Research Report

KSTS/RR-85/005
24 April 1985

Hilbert transforms on one parameter groups of operators

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ABSTRACT

Let X be a complete locally convex space and let $\{U_t : -\infty<t<\infty\}$ be a one parameter group of uniformly bounded operators on X such that there exists $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}U_t x dt$ in X for all $x\in X$, and we assume that operator $H_{\varepsilon,N}$ $(0<\varepsilon<N<\infty)$ on X (where $H_{\varepsilon,N}x=\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{U_t x}{t}dt$ for all $x\in X$) is bounded uniformly concerning to ε and N. Then we prove that, for any $x\in X$, $Hx=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}}H_{\varepsilon,N}x$ exists in X and H is a continuous linear operator on X. Moreover, as the application of this result, we prove the generalized M.Riesz's theorem on a measure preserving flow which was first proved by M.Cotlar [1].

1. INTRODUCTION

In [1], M.Cotlar showed that M.Riesz's theorem could be extended to on a measure preserving flow as well as a real line or a circle. In this paper, more generally, we shall consider Hilbert transform on a one parameter group of operators on a complete locally convex space. For this, we define several terms and prepare some lemmas as follows.

Definition 1. *Let* $\mathbf{R}$ *be a real field and let* X *be a complete locally convex space. Then* $\{U_t; t \in \mathbf{R}\}$ *is said to be a one parameter group of operators on* X, *if the following conditions are satisfied;*

(i) U_t *is a continuous linear operator on* X *for all* $t \in \mathbf{R}$, *and* U_0 *is an identity operator on* X,

(ii) $U_t U_s = U_{t+s}$ *for all* $t, s \in \mathbf{R}$

(iii) *for any* $t \in \mathbf{R}$ *and any* $x \in X$, $(U_{t+h} - U_t)x$ *converges to* 0 *as* $h \to 0$ *in the topology of* X *(for short, in* X*)*

Definition 2. *A continuous linear operator* $H_{\varepsilon,N}$ $(0<\varepsilon<N<\infty)$ *on* X *is defined as follows;*

$$H_{\varepsilon,N}x = \frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{U_t x}{t} dt \qquad (x \in X)$$

(this integral can be well defined since a mapping $t \in \mathbf{R} \to (U_t x)/t \in X$ *is continuous on a compact set* $\{t \in \mathbf{R}; \varepsilon \le |t| \le N\}$ *). Also, if there exists* $\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} H_{\varepsilon,N}x$ *in* X, *we denote it by* Hx *and call it a Hilbert transform of* x. *And a domain of* H *(i.e.* $\{x \in X; Hx \text{ exists}\}$ *) is denoted by* $D(H)$.

Lemma 1. *Let* X *be a complete locally convex space and let* $\{U_t : t \in \mathbf{R}\}$ *be a one parameter group of operators on* X. *Let* x *be any element in* X *represented by*

$$x = \frac{1}{2\delta}\int_{-\delta}^{\delta} U_t v dt$$

where $\delta>0$ *and* $v \in X$.

Then, there exists $\lim_{\varepsilon\to 0+} H_{\varepsilon,1}x$ *in* X.

Proof. Note that

$$\lim_{h\to 0}\frac{U_h x-U_{-h}x}{2h}=\lim_{h\to 0}\frac{1}{2h}[U_h(\frac{1}{\delta}\int_{-\delta}^{\delta}U_t v dt)-U_{-h}(\frac{1}{2\delta}\int_{-\delta}^{+\delta}U_t v dt)]$$

$$=\lim_{h\to 0}\frac{1}{2h}[\frac{1}{2\delta}\int_{h-\delta}^{h+\delta}U_t v dt-\frac{1}{2\delta}\int_{-h-\delta}^{-h+\delta}U_t v dt]$$

$$=\lim_{h\to 0}\frac{1}{2\delta}[\frac{1}{2h}\int_{\delta-h}^{\delta+h}U_t v dt-\frac{1}{2h}\int_{-\delta-h}^{-\delta+h}U_t v dt]$$

$$=\frac{1}{2\delta}(U_\delta v-U_{-\delta}v)$$

Let q be any semi-norm from the system of semi-norms $\{q\}$ defining the topology of X. From above equality, there exists $\eta>0$ such that

$$q(\frac{U_h x-U_{-h}x}{2h}-\frac{U_\delta v-U_{-\delta}v}{2\delta})\leq 1 \qquad \text{for } all\ 0<|h|<\eta$$

