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AN IDEAL METRIC AND THE RATE OF CONVERGENCE TO A SELF-SIMILAR PROCESS

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1. Introduction.

Let $\{X(t), \quad t \geq 0\}$ be a self-similar process with parameter $H > 0$ in the sence that $\{X(ct)\} \stackrel{d}{=} \{c^H X(t)\}$ for any $c > 0$ and $\{Y_j, j = 1, 2, \cdots\}$ be a sequence of random variables belonging to the domain of attraction of $X(t)$ in the sence that

$$(1.1) \qquad X_n(t) \equiv n^{-H} \sum_{j=1}^{[nt]} Y_j \stackrel{d}{\Rightarrow} X(t) ,$$

where $\stackrel{d}{=}$ and $\stackrel{d}{\Rightarrow}$ denote the equality and the convergence of all finite dimensional distributions, respectively. This paper deals with the analysis of the rate of convergence of (1.1). (cf. [1], [3], [4], [6]) Especially, we shall consider the fractional stable process studied in [4], and to attack the problem we shall use the method of probability metrics, (for a general acquaintance with the method we recommend the survey paper [11].)

2. Ideal metrics and their properties.

Denote by $\mathfrak{X}$ the set of real valued random variables defined on some probability space and by $\mathcal{L}(\mathfrak{X})$ the space of laws P_X, $X \in \mathfrak{X}$. In the space $\mathcal{L}(\mathfrak{X})$ the mapping μ: $\mathcal{L}(\mathfrak{X})\times\mathcal{L}(\mathfrak{X}) \to [0, \infty]$ is called a simple probability metric $\mu(X, Y) \equiv \mu(P_X, P_Y)$ in case when it possesses the metric properties of "identification", "symmetry" and "triangle inequality". For shortness we shall say only "metric". The metric μ is called an ideal metric of order $r \geq 0$ if the following properties are fulfilled:

1) (Regularity). For any X, Y and $Z \in \mathfrak{X}$ such that Z is independent of X and Y,

$$\mu(X + Z, Y + Z) \leq \mu(X, Y) .$$

2) (Homogeneity of order $r \geq 0$). For any $X, Y \in \mathfrak{X}$ and any number $c \neq 0$,

$$\mu(cX, cY) = |c|^r \mu(X, Y) .$$

The existence of an ideal metric of a given order $r \geq 0$ was shown by Zolotarev (cf. [7]). Namely, denote by $\mathcal{F} = \{f\}$ the set of all possible real-valued continuous functions on R, and define the set

$$\mathcal{F}_r \equiv \{ f \in \mathcal{F} ; |f^{(m)}(x) - f^{(m)}(y)| \leq |x - y|^\gamma, \quad x, y \in R \},$$

where if $r > 0$, m is a nonnegative integer and $\gamma \in (0, 1]$ such that $r = m + \gamma$, and if $r = 0$, $m = \gamma = 0$, and where

$$f^{(m)}(x) = \frac{d^m}{dx^m} f(x) , \quad m = 1, 2, \cdots; \quad f^{(0)}(x) = f(x) .$$

Then for each $r \geq 0$ the functional

$$\zeta_r(X, Y) \equiv \sup\{|E(f(X) - f(Y))| ; f \in \mathcal{F}_r\}$$

is an ideal metric of order r. This is called the Zolotarev metric of order r. If $E(X^j - Y^j) = 0$, $j = 1, \cdots, m$ and $r > 0$, then

$$\zeta_r(X, Y) \leq [\Gamma(1+\gamma)/\Gamma(1+r)][m\kappa_r(X, Y) + \{\kappa_r(X, Y)\}^\gamma b_r^{1-\gamma}] ,$$

where

$$\kappa_r(X, Y) \equiv r \int_{-\infty}^{\infty} |x|^{r-1} |P(X \leq x) - P(Y \leq x)| dx$$

is the so-called difference pseudomoment of order r and $b_r \equiv b_r(X, Y) \equiv \min(E|X|^r, E|Y|^r)$, (cf. [8], [9]). If r is a positive integer, then

ζ_r can be given in the following integral form

(2.1) $$\zeta_r(X, Y) = \int_{-\infty}^{\infty} |D_r(x)| dx ,$$

where $F_X(x) = P(X \leqq x)$,

$$F_{r,X}(x) \equiv \int_{-\infty}^{x} \frac{(x-t)^{r-1}}{(r-1)!} dF_X(t) ,$$

which is the (r-1)-multiple integral of the distribtution function $F_X(x)$, and where $D_r(x) \equiv D_{r,X,Y}(x) \equiv F_{r,X}(x) - F_{r,Y}(x)$. (cf. [10].)

Next we shall introduce another ideal metric of order $r \geqq 0$ with two advantages;

(A1) it has an integral form like (2.1) for any $r \geqq 0$,

and

(A2) there exists an upper bound of the ideal metric for any $r \geqq 0$, which depends on the difference pseudomoments κ_r but does not depend on b_r.

The advantage (A2) is essential in estimating the rate of convergence of the processes related to stable random variables, where we cannot assume the finiteness of b_r.

For $s = 1, 2, \cdots$, $p \in [1, \infty]$ and $a \geqq 0$, denote

$$G_{XY}(x) \equiv |x|^a |D_{s,X,Y}(x)| ,$$

$$\zeta(X, Y; s, p, a) \equiv \|G_{XY}\|_p ,$$

where $\|\cdot\|_p$ stands for the p-norm;

$$\|g\|_p \equiv [\int_{-\infty}^{\infty} |g(x)|^p dx]^{1/p} , \quad 1 \leqq p < \infty ,$$

$$\|g\|_\infty \equiv \text{ess sup}\{|g(x)|; x \in R\} .$$

Note that

$$\kappa_r(X, Y) = r\, \zeta(X, Y; 1, 1, r-1).$$

Let

$$\mathcal{F}(s, p, a) = \{f : R \to R, \text{ continous}; \ \|f_{s,a}\|_q \leqq 1\} ,$$

where $f_{s,a}(x) = x^{-a}f^{(s)}(x)$ and $1/p + 1/q = 1$. Denote

$$\zeta(X, Y; \mathcal{F}(s, p, a)) \equiv \sup\{|E(f(X) - f(Y))| \ ; \ f \in \mathcal{F}(s, p, a)\} .$$

<u>Lemma 1</u>. <u>If</u> $\zeta(X, Y; \mathcal{F}(s, p, a)) < \infty$, <u>then</u>

(2.2) $$\zeta(X, Y; \mathcal{F}(s, p, a)) = \zeta(X, Y; s, p, a) .$$

<u>Proof</u>. Note that the finiteness of $\zeta(X, Y; \mathcal{F}(s, p, a))$ implies

