpha \mathbf{Z})).$$

§2. Computation of $K_j(C(T^2) \times_\alpha \mathbf{Z})$ $j = 0, 1$

We use the following Pimsner-Voiculescu exact sequence;

$$\begin{array}{ccccc}
K^0(T^2) & \xrightarrow{id-\alpha_*^{-1}} & K^0(T^2) & \longrightarrow & K_0(C(T^2) \times_\alpha \mathbf{Z}) \\
\uparrow & & & & \downarrow \\
K_1(C(T^2) \times_\alpha \mathbf{Z}) & \longleftarrow & K^1(T^2) & \xleftarrow{id-\alpha_*^{-1}} & K^1(T^2).
\end{array}$$

$K_j(C(T^2) \times_\alpha \mathbf{Z}) \cong K^j(T^2) / Im(id - \alpha_*^{-1}) \oplus Ker(id - \alpha_*^{-1})$ $(j = 0, 1)$. We then compute $Im(id - \alpha_*^{-1})$ and $Ker(id - \alpha_*^{-1})$. Let $M_n(C(T^2))$ be the algebra of $n \times n$ matrices with entries in $C(T^2)$ and let $Proj M_n(C(T^2))$ be the set of projections of

$M_n(C(T^2))$ and let $U_n(C(T^2))$ be the unitary group of $M_n(C(T^2))$.We define p_j and q_j in $\cup_{n=1}^{\infty} Proj M_n(C(T^2))$ $j=1,2$ as follows;

$$p_1(s,t)=1$$
$$q_1(s,t)=0$$

and

$$p_2(s,t)=R(t)\begin{bmatrix} e^{-2\pi is} & 0 \\ 0 & 1 \end{bmatrix} R(t)^* \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R(t) \begin{bmatrix} e^{2\pi is} & 0 \\ 0 & 1 \end{bmatrix} R(t)^*$$

$$R(t)=\begin{bmatrix} \cos\frac{\pi}{2}t & -\sin\frac{\pi}{2}t \\ \sin\frac{\pi}{2}t & \cos\frac{\pi}{2}t \end{bmatrix}$$

$$0\le s,t\le 1$$

$$q_2(s,t)=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

And we define u_j in $\cup_{n=1}^{\infty} U_n(C(T^2))$ j=1,2 as follows;

$$u_1(s,t)=e^{2\pi it}$$
$$u_2(s,t)=e^{2\pi is}.$$

Lemma 1. 1)Tow generators of $K^0(T^2)$ are $[p_1]-[q_1]$ and $[p_2]-[q_2]$

2)Two generators of $K^1(T^2)$ are $[u_1]$ and $[u_2]$.

Remark. We identfy $C(T^2)$ with all complex valed continuous functions on $[0,1] \times [0,1]$ such that f(0,t)=f(1,t) and f(s,0)=f(s,1) for s,t$\in$[0,1].

Proof of lemma1. 1) $K^0(T^2)$ is isomorphic to $K^0(T^1) \oplus K^1(T^1)$. The isomorphism is the direct sum of i_* and Φ where i_* is the homomorphism of $K^0(T^1)$ into $K^0(T^2)$ induced by the inclusion map $i; C(T^1) \to C(T^2)$ and Φ is the composed map of the suspension map of $K^1(T^1)$ into $K^0(T^1\times(0,1))$ and the homomorphism of $K_0(T^1\times(0,1))$ into $K^0(T^2)$ induced by the inclusion map of $C_0(T^1\times(0,1))$ into $C(T^2)$. And let $[1_{T^1}]$ be a generator of $K^0(T^1)$ where 1_{T^1} is the identiy of $C(T^1)$ and let $[v]$ be a generator of $K^1(T^1)$ where v is defined by $v(s)=e^{2\pi is}$.Then $i_*([1_{T^1}])$ and $\Phi([v])$ is the generators of $K^0(T^2)$.Therefore we obtain 1).

2) We can prove 2) in the same manner as 1). *Q.E.D.*

Lemma 2.

$$K_j(C(T^2)\times_\alpha \mathbf{Z})\cong \mathbf{Z}^3 \quad j=0,1$$

Proof. We use the Pimsner-Voiculescu exact sequence. Clearly $\alpha_*^{-1}([p_1])=[p_1]$ $\alpha_*^{-1}([q_1])=[q_1]$, $\alpha_*^{-1}([q_2])=[q_2]$.

$$\alpha^{-1}(p_2)(s,t)=R(t)\begin{bmatrix} e^{-2\pi i(s-t)} & 0 \\ 0 & 1\end{bmatrix}R(t)^*\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}R(t)\begin{bmatrix} e^{2\pi i(s-t)} & 0 \\ 0 & 1\end{bmatrix}R(t)^*.$$

Let

$$V(s,t)=R(t)\begin{bmatrix} e^{2\pi i t} & 0 \\ 0 & 1\end{bmatrix}R(t)^*.$$

Then $V\in U_2(C(T^2))$ and $\alpha^{-1}(p_2)(s,t)=V(s,t)p_2(s,t)V(s,t)^*$

Thus $\alpha_*^{-1}([p_2])=[p_2]$. Therefore the homomorphism $id-\alpha_*^{-1}$ of $K^0(T^2)$ into $K^0(T^2)$ is a 0-map.