Then we get that, for any $0<|h|<\eta$,

$$q(\frac{U_h x-U_{-h}x}{2h})\leq q(\frac{U_h x-U_{-h}x}{2h}-\frac{U_\delta v-U_{-\delta}v}{2\delta})+q(\frac{U_\delta v-U_{-\delta}v}{2\delta})$$

$$\leq 1+q(\frac{U_\delta v-U_{-\delta}v}{2\delta}).$$

Hence we see that, for any ε,ε' such that $0<\varepsilon<\varepsilon'<\eta$,

$$q(H_{\varepsilon,1}x-H_{\varepsilon',1}x)=q(\frac{1}{\pi}\int_{\varepsilon<|t|<\varepsilon'}\frac{U_t x}{t}dt)$$

$$\leq q(\frac{2}{\pi}\int_{\varepsilon}^{\varepsilon'}\frac{U_t x-U_{-t}x}{2t}dt)$$

$$\leq\frac{2(\varepsilon'-\varepsilon)}{\pi}(1+q(\frac{U_\delta x-U_{-\delta}x}{2\delta}))$$

which implies that $\{H_{\varepsilon,1}x\}_{\varepsilon>0}$ is a Cauchy net as $\varepsilon \to 0+$. Therefore ,from the completeness of X, there exists $\lim_{\varepsilon\to 0+} H_{\varepsilon,1}x$ in X. This completes the proof.

Lemma 2. *Let X be a complete locally convex space and let $\{U_t : t \in \mathbf{R}\}$ be a one parameter group of operators on X. Let x be any element X represented by*

$$x = z - \frac{1}{2T}\int_{-T}^{T} U_s z ds$$

where $T>0$ and $z \in X$ such that $\{U_t z\}$ is bounded in X uniformly for $t \in \mathbf{R}$.

Then there exists $\lim_{N\to\infty} H_{1,N}x$ in X.

Proof. Since X is complete, it is sufficient to prove that $\{H_{1,N}x\}_{N=1}^{\infty}$ is a Cauchy sequence as $N \to \infty$.

Let q be any semi-norm from the system of semi-norms $\{q\}$ defining the topology of X. Now we get that, for any N, N' such that $0<T<N<N'$,

$$q(H_{1,N'}x - H_{1,N}x)$$

$$= \frac{1}{\pi} q\left(\int_{N<|t|<N'} \frac{U_t x}{t} dt\right)$$

$$= \frac{1}{\pi} q\left(\int_{N<|t|<N'} \frac{1}{t} U_t\left(z - \frac{1}{2T}\int_{-T}^{T} U_s z ds\right) dt\right)$$

$$= \frac{1}{\pi} q\left(\frac{1}{2T}\int_{-T}^{T}\left(\int_{N}^{N'} \frac{(U_t - U_{t+s})z}{t} dt + \int_{-N'}^{-N} \frac{(U_t - U_{t+s})z}{t} dt\right) ds\right)$$

$$\leq \frac{1}{\pi} q\left(\frac{1}{2T}\int_{-T}^{T}\left(\int_{N}^{N'} \frac{(U_t - U_{t+s})z}{t} dt\right) ds\right)$$

$$+ \frac{1}{\pi} q\left(\frac{1}{2T}\int_{-T}^{T}\left(\int_{-N'}^{-N} \frac{(U_t - U_{t+s})z}{t} dt\right) ds\right)$$

$$= I_1 + I_2, \quad say.$$

Since we can, from the boundedness of $\{U_t z : t \in \mathbf{R}\}$, take $M>0$ such that $q(U_t z)<M$ for all $t \in \mathbf{R}$, we see that, for any N,N' such that $0<T\le N<N+T\le N'$,