(2.3) $$E(X^j - Y^j) = 0, \quad j = 0, 1, \cdots, s-1.$$

Really, if $s \geqq 2$ and (2.3) is not fulfilled, then

$$\zeta(X, Y; \mathcal{F}(s,p,a)) \geqq \sup_{c \in R} \left|\int c x^{j_0} \, d(F_X(x) - F_Y(x))\right| = \infty ,$$

where j_0 is the first j for which (2.3) is wrong. If $f \in \mathcal{F}(s, p, a)$, then by (2.3)

$$\int_{-\infty}^{\infty} f(x) d(F_X(x) - F_Y(x))$$

$$\doteq \int_{-\infty}^{\infty} [f(0) + \cdots + \frac{x^{s-1}}{(s-1)!} f^{(s-1)}(0)$$

$$+ \int_0^x \frac{(x-t)^{s-1}}{(s-1)!} f^{(s)}(t)dt] d(F_X(x) - F_Y(x))$$

$$= (-1)^s \int_{-\infty}^{0} f^{(s)}(t) D_s(t)dt + \int_0^{\infty} f^{(s)}(t)\bar{D}_s(t)dt ,$$

where $\bar{D}_s(t) \equiv \bar{D}_{s,X,Y}(t) \equiv \bar{F}_{s,X}(t) - \bar{F}_{s,Y}(t)$ and

$$\bar{F}_{s,X}(t) = \int_t^{\infty} \frac{(u-t)^{s-1}}{(s-1)!} dF_X(u) .$$

Thus, by (2.3) again,

$$\int_{-\infty}^{\infty} f(x)d(F_X(x) - F_Y(x)) = (-1)^s \int_{-\infty}^{\infty} f_{s,a}(t)t^a D_s(t)dt .$$

By the duality of $\mathcal{L}^p \equiv \{f ; \|f\|_p < \infty\}$ and $\mathcal{L}^q$ with $1/p+1/q=1$, we get (2.2). The lemma is thus concluded.

Denote

$$\theta_{s,p}(X, Y) \equiv \zeta(X, Y; \mathcal{F}(s, p, 0)), \quad \text{for } s = 1, 2, \cdots, \quad p \in [1, \infty]$$

and

$$\kappa_{s,p}(X, Y) \equiv s(s-1)[\int_{-\infty}^{0} |x|^{s-2} (\int_{-\infty}^{x} |F_X(u) - F_Y(u)|^p du)^{1/p} dx$$

$$+ \int_{0}^{\infty} x^{s-2} (\int_{x}^{\infty} |F_X(u) - F_Y(u)|^p du)^{1/p} dx] ,$$

$$\text{for } s = 2, 3, \cdots, \quad p \in [1, \infty) .$$

It is easily seen that the metric $\theta_{s,p}$ is an ideal metric of order $r = s - 1 + 1/p$. In the following, we shall state some properties of $\theta_{s,p}$, the proofs of which will be given in the next section. Note that by Lemma 1, if $\theta_{s,p}(X, Y) < \infty$, then

(2.4) $\qquad \theta_{s,p}(X, Y) = \zeta(X, Y; s, p),$

where

$$\zeta(X, Y; s, p) \equiv \zeta(X, Y; s, p, 0) = \begin{cases} [\int_{-\infty}^{\infty} |D_{s,X,Y}(x)|^p dx]^{1/p} & \text{for } 1 \leq p < \infty, \\ \sup\{|D_{s,X,Y}(x)| ; x \in R\} & \text{for } p = \infty. \end{cases}$$

This representation assures the advantage (A1), and the statement (b) in the following Theorem 1 gives us the advantage (A2).

Theorem 1. Suppose that (2.3) is fulfilled and $r = s - 1 + 1/p$.

(a) For $s = 1, 2, \cdots$ and $p \in [1, \infty]$,

(2.5) $$\theta_{s,p}(X, Y) \leqq \zeta_r(X, Y) ,$$

where $\zeta_r(X, Y)$ is the Zolotarev metric.

(b) For $p \in [1, \infty)$,

(2.6) $$\begin{cases} \theta_{1,p}(X, Y) = \{\kappa_1(X, Y)\}^{1/p} , \\ \theta_{s,p}(X, Y) \leqq \{\Gamma(1+1/p)/\Gamma(1+r)\}\kappa_r(X, Y) , \quad s = 2, 3, \cdots. \end{cases}$$

(c) For $s = 2, 3, \cdots$ and $p \in [1, \infty)$,

(2.7) $$\theta_{s,p}(X, Y) \leqq (s!)^{-1}\kappa_{s,p}(X, Y) .$$

(d) For $s = 1, 2, \cdots$, $p \in [1, \infty)$ and $p' \in [1, p]$,

(2.8) $$\theta_{s,p}(X, Y) \leqq \{\max(E|X|^{s-1}, E|Y|^{s-1})\}^{1-p'/p}\{\theta_{s,p'}(X, Y)\}^{p'/p} .$$

In Theorem 1, we suppose that (2.3) is fulfilled. The next theorem shows that (2.3) is necessary for the finiteness of the left hand side of (2.4), and that under some moment conditions (2.3) is also necessary for the finiteness of the right hand side of (2.4).

Theorem 2. Let $s = 2, 3, \cdots$.

(a) If $\theta_{s,p}(X, Y) < \infty$, then (2.3) is satisfied.

(b) If $\kappa_s(X, Y) < \infty$ and $\zeta(X, Y; s+1, \infty) < \infty$, then (2.3) holds.

(c) If $p < \infty$, $\kappa_r(X, Y) < \infty$, $\zeta(X, Y; s, p) < \infty$ and $|D_{s+1,X,Y}(a)| < \infty$ for some $a \in (-\infty, \infty)$, then (2.3) is fulfilled. In particular, if $p < \infty$, $\zeta(X, Y; s, p) < \infty$ and $E|X|^s + E|Y|^s < \infty$, then (2.3) holds.

Let $L(X, Y)$ be the Lévy metric in $\mathfrak{X}$;

$$L(X, Y) \equiv \inf\{\varepsilon > 0 \; ; \; F_X(x-\varepsilon) - \varepsilon \leqq F_Y(x) \leqq F_X(x+\varepsilon) + \varepsilon, \text{ for all } x \in R\},$$

and let

$$\omega_{s,p}(X, Y) \equiv \sup\{|E(f(X) - f(Y))| \; ; \; \|f^{(s)}\|_q \leqq 1, \; \|f\|_\infty \leqq 1\},$$

where $1/p + 1/q = 1$.

Theorem 3. For any $s = 1, 2, \cdots$ and $p \in [1, \infty]$,

(2.9) $$\theta_{s,p}(X, Y) \geqq \omega_{s,p}(X, Y) \geqq K(s)\{L(X, Y)\}^{s+1/p},$$

where $K(s) = 2/[\pi^s s^{s-1}(s+1)]$. In case $s = 1, 2, 3,$ we have more precise estimates;

(2.10) $$\theta_{1,p}(X, Y) \geqq \omega_{1,p}(X, Y) \geqq \{L(X, Y)\}^{1+1/p},$$

(2.11) $$\theta_{2,p}(X, Y) \geqq \omega_{2,p}(X, Y) \geqq (1/4)\{L(X, Y)\}^{2+1/p}$$

and

(2.12) $$\theta_{3,p}(X, Y) \geqq \omega_{3,p}(X, Y) \geqq (1/24)\{L(X, Y)\}^{3+1/p}.$$

In case $p = 1$, the inequalities (2.10)-(2.12) were proved by Grigorevski and Shiganov [2]. In (2.9) the exponent $s + 1/p$ can in general not be replaced by a smaller one, (see Remark 1 in the next section).