$$\alpha^{-1}(u_1)(s,t)=e^{2\pi i t}=u_1(s,t)$$
$$\alpha^{-1}(u_2)(s,t)=e^{2\pi i(s-t)}=e^{2\pi i s}e^{2\pi i t}$$
$$=u_2(s,t)u_1^*(s,t)$$

Hence $\alpha_*^{-1}([u_1])=[u_1]$, $\alpha_*^{-1}([u_2])=-[u_1]+[u_2]$. Therefore the homomorphism $id-\alpha_*^{-1}$ of $K^1(T^2)$ into $K^1(T^2)$ is given by the matrix

$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

It follows by the Pimsner-Voiculescu exact sequence that $K_j(C(T^2)\times_\alpha \mathbf{Z})=\mathbf{Z}^3$ $(j=0,1)$ Q.E.D.

Corollary 1.

$$K_j(C^*(G))\cong \mathbf{Z}^3 \quad j=0,1$$

§3. Computation of $\sigma_*(K_0(C(T^2)\times_\alpha \mathbf{Z}))$

Let $[e_j]-[f_j]$ $j=1,2,3$ be three generators of $K_0(C(T^2)\times_\alpha \mathbf{Z})$. The homomorphism i_* of $K^0(T^2)$ into $K_0(C(T^2)\times_\alpha \mathbf{Z}))$ is injective since

$$id-\alpha_*^{-1};K^0(T^2)\to K^0(T^2)$$

is a 0-map. Hence two generators are given as follows;

$$e_1(m,s,t)=\begin{cases}1 & \text{if } m=0\\ 0 & \text{if } m\neq 0\end{cases}$$

$$f_1(m,s,t)=0$$

$$e_2(m,s,t)=\begin{cases}R(t)\begin{bmatrix}e^{-2\pi is} & 0\\ 0 & 1\end{bmatrix}R(t)^*\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}R(t)\begin{bmatrix}e^{2\pi is} & 0\\ 0 & 1\end{bmatrix}R(t)^* & \text{if } m=0\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \text{if } m\neq 0\end{cases}$$

$$f_2(m,s,t)=\begin{cases}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} & \text{if } m=0\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \text{if } m\neq 0\end{cases}.$$

The generator $[e_3]-[f_3]$ is the element of $K_0(C(T^2)\times_\alpha\mathbf{Z})$ satisfying that

$$d_0([e_3]-[f_3])=[u_1]$$

where d_0 is the connecting map of $K_0(C(T^2)\times_\alpha\mathbf{Z})$ into $K^1(T^2)$.

Let g be the function on T^2 defined by $g(s,t)=\cos\frac{\pi}{2}t$ and let h be the function on T^2 defined by $h(s,t)=\sin\frac{\pi}{2}t$. We regard $C(T^2)$ as a C^*-subalgebra of $C(T^2)\times_\alpha\mathbf{Z}$. Then let

$$e_3=\begin{bmatrix}\delta_{-1} & 0\\ 0 & \delta_{-1}\end{bmatrix}\begin{bmatrix}g^2h^2 & -g^3h\\ gh^3 & -g^2h^2\end{bmatrix}+\begin{bmatrix}g^4+h^4 & g^3h-gh^3\\ g^3h-gh^3 & 2g^2h^2\end{bmatrix}+\begin{bmatrix}g^2h^2 & gh^3\\ -g^3h & -g^2h^2\end{bmatrix}\begin{bmatrix}\delta_1 & 0\\ 0 & \delta_1\end{bmatrix}$$

and

$$\delta_1(m)=\begin{cases}1_{T^2} & \text{if } m=1\\ 0 & \text{if } m\neq 1\end{cases}$$

$$\delta_{-1}(m)=\delta_1^*(m)=\begin{cases}1_{T^2} & \text{if } m=-1\\ 0 & \text{if } m\neq -1\end{cases}.$$

We see that e_3 is a *Rieffel projection* in $M_2(C(T^2)\times_\alpha\mathbf{Z})$ by computation.

Remark. Let A be a unital C^*-algebra and $(A,\mathbf{Z},\beta)$ a C^*-dynamical system. A projection in $A\times_\beta\mathbf{Z}$ satisfying the following condition is called a *Rieffel projection*;

1) $p=u^*x_1^*+x_0+x_1u\quad x_0,x_1\in A$

2)u is a unitary element in A satisfying that $Adu=\beta$.

Lemma 3. With the above notation let ε be the left support projection of x_1 in the enveloping von Neumann algebra of A .Then the unitary $exp(2\pi i x_0 \varepsilon)$ is in A and

$$d_0([p])=[exp(2\pi i x_0 \varepsilon)]$$

where d_0 is the connecting map of $K_0(A\times_\beta \mathbf{Z})$ into $K_1(A)$.