$$I_1=\frac{1}{\pi}q(\frac{1}{2T}\int_{-T}^{T}(\int_{N}^{N'}\frac{U_t z}{t}dt-\int_{N+s}^{N'+s}\frac{U_t z}{t-s}dt)ds)$$

$$\le\frac{1}{2\pi T}q(\int_{0}^{T}(\int_{N}^{N'}\frac{U_t z}{t}dt-\int_{N+s}^{N'+s}\frac{U_t z}{t-s}dt)ds)$$

$$+\frac{1}{2\pi T}q(\int_{-T}^{0}(\int_{N}^{N'}\frac{U_t z}{t}dt-\int_{N+s}^{N'+s}\frac{U_t z}{t-s}dt)ds)$$

$$=\frac{1}{2\pi T}q(\int_{0}^{T}(\int_{N}^{N+s}\frac{U_t z}{t}dt+\int_{N+s}^{N'}(\frac{1}{t}-\frac{1}{t-s})U_t z dt-\int_{N'}^{N'+s}\frac{U_t z}{t-s}dt)ds)$$

$$+\int_{-T}^{0}(-\int_{N+s}^{N}\frac{U_t z}{t-s}dt+\int_{N}^{N'+s}(\frac{1}{t}-\frac{1}{t-s})U_t z dt+\int_{N'+s}^{N'}\frac{U_t z}{t}dt)ds)$$

$$\le\frac{M}{2\pi T}\int_{0}^{T}(\int_{N}^{N+s}\frac{1}{t}dt+\int_{N+s}^{N'}(\frac{1}{t-s}-\frac{1}{t})dt+\int_{N'}^{N'+s}\frac{1}{t-s}dt)ds$$

$$+\frac{M}{2\pi T}\int_{-T}^{0}(\int_{N+s}^{N}\frac{1}{t-s}dt+\int_{N}^{N'+s}(\frac{1}{t}-\frac{1}{t-s})dt+\int_{N'+s}^{N'}\frac{1}{t}dt)ds$$

$$\le\frac{M}{2\pi T}\int_{0}^{T}(\log\frac{N+s}{N}+\log\frac{(N'-s)(N+s)}{NN'}+\log\frac{N'+s}{N'})ds$$

$$+\frac{M}{2\pi T}\int_{-T}^{0}(\log\frac{N}{N+s}+\log\frac{(N'+s)(N-s)}{NN'}+\log\frac{N'}{N'+s})ds$$

$(*)\quad \to 0 \quad (as\ \ N,N'\to\infty)$.

Also we see, by (*), that, for any N,N' such that $0<T\le N\le N'\le N+T$ $(<N+2T)$,

$$I_1=\frac{1}{\pi}q(\frac{1}{2T}\int_{-T}^{T}(\int_{N}^{N'}\frac{(U_t-U_{t+s})z}{t}dt)ds)$$

$$\leq \frac{1}{\pi} q\left(\frac{1}{2T}\int_{-T}^{T}\left(\int_{N}^{N+2T}\frac{(U_t-U_{t+s})z}{t}dt\right)ds\right)$$

$$+\frac{1}{\pi} q\left(\frac{1}{2T}\int_{-T}^{T}\left(\int_{N'}^{N+2T}\frac{(U_t-U_{t+s})z}{t}dt\right)ds\right)$$

$$\to 0 \quad (as \ \ N,N' \to \infty)$$

From this and (*), we see that $I_1 \to 0$ as $N,N' \to \infty$. In a similar way, we can also see that $I_2 \to 0$ as $N,N' \to \infty$. Hence this implies that $\{H_{1,N}x\}_{N=1}^{\infty}$ is a Cauchy sequence in X. This completes the proof.

2. MAIN THEOREMS

Now we can show the following Theorems by Lemma 1 and Lemma2.

Theorem 1. *Let X be a complete locally convex space and let $\{U_t : t \in \mathbf{R}\}$ be a one parameter group of operators on X. Let x be any element in X represented by*

$$x = \frac{1}{2\delta}\int_{-\delta}^{\delta} U_s\left(z - \frac{1}{2T}\int_{-T}^{T} U_t z dt\right)ds + v$$

where $\delta, T > 0$, $z \in X$ and $v \in X$ such that $\{U_t z\}$ is bounded in X uniformly for $t \in \mathbf{R}$ and $U_t v = v$ for all $t \in \mathbf{R}$.