It is also well-known that if X has a unformly bounded density $p_X(x)$, then

(2.13) $$L(X, Y) \geqq (1 + \sup\{p_X(x) \; ; \; x \in R\})^{-1} \rho(X, Y),$$

where $\rho(X, Y)$ is the ordinary uniform metric;

$$\rho(X, Y) \equiv \sup\{|F_X(x) - F_Y(x)|; \; x \in R\}.$$

Using (2.9)-(2.13), we obtain a lower estimate of $\theta_{s,p}$ by ρ.

3. Proofs of Theorems 1 - 3.

Proof of Theorem 1. (a) Let $p < \infty$. For every $x, y \in R$ and f with $\|f^{(s)}\|_q \leqq 1$, we have

$$|f^{(s-1)}(x) - f^{(s-1)}(y)| \leqq \|f^{(s)}\|_q \, |x - y|^{1/p} \leqq |x - y|^{1/p} .$$

Since $r = s - 1 + 1/p$, then $\mathcal{F}(s, p, 0) \subset \mathcal{F}_r$ and hence (2.5) holds.

Let $p = \infty$. As in the proof of Lemma 1, we see that (2.3) implies that for $s = 2, 3, \cdots$,

$$\theta_{s,\infty}(X, Y) \leqq \sup_{t\in R}|D_s(t)| = \sup_{t\in R}\left|\int_{-\infty}^{t} D_{s-1}(x)dx\right| \leqq \int_{-\infty}^{\infty}|D_{s-1}(x)|dx$$

$$\leqq \begin{cases} \infty , & \text{if } \theta_{s-1,1}(X, Y) = \infty , \\ \theta_{s-1,1}(X, Y) = \zeta_r(X, Y) , & \text{if } \theta_{s-1,1}(X, Y) < \infty \end{cases}$$

by Lemma 1. As to $s = 1$, since $|f(x) - f(y)| \leqq \|f'\|_1$, then $\theta_{1,\infty}(X, Y) \leqq \zeta_0(X, Y)$.

(b) The first assertion is trivial. If $s \geqq 2$, then by (2.3) and (2.4),

$$\theta_{s,p}(X, Y) = [\int_{-\infty}^{\infty} |D_s(x)|^p \, dx]^{1/p} \tag{3.1}$$

$$\leqq [\int_{-\infty}^{0} |D_s(x)|^p \, dx]^{1/p} + [\int_{0}^{\infty} |\bar{D}_s(x)|^p \, dx]^{1/p} \equiv I_1 + I_2 ,$$

say. Let $I[\cdot]$ be the indicator function. Then by the Minkovski inequality

$$I_1 = [\int_{-\infty}^{0} |\int_{-\infty}^{0} D_{s-1}(u)I[u \leqq x] \, du|^p \, dx]^{1/p}$$

$$\leqq \int_{-\infty}^{0} |u|^{1/p}|D_{s-1}(u)|du \equiv J_{s,p} ,$$

say, where

$$J_{2,p} = \int_{-\infty}^{0} |u|^{1/p}|F_X(u) - F_Y(u)|du$$

and if $s \geqq 3$,

$$J_{s,p} = \int_{-\infty}^{0} |u|^{1/p} \left| \int_{-\infty}^{u} \frac{(u-t)^{s-3}}{(s-3)!} (F_X(t) - F_Y(t))dt \right| du$$

$$\leqq \{(1+\tfrac{1}{p})(2+\tfrac{1}{p})\cdots((s-2)+\tfrac{1}{p})\}^{-1} \int_{-\infty}^{0} |t|^{s-2+1/p} |F_X(t) - F_Y(t)|dt .$$

Analogously we can estimate I_2 and get (2.6).

(c) Note that

(3.2) $\quad [\int_A |\int_B \phi(u)du|^p dx]^{1/p} \leqq \int_A [\int_B |\phi(u)|^p du]^{1/p} dx , \quad p \geqq 1 .$

where $A = (-\infty, t]$ and $B = (-\infty, x]$, $x \leqq t \leqq 0$, or $A = [t, \infty)$ and $B = [x, \infty)$, $x \geqq t \geqq 0$. Using (3.1) and (3.2), we have

$$I_1 = [\int_{-\infty}^{0} |\int_{-\infty}^{t} D_{s-1}(x)dx|^p \, dt]^{1/p}$$

$$\leqq \int_{-\infty}^{0} [\int_{-\infty}^{t} |D_{s-1}(x)|^p \, dx]^{1/p} \, dt$$

$$= \int_{-\infty}^{0} [\int_{-\infty}^{t} |\int_{-\infty}^{x} D_{s-2}(u)du|^p \, dx]^{1/p} \, dt$$

$$\leqq \int_{-\infty}^{0} \int_{-\infty}^{t} [\int_{-\infty}^{x} |D_{s-2}(u)|^p \, du]^{1/p} \, dxdt$$

$$\leqq \int_{-\infty}^{0} \frac{|x|^{s-2}}{(s-2)!} [\int_{-\infty}^{x} |D_1(u)|^p \, du]^{1/p} \, dx ,$$

and analogously

$$I_2 \leqq \int_{0}^{\infty} \frac{x^{s-2}}{(s-2)!} [\int_{x}^{\infty} |\bar{D}_1(u)|^p \, du]^{1/p} \, dx .$$

So, we get (2.7).

(d) Let $B_s \equiv \max\{E|X|^{s-1}, E|Y|^{s-1}\}$. Then

$$\theta_{s,p}(X, Y) = B_s[\int_{-\infty}^{0} |B_s^{-1}D_s(x)|^p\, dx + \int_0^{\infty} |B_s^{-1}\bar{D}_s(x)|^p\, dx]^{1/p}$$

$$\leqq B_s[\int_{-\infty}^{0} |B_s^{-1}D_s(x)|^{p'}\, dx + \int_0^{\infty} |B_s^{-1}\bar{D}_s(x)|^{p'}\, dx]^{1/p}$$

$$= B_s^{1-p'/p}\{\theta_{s,p'}(X, Y)\}^{p'/p} .$$

Proof of Theorem 2. (a) As in the proof of Lemma 1, if (2.3) is not true, then $\theta_{s,p}(X, Y) = \infty$.

(b) Let

$$\kappa_s(X, Y) = s\int_{-\infty}^{\infty} |x|^{s-1}|F_X(x) - F_Y(x)|dx < \infty \tag{3.3}$$

and

$$\theta_{s+1,\infty}(X, Y) = \sup\{|\int_{-\infty}^{x} \frac{(x-t)^{s-1}}{(s-1)!} (F_X(t) - F_Y(t))dt| \ ; x \in R\} < \infty. \tag{3.4}$$

By (3.4)

$$\limsup_{x\to\infty}|\sum_{j=0}^{s-1} \binom{s-1}{j} x^{s-1-j} \int_{-\infty}^{x} (-t)^j (F_X(t) - F_Y(t))dt| < \infty$$

and by (3.3)

$$\limsup_{x\to\infty}|\int_{-\infty}^{x} (-t)^{s-1}(F_X(t) - F_Y(t))dt| \leqq \frac{1}{s}\, \kappa_s(X, Y) < \infty .$$

Thus

$$(3.5)\qquad \limsup_{x\to\infty}\left|\sum_{j=0}^{s-2}\binom{s-1}{j} x^{s-2-j}\int_{-\infty}^{x}(-t)^j(F_X(t)-F_Y(t))dt\right| = 0 .$$