Proof. See Pimsner-Voiculescu [4]. Q.E.D.

Lemma 4. $[e_3]-[f_3]$ is a generator of $K_0(C(T^2)\times_\alpha \mathbf{Z})$ where

$$f_3(m,s,t)=\begin{cases}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} & \text{if } m=0\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \text{if } m\neq 0\end{cases}.$$

Proof. It is clear that $d_0([f_3])=0$. So we show that $d_0([e_3])=[u_1]$. Let

$$x_0=\begin{bmatrix}g^4+h^4 & g^3h-gh^3\\ g^3h-gh^3 & 2g^2h^2\end{bmatrix}$$

$$x_1=\begin{bmatrix}g^2h^2 & gh^3\\ -g^3h & -g^2h^2\end{bmatrix}.$$

Let ε be the left support projection of x_1 in the enveloping von Neumann algebra of $C(T^2)$.Since $\varepsilon=[x_1]=[x_1x_1^*]=s-lim_{n\to\infty}(\frac{1}{n}+x_1x_1^*)^{-1}x_1x_1^*$,where $[x_1]$ and $[x_1x_1^*]$ are the range projections of x_1 and $x_1x_1^*$ respectively,by the trivial calculation we see that

$$\varepsilon(s,t)=\begin{cases}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \text{if } t=0\\ \begin{bmatrix}h^2(s,t) & -g(s,t)h(s,t)\\ -g(s,t)h(s,t) & g^2(s,t)\end{bmatrix} & \text{if } 0<t\leq 1\end{cases}.$$

Hence we obtain that

$$exp(2\pi i x_0\varepsilon)=exp(2\pi i h^2\begin{bmatrix}h^2 & -gh\\ -gh & g^2\end{bmatrix}).$$

Let

$$F(c,s,t)=exp(2\pi i h^2(s,t)\begin{bmatrix} h^2(s,ct) & -g(s,ct)h(s,ct) \\ -g(s,ct)h(s,ct) & g^2(s,ct) \end{bmatrix}).$$

$$0\leq c\leq 1$$

Then

$$F=(c,s,0)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F(c,s,1)=exp(2\pi i\begin{bmatrix} h^2(s,c) & -g(s,c)h(s,c) \\ -g(s,c)h(s,c) & g^2(s,c) \end{bmatrix})$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\Sigma_{n=1}^{\infty}\frac{(2\pi i)^n}{n!}\begin{bmatrix} h^2(s,c) & -g(s,c)h(s,c) \\ -g(s,c)h(s,c) & g^2(s,c) \end{bmatrix}^n$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\Sigma_{n=1}^{\infty}\frac{(2\pi i)^n}{n!}\begin{bmatrix} h^2(s,c) & -g(s,c)h(s,c) \\ -g(s,c)h(s,c) & g^2(s,t) \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+(e^{2\pi i}-1)\begin{bmatrix} h^2(s,c) & -g(s,c)h(s,c) \\ -g(s,c)h(s,c) & g^2(s,c) \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

since $\begin{bmatrix} h^2(s,c) & -g(s,c)h(s,c) \\ -g(s,c)h(s,c) & g^2(s,c) \end{bmatrix}$ is a projection.

Therefore F is a continuous function of the interval $[0,1]$ into $U_2(C(T^2))$. Hence

$$d_0([e_3])=[F(1)]=[F(0)]=[e^{2\pi i h^2}]=[u_1]$$

by lemma3. Thus we obtain lemma4. Q.E.D.

Theorem 1.

$$\sigma_*(K_0(C(T^2)\times_\alpha \mathbf{Z}))=\mathbf{Z}$$

where σ_* is the homomorphism of $K_0(C(T^2)\times_\alpha \mathbf{Z})$ into $\mathbf{R}$ induced by the trace σ defined in §1.

Proof.

$$\sigma_*([e_1])=1$$
$$\sigma_*([f_1])=0$$
$$\sigma_*([e_2])=\int_0^1\int_0^1 Tr(e_2(0,s,t))dsdt=1$$
$$\sigma_*([f_2])=1$$

$$\sigma_*([e_3]) = \int_0^1\int_0^1 Tr(e_3(0,s,t))dsdt$$
$$= \int_0^1\int_0^1 (g^4(s,t)+h^4(s,t)+2g^2(s,t)h^2(s,t))dsdt = 1$$
$$\sigma_*([f_3]) = 1$$

where Tr is the canonical trace on the matrix algebra $M_2(\mathbf{C})$. Since σ_* is the homomorphism, we obtain that

$$\sigma_*(K_0(C(T^2)\times_\alpha \mathbf{Z})) = \mathbf{Z}.$$

Q.E.D.

Corollary 2.

$$\tau_*(K_0(C^*(G))) = \mathbf{Z}$$

where τ_* is the homomorphism of $K_0(C^*(G))$ into $\mathbf{R}$ induced by the trace defined in §1.

Remark. The above collorary shows that $C^*(G)$ has no nontrivial projection although it is not simple.

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