Then $\lim_{\substack{\varepsilon \to 0+ \\ N \to \infty}} H_{\varepsilon,N} x$ exists in X (i.e. $x \in D(H)$).

Proof. Since it is clear that $H_{\varepsilon,N} v = 0$, we see

$$H_{\varepsilon,N} x = H_{\varepsilon,N}\left(\frac{1}{2\delta}\int_{-\delta}^{\delta} U_s\left(z - \frac{1}{2T}\int_{-T}^{T} U_t z dt\right)ds\right)$$

$$= H_{\varepsilon,1}\left(\frac{1}{2\delta}\int_{-\delta}^{\delta} U_s\left(z - \frac{1}{2T}\int_{-T}^{T} U_t z dt\right)ds\right)$$

$$+ H_{1,N}\left(\frac{1}{2\delta}\int_{-\delta}^{\delta} U_s z ds - \frac{1}{2T}\int_{-T}^{T} U_t\left(\frac{1}{2\delta}\int_{-\delta}^{\delta} U_s z ds\right)dt\right)$$

which implies, from Lemma 1 and 2, that $\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} H_{\varepsilon,N}x$ exists in X, since $\{U_t(\frac{1}{2\delta}\int_{-\delta}^{\delta} U_s z ds)\}$ is clearly bounded in X uniformly for $t\in\mathbf{R}$

Theorem 2. *Let X be a complete locally convex space and let $\{U_t : t\in\mathbf{R}\}$ be a one parameter group of operators on X such that a set $\{x\in X$: $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} U_t x dt$ (denoted by $\bar{x}$) exists in X and $\{U_t x\}$ is bounded in X uniformly for $t\in\mathbf{R}\}$ is dense in X. Then the domain of H (denoted by $D(H)$) is a dense set in X.*

Proof. Let u be any element in X and let V be any balanced convex neighborhood of 0 in X. Then, we can take $x\in X$ and $\delta, T>0$ such that

$$u-x\in\frac{V}{3}, \qquad x-\frac{1}{2\delta}\int_{-\delta}^{\delta} U_s x ds \in \frac{V}{3}$$

and

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} U_s(\frac{1}{2T}\int_{-T}^{T} U_t x dt) ds - \bar{x} \in \frac{V}{3}.$$

Then, we see that

$$u-[\frac{1}{2\delta}\int_{-\delta}^{\delta} U_s(x-\frac{1}{2T}\int_{-T}^{T} U_t x dt) ds + \bar{x}]$$

$$=[u-x]+[x-\frac{1}{2\delta}\int_{-\delta}^{\delta} U_s x ds]+[\frac{1}{2\delta}\int_{-\delta}^{\delta} U_s(\frac{1}{2T}\int_{-T}^{T} U_t x dt) ds - \bar{x}]$$

$$(*) \qquad \in\frac{V}{3}+\frac{V}{3}+\frac{V}{3}=V$$

Also, since we can easily see that $U_t\bar{x}=\bar{x}$ for all $t\in\mathbf{R}$, we get, by Theorem 1, that

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} U_s(x-\frac{1}{2T}\int_{-T}^{T} U_t x dt)ds+\bar{x} \in D(H).$$

From this and (*), it follows that $D(H)$ is dense in X, since $u(\in X)$ and neighbourhood V of 0 in X are arbitrary.

Theorem 3. *Let X be a complete locally convex space and let $\{U_t : t\in\mathbf{R}\}$ be a one parameter group of operators on X such that a set $\{x\in X:\ \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} U_t x dt$ (denoted by $\bar{x}$) exists in X and $\{U_t x\}$ is bounded in X uniformly for $t\in\mathbf{R}\}$ is dense in X. Assume that, for any neighbourhood V of 0 in X, there exists a neighbourhood W of 0 in X such that*

$$H_{\varepsilon,N} z\in V \quad \text{for all } z\in W \text{ and } 0<\varepsilon<N<\infty.$$

Then, for any $x\in X$, there exists Hx. Moreover, H is a continuous linear operator on X.