Since

$$\limsup_{x\to\infty}\left|\int_{-\infty}^{x}(-t)^{s-2}(F_X(t)-F_Y(t))dt\right|$$

$$\leqq (s-1)^{-1}\nu_{s-1}(X, Y) = 2+s^{-1}\nu_s(X, Y) < \infty ,$$

by (3.5) we have

$$(3.6)\qquad \limsup_{x\to\infty}\left|\sum_{j=0}^{s-3}\binom{s-1}{j} x^{s-3-j}\int_{-\infty}^{x}(-t)^j(F_X(t)-F_Y(t))dt\right| = 0 .$$

As (3.5) and (3.6) we obtain

$$(3.7)\qquad \limsup_{x\to\infty}\left|\sum_{j=0}^{s-k}\binom{s-1}{j} x^{s-k-j}\int_{-\infty}^{x}(-t)^j(F_X(t)-F_Y(t))dt\right| = 0$$

for all $k = 2, 3, \cdots, s$. In case $k = s$, we have

$$0 = \limsup_{x\to\infty}\left|\int_{-\infty}^{x}(F_X(t)-F_Y(t))dt\right|$$

$$= \limsup_{x\to\infty}\left|\int_{-\infty}^{x} t\, d(F_X(t)-F_Y(t))\right| ,$$

namely $E(X - Y) = 0$. Put $k = s-1$ in (3.7), then we have

$$0 = \limsup_{x\to\infty}\left|x\int_{-\infty}^{x}(F_X(t)-F_Y(t))dt + (s-1)\int_{-\infty}^{x}(-t)(F_X(t)-F_Y(t))dt\right| .$$

Since

$$\limsup_{x\to\infty}\left|x\int_{-\infty}^{x}(F_X(t)-F_Y(t))dt\right|$$

$$= \limsup_{x\to\infty}\left|x\int_{x}^{\infty}(F_X(t)-F_Y(t))dt\right|$$

$$\leq \limsup_{x\to\infty} \int_x^\infty |t||F_X(t) - F_Y(t)|dt = 0 ,$$

then

$$0 = \limsup_{x\to\infty} |(s-1)\int_{-\infty}^x (-t)(F_X(t) - F_Y(t))dt|$$

and hence $E(X^2 - Y^2) = 0$. In the same way we obtain that $E(X^j - Y^j) = 0$, $j = 1, \cdots, s-1$.

(c) Let $\kappa_r(X, Y) < \infty$, $\zeta(X, Y; s, p) = \|D_s\|_p < \infty$ and $|D_{s+1}(a)| < \infty$ for some $a \in (-\infty, \infty)$. Then

$$\infty > \|D_s\|_p \geq \limsup_{x\to\infty} |x - a|^{-1/q} |D_{s+1}(x) - D_{s+1}(a)|$$

$$\geq \limsup_{x\to\infty} x^{-1/q} |D_{s+1}(x)| .$$

So,

$$\limsup_{x\to\infty} |\sum_{j=0}^{s-1} \binom{s-1}{j} x^{s-1-j-1/q} \int_{-\infty}^x (-t)^j (F_X(t) - F_Y(t))dt| < \infty$$

and

$$\limsup_{x\to\infty} |x^{-1/q} \int_{-\infty}^x (-t)^{s-1}(F_X(t) - F_Y(t))dt|$$

$$\leq \lim_{x\to\infty} x^{-1/q} \int_{-\infty}^0 (-t)^{s-1}|F_X(t) - F_Y(t)|dt$$

$$+ \lim_{x\to\infty} x^{-1/q} \int_0^x t^{s-1}|F_X(t) - F_Y(t)|dt$$

$$\leq [(s-1)^{-1} + r^{-1}]\kappa_r(X, Y) < \infty .$$

Hence, for any $\varepsilon \in (0, 1-1/q)$

$$\limsup_{x\to\infty} |\sum_{j=0}^{s-2} \binom{s-1}{j} x^{s-1-j-1/q-\varepsilon} \int_{-\infty}^x (-t)^j (F_X(t) - F_Y(t))dt| = 0 .$$

Following the proof of (b), we conclude (2.3).

If $E|X|^s + E|Y|^s < \infty$, then

$$\kappa_r(X, Y) \leq E|X|X|^{r-1} - Y|Y|^{r-1}| \leq E|X|^r + E|Y|^r < \infty$$

(see [7]) and

$$|D_{s+1}(a)| \leq \int_{-\infty}^{a} \frac{|a - x|^s}{s!} d(F_X(t) + F_Y(t))$$

$$\leq E|X|^s + E|Y|^s < \infty .$$

Proof of Theorem 3. As in [2], we define

$$(3.8) \qquad f_0(x) = \begin{cases} 1, & \text{if } x \leq z, \\ -1, & \text{if } x > z + \varepsilon, \\ \sum_{j=0}^{m} a_j \sin[(2j+1)(\pi/\varepsilon)(\varepsilon/2 - x + z)], & \text{if } z < x \leq z+\varepsilon, \end{cases}$$

where $m = [(s-1)/2]$ and the constants $\{a_j\}$ are chosen in such a way that $f_0^{(s-1)}(x)$ is a continuous function, for example,

$$a_j = [(2m+1)!!]^2/[2^{2m}(2j+1)(m+j+1)!(m-j)!] , \quad j = 0, 1, \cdots, m .$$

Then

$$||f_0^{(s)}||_q = \{\int_z^{z+\varepsilon} |\sum_{j=0}^{m} a_j \frac{d^s}{dx^s} \sin[(2j+1)(\pi/\varepsilon)(\varepsilon/2 - x + z)]|^q dx\}^{1/q}$$

$$\leq (\pi/\varepsilon)^s \varepsilon^{1/q} \sum_{j=0}^{m} a_j (2j+1)^s \leq C(s)\varepsilon^{-s+1-1/p} ,$$

where $C(s) = \pi^s s^{s-1}(s+1)$, (see [2]). Note that since

$$\|f_0\|_\infty = 1 \leqq C(s)\varepsilon^{-s+1-1/p} ,$$

then

$$M_{s,p} \equiv \max(\|f_0\|_\infty, \|f_0^{(s)}\|_q) \leqq C(s)\varepsilon^{-s+1-1/p} .$$

From (3.8) and the definition of $\omega_{s,p}$ it follows that

$$\begin{aligned} (3.9) \qquad \omega_{s,p}(X, Y) &\geqq \left|\int_{-\infty}^{\infty} M_{s,p}^{-1} f_0(x)\, d(F_X(x) - F_Y(x))\right| \\ &\geqq M_{s,p}^{-1} \left[\int_{-\infty}^{z} (f_0(x) + 1)dF_X(x) - \int_{-\infty}^{z+\varepsilon} (f_0(x) + 1)dF_Y(x)\right] \\ &\geqq 2\varepsilon^{s-1+1/p}C(s)^{-1} [F_X(z) - F_Y(z+\varepsilon)] . \end{aligned}$$

Let $L(X, Y) > \varepsilon$. By the symmetry in (3.9) we have

$$\omega_{s,p}(X, Y) \geqq 2C(s)^{-1} \varepsilon^{s+1/p} .$$

Letting $\varepsilon \to L(X, Y)$, we have

$$\omega_{s,p}(X, Y) \geqq 2C(s)^{-1}\{L(X, Y)\}^{s+1/p} .$$

Hence

$$\theta_{s,p}(X, Y) \geqq \omega_{s,p}(X, Y) \geqq 2C(s)^{-1}\{L(X, Y)\}^{s+1/p} .$$

The proof of (2.9) is thus completed. We omit the proofs of (2.10)-(2.12), since the idea of their proofs is the same as that in the general case, (see also [2]).