Proof. Let x be any element in X. It is sufficient to prove that $\{H_{\varepsilon,N}x\}$ is a Cauchy net as $\varepsilon\to 0+, N\to\infty$. Let V be any balanced convex neighbourhood of 0 in X. Take a balanced convex neighbourhood W of 0 in X such that

$$H_{\varepsilon,N} z\in \frac{V}{3} \qquad \text{for all } z\in W \text{ and } 0<\varepsilon<N<\infty.$$

From Theorem 2, there exist y in $D(H)$ and $0<\varepsilon_0<N_0<\infty$ such that

$$x-y\in W$$

and

$$H_{\varepsilon,N} y-H_{\varepsilon',N'} y\in\frac{V}{3}$$

for all $\varepsilon,\varepsilon',N$ and N' such that $0<\varepsilon,\varepsilon'<\varepsilon_0$ and $N_0<N,N'<\infty$.

Hence we see that, for any $\varepsilon,\varepsilon',N$ and N' such that $0<\varepsilon,\varepsilon'<\varepsilon_0$ and $N_0<N,N'<\infty$,

$$H_{\varepsilon,N}x-H_{\varepsilon',N'}x$$

$$=(H_{\varepsilon,N}y-H_{\varepsilon',N'}y)+H_{\varepsilon,N}(x-y)-H_{\varepsilon',N'}(x-y)$$

$$\in \frac{V}{3}+\frac{V}{3}+\frac{V}{3}= V,$$

which implies that $\{H_{\varepsilon,N}x\}$ is a Cauchy net as $\varepsilon\to 0+$ and $N\to\infty$. Then we get, from the completeness of X, that Hx exists.

Next we shall prove that H is a continuous linear operator on X. Since the linearity of H trivially follows, it is sufficient to prove the continuity of H at 0 in X. Let K be any balanced convex neibourhood of 0 in X. Then there exists a balanced convex neighbourhood G of 0 in X such that

(*) $\qquad H_{\varepsilon,N}z \in \frac{K}{2}$ for *all* $z\in G$ and $0<\varepsilon<N<\infty$.

Let u be any element in G. Since $\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} H_{\varepsilon,N}u = Hu$ in X, there exist ε_1 and N_1 such that

$$Hu - H_{\varepsilon_1,N_1}u \in \frac{K}{2}$$

Hence we see, from this and (*), that

$$\begin{aligned} Hu &= Hu - H_{\varepsilon_1,N_1}u + H_{\varepsilon_1,N_1}u \\ &\in \frac{K}{2}+\frac{K}{2} = K \end{aligned}$$

which implies the continuity of H at 0 in X. Therefore we have that H is a continuous linear operator on X. This completes the proof.

Corollary 1. *Let X be a Banach space and let $\{U_t : t\in\mathbf{R}\}$ be a one parameter group of operators on X such that*

(i) *a set $\{x\in X$: $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}U_t x\,dt$ (denoted by $\bar{x}$) exists in X and $\{U_t x\}$ is bounded in X uniformly* for $t\in\mathbf{R}\}$ *is dense in X*

and

(ii) *there exists a $C>0$ such that*
$$\|H_{\varepsilon,N}x\| \le C\|x\| \qquad \text{for } all\ x\in X \text{ and } 0<\varepsilon<N<\infty .$$

Then, for any $x\in X$, there exists Hx. Moreover, H is a continuous linear operator *on X.*

Proof. The proof immediately follows from Theorem 3.

Corollary 2. *Let X be a Hilbert space and let $\{U_t : t \in \mathbf{R}\}$ be a one parameter group of unitary operators on X (i.e. $U_t^* = U^{-t}$ for all $t \in \mathbf{R}$). Then a Hilbert transfrm H is a continuous linear operator on X.*

Proof. A first part of condition (i) in Corollary 1 is satisfied in a Hilbert space from von Neumann's ergodic theorem. And a second part of condition (i) in Corollary 1 is clearly satisfied since $\|U_t\|=1$ for all $t \in \mathbf{R}$. Therefore it is sufficient to prove that the condition (ii) in Corollary 1 is satisfied in a Hilbert space X. This is assured in the following Lemma.