Remark 1. In the following, we shall show by examples that the exponetnts of L in the estimates (2.10)-(2.12) are precise.

Let $s = 1$, and let $P(X = 0) = 1$, $P(Y = 0) = 1 - \varepsilon$ and $P(Y = \varepsilon) = \varepsilon$. Then $L(X, Y) = \varepsilon$ and $\theta_{1,p}(X, Y) = \varepsilon^{1+1/p}$.

Let $s = 2$, and let $P(X = 0) = \varepsilon$, $P(X = 2\varepsilon) = 1 - \varepsilon$, $P(Y = \varepsilon) = 2\varepsilon$ and $P(Y = 2\varepsilon) = 1-2\varepsilon$. Then $L(X, Y) = \varepsilon$ and $\theta_{s,p}(x, Y) = \{2/(p+1)\}^{1/p}\varepsilon^{2+1/p}$.

Let $s = 3$, and let

$$F_X(x) = \begin{cases} 0, & \text{if } x \leqq -\varepsilon, \\ x + 1/2, & \text{if } -\varepsilon < x \leqq \varepsilon, \\ 1, & \text{if } x > \varepsilon \end{cases}$$

and

$$F_Y(x) = \begin{cases} 0, & \text{if } x \leqq -\varepsilon, \\ 1/2-\varepsilon, & \text{if } -\varepsilon < x \leqq -\varepsilon/\sqrt{3}, \\ 1/2, & \text{if } -\varepsilon/\sqrt{3} < x \leqq \varepsilon/\sqrt{3}, \\ 1/2+\varepsilon, & \text{if } \varepsilon/\sqrt{3} < x \leqq \varepsilon, \\ 1, & \text{if } x > \varepsilon. \end{cases}$$

Then $L(X, Y) = \varepsilon/2\sqrt{3}$ and the standard calculation gives us $\theta_{3,p}(X,Y) \leqq \text{const.}\times\varepsilon^{3+1/p}$.

<u>Remark 2</u>. In case $p = 1$, Grigorevski and Shiganov [2] proved

$$\theta_{s,1}(X, Y) \geqq K(s)\{\pi(X, Y)\}^{s+1} , \tag{3.10}$$

where

$$\pi(X, Y) \equiv \inf\{\varepsilon > 0 \ ; \ P(X \in A) - P(Y \in A^{\varepsilon}) < \varepsilon, \\ P(Y \in A) - P(X \in A^{\varepsilon}) < \varepsilon \quad \text{for all Borel set } A\} ,$$

A^{ε} being $\{x \ ; \ |x - y| < \varepsilon, \ y \in A\}$, is the ordinary Lévy-Prokhorov metric. Since $\pi > L$, then in case $p = 1$, (3.10) is better than (2.9). However, for $p \in (1, \infty]$, it is impossible to find the estimation

$$\theta_{s,p} \geqq \sigma(\pi),$$

for some nondecreasing function $\sigma(t)$ of $t \geqq 0$ such that $\sigma(0) = 0$ and $\sigma(t) > 0$ for $t > 0$. The following example shows it.

Define $\{X_n, Y_n, n = 1, 2, \cdots\}$ by

$$P(X_n = 2j) = P(Y_n = 2j + 1) = 1/n, \ j = 1, \cdots, n.$$

Then $\pi(X_n, Y_n) = 1$, but

$$\theta_{1,p}(X_n, Y_n) = [\int_{-\infty}^{\infty} |F_{X_n}(x) - F_{Y_n}(x)|^p \, dx]^{1/p} = n^{1/p-1} \to 0$$

for $n \to \infty$ when $p > 1$.

4. Closeness between the elements in the domain of attraction of the self-similar process.

In this section, we shall study the domain of attraction of the fractional stable process considered in [4]. This process is self-similar with parameter $H \in (0, 1)$. We shall recall briefly the main result of [4].

Let $\{X_j, j \in Z\}$ be a sequence of independent and identically distributed random variables belonging to the domain of normal attraction of a strictly stable distribution of index $\alpha < 2$ with characteristic function

(4.1) $\exp\{-|z|^{\alpha}(A_1 + iA_2 \mathrm{sgn} z)\}$

for some $0 < A_1 < \infty$, $|A_1^{-1}A_2| \leq \tan(\alpha\pi/2)$, where $\mathrm{sgn} z = +1, 0$ or -1, according as $z > 0$, $= 0$ or < 0. Take $\beta \in (1/\alpha - 1, 1/\alpha)$, $\beta \neq 0$ and consider the random variables

$$Y_k \equiv \sum_{j \in Z} c_j X_{k-j}, \quad k = 1, 2, \cdots,$$

where

$$c_j = \begin{cases} 0, & \text{if } j = 0, \\ j^{-\beta-1}, & \text{if } j > 0, \\ -|j|^{-\beta-1}, & \text{if } j < 0. \end{cases}$$

For $t \in [0, 1]$, we define

$$\Delta_n(t) \equiv |\beta| n^{-H}\Big(\sum_{k=1}^{[nt]} Y_k + (nt - [nt])Y_{[nt]+1}\Big),$$

where $[a]$ is the integer part of a, $\Sigma_{k=1}^{0}$ means 0 and $H = 1/\alpha - \beta$. Consider two independent stable processes $\{Z_+(t), t \geq 0\}$ and $\{Z_-(t), t \geq 0\}$, both having characteristic functions

(4.2) $E\{e^{izZ_{\pm}(t)}\} = \exp\{-t|z|^{\alpha}(A_1 + iA_2 \mathrm{sgn}\theta)\}$.

Define the fractional stable process by

$$\Delta(t) = \int_{-\infty}^{\infty} (|t - s|^{-\beta} - |s|^{-\beta}) dZ(s) \ , \quad t \in [0, 1] \ ,$$

where $\Delta(0) = 0$ a.s. and $Z(s) = Z_+(s)\, I[s \geqq 0] + Z_-(-s) I[s \leqq 0]$.

<u>Theorem A</u>. ([4]). <u>As</u> $n \to \infty$,

$$\Delta_n(t) \overset{d}{\Rightarrow} \Delta(t) \ .$$

Let $\{X_j', j \in Z\}$ be another sequence with the same properties as $\{X_j, j \in Z\}$. Suppose $\{X_j', j \in Z\}$ and $\{X_j, j \in Z\}$ be mutually independent. Define the process $\Delta_n'(t)$ corresponding to $\{X_j', j \in Z\}$ in the same way as $\Delta_n(t)$. The results in this section are the following.