Lemma 3. *X and $\{U_t : t \in \mathbf{R}\}$ are defined as in Corollary 2. Then, it follows that*

$$\|H_{\varepsilon,N}x\| \leq \|x\| \qquad \text{for all } x \in X \text{ and } 0<\varepsilon<N<\infty.$$

Proof. We see, from Stone's Theorem, that

$$\|H_{\varepsilon,N}x\|^2 = \left\|\frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{U_t x}{t}dt\right\|^2$$

$$= \left\|\frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{1}{t}\left(\int_{-\infty}^{\infty} e^{it\lambda}dE(\lambda)x\right)dt\right\|^2$$

$$= \left\|\int_{-\infty}^{\infty} g_{\varepsilon,N}(\lambda)dE(\lambda)x\right\|^2$$

$$= \int_{-\infty}^{\infty} |g_{\varepsilon,N}(\lambda)|^2 d\|E(\lambda)x\|^2$$

where $\{E(\lambda):\lambda\in\mathbf{R}\}$ is a spectral family of a one parameter group of unitary operators $\{U_t : t \in \mathbf{R}\}$ and $g_{\varepsilon,N}(\lambda) = \frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{e^{i\lambda t}}{t}dt$.

Since we can easily show that $|g_{\varepsilon,N}(\lambda)| \leq 1$ for all $\lambda\in\mathbf{R}$ and $0<\varepsilon<N<\infty$, we see that

$$\|H_{\varepsilon,N}x\|^2 \leq \int_{-\infty}^{\infty} d\|E(\lambda)x\|^2$$

$$\leq \|x\|^2.$$

for all $x \in X$ and $0<\varepsilon<N<\infty$. Hence this completes the proof.

3. APPLICATION

Let (Ω, B, μ) be a σ-finite measure space and let $L^p(\Omega)$ $(1 \leq p < \infty)$ be a set of all p-ordered integrable functions on Ω with norm $\|.\|_p$. We define $\{T_t : t \in \mathbf{R}\}$ as a measure preserving flow on Ω, that is,

(i) *for any* $t \in \mathbf{R}$, T_t *is a measure preserving transformation on* Ω *and* T_0 *is an identity on* Ω,

(ii) $T_t T_s = T_{t+s}$ *for all* $t, s \in \mathbf{R}$,

(iii) *a mapping* $(t, \omega) \in \mathbf{R} \times \Omega \to T_t \omega \in \Omega$ *is measurable.*

Now we can define a one parameter group of operators $\{U_t : t \in \mathbf{R}\}$ on $L^p(\Omega)$ such that for all $f \in L^p(\Omega)$.

$$(U_t f)(\omega) = f(T_t \omega) \quad \text{for all } t \in \mathbf{R} \text{ and } \omega \in \Omega.$$

It is well known that $\{U_t : t \in \mathbf{R}\}$ satisfies all conditions in Definition 1 and that $U_t^* = U_{-t}$ for all $t \in \mathbf{R}$ in $L^2(\Omega)$

Under these preparations, we see the following Proposition in [2],

Proposition 1. *There exists a constant $C>0$ (independent of ε, N and λ) such that*

$$\mu\{\omega \in \Omega : \left| \frac{1}{\pi} \int_{\varepsilon<|t|<N} \frac{f(T_t \omega)}{t} dt \right| > \lambda\} \leq \frac{C}{\lambda} \|f\|_1$$

for all $0<\varepsilon<N<\infty$, $0<\lambda<\infty$ and all $f \in L^1(\Omega)$.

Proof. See [2].

Now we have the following generalized M.Riesz's theorem which was first proved by M.Cotlar [1]. Our proof is based on Corollary 1.

Theorem 4. *Let (Ω, B, μ) be a σ-finite measure space and let $\{T_t : t \in \mathbf{R}\}$ be a measure preserving flow on Ω. Let p be any real such that $1<p<\infty$.*

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Then, it follows that

(i) *for any* $f \in L^p(\Omega)$, *there exists* $\lim_{\substack{\varepsilon\to 0+ \\ N\to\infty}} \frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{f(T_t\omega)}{t}dt$ (*denoted by* Hf) *in the norm topology of* $L^p(\Omega)$,

(ii) H *is a continuous linear operator on* $L^p(\Omega)$.