<u>Theorem 4</u>. <u>Let</u> $1 \leqq \alpha < 2$ <u>and take</u> $r > 1$ <u>with</u> $1 \leqq \alpha < r < 2\alpha$. <u>Further take</u> β <u>such that</u> $2/r - 1 \leqq \beta < 1/\alpha - 1 + 1/r$, $\beta \neq 0$. <u>Suppose</u> $\theta_{s,p}(X_0, X_0') < \infty$ <u>for</u> s, p <u>with</u> $s - 1 + 1/p = r$. <u>Then for each</u> $t \in [0, 1]$

$$\theta_{s,p}(\Delta_n(t), \Delta_n'(t)) \leqq 4|\beta|^r \theta_{s,p}(X_0, X_0') \ (n+1)^{-Hr+r-1} C_{\beta,r} \ , \tag{4.3}$$

<u>where</u>

$$C_{\beta,r} = \begin{cases} 3 + \ln(n+2), & \underline{\text{if}} \ \ \beta = 2/r - 1, \\ 1 + 3\{(\beta+1)r-2\}^{-1}, & \underline{\text{if}} \ \ 2/r - 1 < \beta < 1/\alpha - 1 + 1/r \ . \end{cases}$$

<u>Theorem 5</u>. <u>Let</u> $0 < \alpha < 1$ <u>and take</u> r <u>with</u> $\alpha < r < \alpha/(1-\alpha)$. <u>Further take</u> β <u>such that</u> $1/r < \beta < 1/\alpha$. <u>Suppose</u> $\theta_{s,p}(X_0, X_0') < \infty$ <u>for</u> s, p <u>with</u> $s - 1 + 1/p = r$. <u>Then for each</u> $t \in [0, 1]$

(4.4) $\theta_{s,p}(\Delta_n(t), \Delta_n'(t)) \leqq 4\theta_{s,p}(X_0, X_0')n^{-Hr}[(\beta + 1)^r + 1 + (\beta r - 1)^{-1}]$.

Theorem 6. Let $0 < \alpha \leqq 1/2$ _and take_ $r < 1$ _with_ $2\alpha/(\alpha+1) < r < 2\alpha \leqq 1$. _Further take_ β _such that_ $2/r - 1 < \beta < 1/\alpha$. _Suppose_ $\theta_{s,p}(X_0, X_0') < \infty$ _for_ s, p _with_ $s - 1 + 1/p = r$. _Then for each_ $t \in [0, 1]$

(4.5) $\theta_{s,p}(\Delta_n(t), \Delta_n'(t)) \leqq 4|\beta|^r\theta_{s,p}(X_0, X_0')n^{-Hr}(1 + 3\{(\beta+1)r-2\}^{-1})$.

By (2.6) and (2.7), we can find out when $\theta_{s,p}(X_0, X_0') < \infty$ in term of the difference pseudomoment $\kappa_r(X_0,X_0')$ or $\kappa_{s,p}(X_0,X_0')$ which is also related to the difference pseudomoment. (Recall our advantage (A2) for $\theta_{s,p}$.) Furthermore, by (2.9)-(2.13), we can estimate the difference between $\Delta_n(t)$ and $\Delta_n'(t)$ in term of the uniform metric ρ.

5. Proofs of Theorems 4 - 6.

Proof of Theorem 4. Define for $t \geq 0$

$$w(t) \equiv \sum_{k=1}^{[t]} Y_k + (t - [t])Y_{[t]+1}$$

and

$$w'(t) \equiv \sum_{k=1}^{[t]} Y_k' + (t - [t])Y_{[t]+1}' .$$

Since $\theta_{s,p}$, $s = 1, 2, \cdots$, $p \in [1, \infty]$, is an ideal metric of order $r = s - 1 + 1/p$, then

(5.1) $$\theta_{s,p}(D_n(t), D_n'(t)) = \theta_{s,p}(|\beta|n^{-H}w(nt), |\beta|n^{-H}w'(nt))$$
$$= |\beta|^r n^{-Hr}\theta_{s,p}(w(nt), w'(nt)) .$$

It is not difficult to check that

$$w(nt) = \sum_{m \in Z} \xi_m(nt) X_m ,$$

where

$$(5.2) \qquad \xi_m(nt) = \sum_{j=1-m}^{[nt]-m} c_j + (nt - [nt]) c_{[nt]+1-m} .$$

Using the properties of homogeneity and regularity of the ideal metric $\theta_{s,p}$, we obtain for the deviation between the distributions of $w(nt)$ and $w'(nt)$ the following estimations:

$$(5.3) \qquad \theta_{s,p}(w(nt), w'(nt)) \leqq \theta_{s,p}(X_0, X_0') \sum_{m \in Z} |\xi_m(nt)|^r$$

(cf. [7], [10]). By the definition of $\{c_j\}$ it follows that

$$(5.4) \qquad \sum_{m \geqq 0} \eta_m(nt)^r \leqq \sum_{m \in Z} |\xi_m(nt)|^r \leqq 4 \sum_{m \geqq 0} \eta_m(nt)^r ,$$

where

$$\eta_m(nt) = \sum_{j=1+m}^{[nt]+1+m} j^{-\beta-1} , \quad m \geqq 0 .$$

Later we need the following estimation of $J(r, \beta) \equiv \Sigma_{m=0}^{\infty} \eta_m(nt)^r$:

$$(5.5) \qquad J(r, \beta) = \infty \quad \text{for } \beta \leqq 1/r - 1 ,$$

$$(5.6) \qquad J(r, \beta) \leqq ([nt] + 1)^{(r-1)_+} (1 + \{(\beta+1)r - 1\}^{-1} + ([nt]+2)^{2-(\beta+1)r} \times\{(\beta+1)r-1\}^{-1}\{2-(\beta+1)r\}^{-1}) \quad \text{for } 1/r-1 < \beta < 2/r-1 ,$$

where $(a)_+ = \max(a, 0)$,

$$(5.7) \qquad J(r, \beta) \leqq ([nt] + 1)^{(r-1)_+} \{3 + \ln([nt] + 2)\} \quad \text{for } \beta = 2/r-1 ,$$

$$(5.8) \qquad J(r, \beta) \leqq ([nt] + 1)^{(r-1)_+} \{1 + \frac{3}{(\beta+1)r - 2}\} \quad \text{for } \beta > 2/r-1$$

and

(5.9) $J(r, \beta) \leqq \beta^{-r}\{(\beta+1)^r + 1 + (\beta r-1)^{-1}\}$ for $\beta > 1/r$

If $(\beta+1)r \leqq 1$, then

$$\eta_m(nt)^r \geqq \{[nt]/([nt]+m)^{1/r}\}^r ,$$

which proves (5.5).

Let $r \geqq 1$. Then

$$\sum_{m\geqq 0} \eta_m(nt)^r \leqq \sum_{m\geqq 0} \Big(\sum_{j=1+m}^{[nt]+1+m} j^{-(\beta+1)r} \Big) ([nt] + 1)^{r/r'} .$$

where $1/r + 1/r' = 1$. Hence

$$J(r, \beta) \leqq ([nt] + 1)^{r/r'} \Big\{ \sum_{j=1}^{[nt]+1} j^{-(\beta+1)r} + \sum_{m=1}^{\infty} \int_m^{[nt]+1+m} x^{-(\beta+1)r} dx \Big\}$$

$$\leqq ([nt] + 1)^{r-1} \Big\{1 + 2\int_1^{[nt]+2} x^{-(\beta+1)r} dx + \int_1^{\infty} \int_y^{[nt]+1+y} x^{-(\beta+1)r} dx\, dy\Big\} .$$

Hence if $(\beta+1)r > 1$,

$$J(r, \beta) \leqq ([nt + 1)^{r-1} \Big\{1 + \{(\beta+1)r-1\}^{-1} + ([nt]+2)^{-(\beta+1)r+2}\{(\beta+1)r-1\}^{-1} + \int_1^{[nt]+2} x^{-(\beta+1)r+1} dx\Big\}$$

and considering the cases $1/r - 1 < \beta < 2/r - 1$, $\beta = 2/r - 1$ and $\beta > 2/r - 1$, we obtain (5.6), (5.7) and (5.8), respectively. If $0 < r < 1$, then the formulas (5.6)-(5.8) are proved in the same way as $r \geq 1$. Let $\beta > 1/r$. Then

$$\sum_{m=0}^{\infty} \eta_m(nt)^r \leq \sum_{m=0}^{\infty} \Big(\sum_{j=m+1}^{\infty} j^{-(\beta+1)} \Big)^r$$

$$\leq \beta^{-r}\{(\beta+1)^r + 1 + (\beta r-1)^{-1}\} ,$$

which proves (5.9).

Let us return to the proof of the theorem. By (5.1), (5.3), (5.4), (5.7) and (5.8),

$$\theta_{s,p}(\Delta_n(t), \Delta_n'(t)) \leq 4|\beta|^r \theta_{s,p}(X_0, X_0')(n + 1)^{-Hr+r-1}$$

$$\times \begin{cases} 3 + \ln(n + 2), & \text{for } \beta = 2/r - 1 , \\ 1 + 3\{(\beta+1)-2\}^{-1}, & \text{for } \beta > 2/r - 1. \end{cases}$$

By the assumptions $1 \leq \alpha < r < 2\alpha$ and $\beta < 1/\alpha - 1 + 1/r$, we have $-Hr+r-1 < 0$. The proof of Theorem 4 is thus completed.

Proof of Theorem 5. Let $1/\alpha - 1 \leq 1/r < 1/\alpha$ and $1/r < \beta < 1/\alpha$. Then by (5.1), (5.3), (5.4) and (5.9), we get (4.4).

Proof of Theorem 6. By (5.1), (5.3), (5.4), (5.8), we obtain (4.5).

6. Bounds on deviation from the fractional stable process $\Delta(t)$ of the elements in the domain of attraction.

For simplicity, we assume $A_1 = 1$ and $A_2 = 0$ in (4.1) and (4.2). Let $\{X_j^*, j \in Z\}$ be a sequence of independent and identically distributed strictly stable random variable having characteristic function (4.1) with $A_1 = 1$ and $A_2 = 0$. Hence $\{X_j^*, j \in Z\}$ belongs to the domain of normal attraction of X_0^*. Define $\Delta_n^*(t)$ corresponding to $\{X_j^*, j \in Z\}$ in the same way as $\Delta_n(t)$. Our results on the rate of convergence of $\Delta_n(t)$ to $\Delta(t)$ are the following.

Theorem 7.

$$\rho(\Delta_n^*(t),\ \Delta(t)) \leqq (K\pi)^{-1}\psi(n)\ ,$$

where

$$K \equiv K(\alpha,\ \beta) \equiv \int_{-\infty}^{\infty} \big|\,|t-s|^{-\beta} - |s|^{-\beta}\big|^{\alpha}\, ds < \infty$$

and

$$\psi(n) \equiv \begin{cases} C_1 n^{-1}, & \text{for } \beta < 0, \\ C_2(n,t)n^{-H\alpha}, & \text{for } \beta > 0. \end{cases}$$

Here C_1 depends only on α and β, $C_2(0) = 0$, $C_2(n,t) < C_3$ if $nt > C_4$, and C_3 and C_4 are constants depending on α and β.

Theorem 8. Let $\alpha > 1$, Under the situation of Theorem 4, if

$$\theta_{2,p}(X_0,\ X_0^*) < \infty\ ,$$

then for each $t \in [0,\ 1]$

$$\rho(\Delta_n(t),\ \Delta(t)) \leqq (1 + C_5)\{4\theta_{2,p}(\Delta_n(t),\ \Delta_n^*(t))\}^{(2+p^{-1})^{-1}} + (K\pi)^{-1}\psi(n), \tag{6.1}$$

where

$$C_5 \equiv C_5(\alpha,\ \beta) \equiv (2\pi)^{-1}\int_{-\infty}^{\infty} \exp(-2^{-1}K|z|^{\alpha})dz < \infty\ .$$

Theorem 9. _Let_ $\alpha \leq 1$. _Under each situation of Theorem 4, 5 or 6, if_

$$\theta_{1,p}(X_0, X_0^*) < \infty ,$$

then for each $t \in [0, 1]$

$$(6.2) \qquad \rho(\Delta_n(t), \Delta(t)) \leqq (1 + C_5)\{\theta_{1,p}(\Delta_n(t), \Delta_n^*(t))\}^{(1+p^{-1})^{-1}} + (K\pi)^{-1}\psi(n).$$

To get the rate of convergence of $\Delta_n(t)$ to $\Delta(t)$ in Theorems 8 and 9, it is enough to apply Theorems 4 - 6 to $\theta_{s,p}(\Delta_n(t), \Delta_n^*(t))$ in the first term on the right hand side of (6.1) and (6.2). We start with the proofs of Theorems 8 and 9.

Proof of Theorem 8. We have

$$\rho(\Delta_n(t), \Delta(t)) \leqq \rho(\Delta_n(t), \Delta_n^*(t)) + \rho(\Delta_n^*(t) , \Delta(t)) .$$

By (2.13)

$$\rho(\Delta_n(t), \Delta_n^*(t)) \leqq (1 + m_n) L(\Delta_n(t), \Delta_n^*(t)) ,$$

where

$$m_n = \sup\{|\tfrac{d}{dx}P(\Delta_n^*(t) \leqq x)|; x \in R\}.$$

Since

$$h_n(z) \equiv E\{\exp(iz\Delta_n^*(t))\} = \exp\{-|z|^\alpha|\beta|^\alpha n^{-H\alpha} \sum_{m\in Z} |\xi_m(nt)|^\alpha\} ,$$

where $\xi_m(nt)$ is the one defined in (5.2), we write

$$h_n(z) = \exp\{- K_n|z|^\alpha\} ,$$

where

$$K_n \equiv K_n(\alpha, \beta) \equiv |\beta|^\alpha n^{-H\alpha} \sum_{m\in Z} |\xi_m(nt)|^\alpha .$$

In [4], it was shown that $K_n \to K$ as $n \to \infty$. Therefore

$$\left|\frac{d}{dx}P(\Delta_n^*(t) \leqq x)\right| \leqq (2\pi)^{-1}\int|h_n(z)|dz = (2\pi)^{-1}\int_{-\infty}^{\infty}\exp(-2^{-1}K|z|^{\alpha})dz$$

for large n. Thus $m_n \leqq C_5$ for large n. Since $\alpha > 1$, $EX_0 = EX_0^* = 0$. Hence we can take $s = 2$ and apply (2.11) to get

$$L(\Delta_n(t),\ \Delta_n^*(t)) \leqq 4\{\theta_{2,p}(\Delta_n(t),\ \Delta_n^*(t))\}^{(2+p^{-1})^{-1}} .$$

As to the second term on the right hand side of (6.1), it will be proved in Theorem 7.