Proof. As the previous arguments, we define a one parameter group of operators $\{U_t; t\in\mathbf{R}\}$ on $L^p(\Omega)$ such that, for any $f\in L^p(\Omega)$,

$$(U_t f)(\omega) = f(T_t\omega) \quad \text{for } all\ t\in\mathbf{R} \text{ and } \omega\in\Omega.$$

First we see, from von Neumann's and Yosida's ergodic theorem, that the first part of condition (i) in Corollary 1 is satisfied, that is, there exists $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}U_t x dt$ exists in X for all $x\in X$. And the second part of condition (i) in Corollary 1 is clearly satisfied since $\|U_t f\|_p = \|f\|_p$ for all $f\in L^p(\Omega)$. Therefore it is sufficient to show that the condition (ii) in Corollary 1 is satisfied.

By Proposition 1 and Lemma 3, there exists a constant C>0 such that, for any $0<\varepsilon<N<\infty$,

$$\mu\{\omega\in\Omega; |H_{\varepsilon,N}f| > \lambda\} \leq \frac{C}{\lambda}\|f\|_1 \qquad \text{for } all\ f\in L^1(\Omega)$$

and

$$\|H_{\varepsilon,N}f\|_2 \leq \|f\|_2 \qquad \text{for } all\ f\in L^2(\Omega).$$

This implies, from Marcinkiewicz's interpolation theorem, that, for any $1<p\leq 2$, there exists a constant $C_p > 0$ such that

$$(*) \qquad \|H_{\varepsilon,N}f\|_p \leq C_p\|f\|_p \qquad \text{for } all\ 0<\varepsilon<N<\infty \text{ and } f\in L^p(\Omega).$$

In the case of $2\leq p<\infty$, put $q=\frac{p}{p-1}$. Then, we see that, for any $0<\varepsilon<N<\infty$, $f\in L^p(\Omega)$ and $g\in L^q(\Omega)$,

$$\int_\Omega H_{\varepsilon,N}f\cdot g d\mu = \int_\Omega\left(\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{U_t f}{t}dt\right)g d\mu$$

$$=\int_\Omega\left(\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{f(T_t\omega)}{t}dt\right).g(\omega)d\mu$$

$$=\int_{\varepsilon<|t|<N}\frac{1}{\pi t}\left(\int_\Omega f(T_t\omega)g(\omega)d\mu\right)dt$$

$$=\int_{\varepsilon<|t|<N}\frac{1}{\pi t}\left(\int_\Omega f(\omega)g(T_{-t}\omega)d\mu\right)dt$$

$$=-\int_\Omega f\left(\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{U_t g}{t}dt\right)d\mu$$

$$=-\int_\Omega f.H_{\varepsilon,N}g\,d\mu$$

which implies, by Hörder's inequality and (*), that

$$\|H_{\varepsilon,N}f\|_p=sup\{|\int_\Omega H_{\varepsilon,N}f.gd\mu| \;:\; \|g\|_q\le 1\}$$

$$=sup\{|\int_\Omega f.H_{\varepsilon,N}gd\mu| \;:\; \|g\|_q\le 1\}$$

$$\le sup\{\|f\|_p\|H_{\varepsilon,N}g\|_q \;:\; \|g\|_q\le 1\}$$

$$\le C_q\|f\|_p = C_{p/(p-1)}\|f\|_p.$$

It follows, from this and (*), that, for any p such that $1<p<\infty$, there exists $C_p'>0$ such that

$$\|H_{\varepsilon,N}f\|_p\le C_p'\|f\|_p \quad \text{for } all\ 0<\varepsilon<N<\infty \text{ and } f\in L^p(\Omega)$$

This shows that the condition (ii) in Corollary 1 is satisfied. Therefore, by Corollary 1, the proof is completed.

The author wishes to express his sincere thanks to Professor S.Koizumi.

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