<u>Proof of Theorem 9</u>. Since $\alpha \leqq 1$, we have to take $s = 1$, and apply (2.10). The statement is concluded by exactly the same argument as above.

<u>Proof of Theorem 7</u>. Recall that

$$h_n(z) = E\{\exp(iz\Delta_n^*(t))\} = \exp(-K_n|z|^{\alpha})$$

and

$$h(z) \equiv E\{\exp(iz\Delta(t))\} = \exp(-K|z|^{\alpha})$$

(see [4]). Hence

$$|h_n(z) - h(z)| \leqq \exp(-K|z|^{\alpha})|z|^{\alpha}|K_n - K| .$$

By the Esseen inequality (cf. [5] p.109),

$$\rho(\Delta_n^*(t),\ \Delta(t)) \leqq b\int_{-T}^{T}|(h_n(z)-h(z))/z|dz + r(b)CT^{-1} ,$$

where T is an arbitrary positive number, b is any positive number greater than $(2\pi)^{-1}$, r(b) is a positive constant depending only on b and $C = \sup\{|\frac{d}{dx}P(\Delta(t) \leqq x)| ;\ x \in R\}$. Hence

$$\rho(\Delta_n^*(t),\ \Delta(t)) \leqq b|K_n - K| \int_{-T}^{T}\exp(-K|z|^{\alpha})|z|^{\alpha-1}dz + r(b)CT^{-1}$$

Since $\exp(-K|z|^{\alpha})|z|^{\alpha-1}$ is integrable over $(-\infty, \infty)$ for $0 < \alpha < 2$, then letting $T \to \infty$ implies that

$$\rho(\Delta_n^*(t),\ \Delta(t)) \leqq (K\pi)^{-1}|K_n - K| \ .$$

Hence, to complete the proof of the theorem, it remains to show

$$(6.3) \qquad |K_n - K| \leqq \psi(n) \ .$$

The proof of (6.3) will be devided into the following four parts:

$$\text{(a)} \qquad \Big| \sum_{m=-\infty}^{0} |\beta n^{-H}\xi_m(nt)|^\alpha - \int_{-\infty}^{0} ||t-s|^{-\beta} - |s|^{-\beta}|^\alpha ds\Big| \leqq \psi(n) \ ,$$

$$\text{(b)} \qquad \Big| \sum_{m=[nt]}^{\infty} |\beta n^{-H}\xi_m(nt)|^\alpha - \int_{t}^{\infty} ||t-s|^{-\beta} - |s|^{-\beta}|^\alpha ds\Big| \leqq \psi(n) \ ,$$

$$\text{(c)} \qquad \Big| \sum_{m=1}^{[[nt]/2]} |\beta n^{-H}\xi_m(nt)|^\alpha - \int_{0}^{t/2} ||t-s|^{-\beta} - |s|^{-\beta}|^\alpha ds\Big| \leqq \psi(n)$$

and

$$\text{(d)} \qquad \Big| \sum_{m=[[nt]/2]+1}^{[nt]-1} |\beta n^{-H}\xi_m(nt)|^\alpha - \int_{t/2}^{t} ||t-s|^{-\beta} - |s|^{-\beta}|^\alpha ds\Big| \leqq \psi(n) \ .$$

However, since the proofs of (a)-(d) are carried out in the simialr manner, we shall show only (a) and omit the proofs of the others.

<u>Proof of (a)</u>. Denote

$$\delta_n \equiv \Big| \sum_{m=-\infty}^{-2} |\beta n^{-H}\xi_m(nt)|^\alpha - \int_{-\infty}^{-3/n} ||t-s|^{-\beta} - |s|^{-\beta}|^\alpha ds\Big| \ .$$

Since

$$\xi_m(nt) = \sum_{j=1-m}^{[nt]-m} j^{-\beta-1} + (nt - [nt])([nt] + 1 - m)^{-\beta-1} \qquad \text{for} \quad m \leqq 0$$

and $\beta + 1 > 0$, then

$$0 \leqq \int_{1-m}^{nt+1-m} x^{-\beta-1}dx \leqq \xi_m(nt) \leqq \int_{-m}^{nt-m} x^{-\beta-1}dx \ .$$

Hence

$$\underline{\lambda}_n \equiv \sum_{m=-\infty}^{-2} n^{-1}|\{(1-m)/n\}^{-\beta} - \{t+(1-m)/n\}^{-\beta}|^\alpha$$

$$\leq \sum_{m=-\infty}^{-2} |\beta n^{-H} \xi_m(nt)|^{\alpha}$$

$$\leq \sum_{m=-\infty}^{-2} n^{-1} |(-m/n)^{-\beta} - (t-m/n)^{-\beta}|^{\alpha} \equiv \bar{\lambda}_n .$$

Denote

$$g(a) \equiv n^{-1} ||t-a|^{-\beta} - |a|^{-\beta}|^{\alpha} , \quad a \in (-\infty, \infty) .$$

Then

$$\underline{\lambda}_n \geq \int_{-\infty}^{-3/n} ||t-s|^{-\beta} - |s|^{-\beta}|^{\alpha} ds \equiv L_n$$

and

$$\bar{\lambda}_n \leq g(3/n) + g(2/n) + L_n .$$

Hence

$$\delta_n \leq g(2/n) + g(3/n) .$$

For any $a \in (-\infty, \infty)$,

$$g(a/n) \leq n^{-1} \max\{|t-a/n|^{-\alpha\beta}, |a/n|^{-\alpha\beta}\} \leq \psi(n) .$$

Thus

(6.4) $$\delta_n \leq \psi(n) .$$

Let $t > 0$ (if $t = 0$, $K_n = K = 0$) and $1/n < t$. Since $0 < t \leq 1$, we have for some $C > 0$

(6.5) $$\int_0^{1/n} |u^{-\beta} - (t+u)^{-\beta}|^{\alpha} du \leq C\{(1/n)^{1-\alpha\beta} + (t+1/n)^{1-\alpha\beta} - t^{1-\alpha\beta}\} \leq \psi(n).$$

So

(6.6) $$\int_{-3/n}^{0} ||t-s|^{-\beta} - |s|^{-\beta}|^{\alpha} ds \leq \psi(n) .$$

If $m = 0, -1$.

(6.7) $$|\beta n^{-H}\xi_{-1}(nt)|^{\alpha}$$

$$= |\beta n^{-H}|^{\alpha}|\sum_{j=1}^{[nt]+1} j^{-\beta-1} + (nt - [nt])([nt] + 2)^{-\beta-1}|^{\alpha}$$

$$\leqq n^{-1}|(t + 1/n)^{-\beta} - (1/n)^{-\beta}|^{\alpha}$$

$$\leqq n^{-1} \max\{(t + 1/n)^{-\alpha\beta}, (1/n)^{-\alpha\beta}\} \leqq \psi(n)$$

and similarly

(6.8) $$|\beta n^{-H}\xi_{0}(nt)|^{\alpha} \leqq \psi(n) .$$

By (6.4), (6.6), (6.7) and (6.8), we obtain (a).

The estimates (b)-(d) can be shown similarly, and hence Theorem 7 is proved.

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