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Rational approximations to the values of certain hypergeometric functions

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Rational approximations to the values of the exponential function have been studied by many authers. We refer among others to the following theorem due to Bundschuh[1], Mahler[7], and Durand[5]: Let a, b be positive integers. Then there are explicit positive constants C, B such that

$$(1) \qquad \left| e^{a/b} - \frac{p}{q} \right| > C q^{-2-B/\log\log q}$$

for all integers p, q with $q \geqslant 3$. Especially for e the very precise estimate was obtained by Davis[4]; namely, $|e-p/q| > (\frac{1}{2}-\varepsilon)q^{-2}\log\log q/\log q$ for all integers p, q with $q > q_0(\varepsilon)$, and $< (\frac{1}{2}+\varepsilon)\, q^{-2}\log\log q/\log q$ for infinitely many integers p, q. His proof is given by constructing explicitly the convergents of the simple continued fraction of e found by Euler. The method of Mahler and Durand depends on the classical formula by Hermite. Bundschuh used Kummer's relation satisfied by a particuler class of hypergeometric functions to construct rational approximations to e^x. He also obtained similar results for tanh x and the retio $J_{\lambda+1}(x)/J_\lambda(x)$ of the Bessel functions of the first kind.

In this paper we give a new proof of the inequality (1) with an improved constant B for the values of not only for the exponential function but also for certain confluent hypergeometric functions including those obtained in [1] and [2] mentioned above. Our method is based on the continued fractions of Gauss.

We denote by

$$X = G_\gamma(Y), \quad Y > 0,$$

the inverse function for the function

$$Y = F_\gamma(X) = X \log X + \gamma X, \quad X > e^{-\gamma},$$

where γ is a given real number.

Theorem 1. Let a, b be positive integers. Then there is a positive constant C depending only on a, b such that

$$\left| e^{a/b} - \frac{p}{q} \right| > C\, q^{-2-2\log a \cdot G_{\gamma}(\log q)/\log q\ \frac{\log\log q}{\log q}}$$

for all integers p,q with $q \geqslant 3$, where $\gamma = \log((4b)/(ea^2))$.

Corollary 1. For any positive ε, ther is a positive constant C depending at most on a, b, and ε such that

$$\left| e^{a/b} - \frac{p}{q} \right| > C\, q^{-2-(2\log a + \varepsilon)/\log\log q}$$

for all integers p, q with $q \geqslant 3$.

Corollary 2. Let b be a positive integer. Then there is a positive constant C depending only on b such that

$$\left| e^{1/b} - \frac{p}{q} \right| > C\, q^{-2} \frac{\log\log q}{\log q}$$

for all integers p, q with $q \geqslant 3$.

For the proof we need the following

Lemma 1. Let

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$

be a continued fraction with real partial denominators which represents an irrational number. Assume that

$$(2) \qquad \sum_{n=1}^{\infty} |a_n a_{n+1}|^{-1} < \infty .$$

Then the ratios $p_n/(a_2 a_3 \cdots a_n)$ and $q_n/(a_1 a_2 \cdots a_n)$ converge to finite non-zero limits as $n \to \infty$. Furthermore

$$\lim_{n\to\infty} a_{n+1} \alpha_n = 0,$$

where

$$\alpha_n = \frac{1}{a_{n+1}} + \frac{1}{a_{n+2}} + \frac{1}{a_{n+3}} + \cdots .$$

Proof. By (2) there is a positive integer n_0 such that

$$(3) \qquad -\frac{2}{9} < \frac{1}{|a_n a_{n+1}|} < \frac{2}{3}, \qquad n \geqslant n_0 .$$

Let $n \geqslant n_0$ be fixed. Denote by $p_{n,k}/q_{n,k}$ the kth convergent of the continued fraction α_n, i.e.

$$\frac{p_{n,k}}{q_{n,k}} = \frac{1}{a_{n+1}} + \frac{1}{a_{n+2}} + \cdots + \frac{1}{a_{n+k}}, \quad k \geqslant 1.$$

Then

$$q_{n,1} = a_{n+1}, \quad q_{n,2} = a_{n+1}a_{n+2}\left(1+\frac{1}{a_{n+1}a_{n+2}}\right)$$

and

$$q_{n,3} = a_{n+1}a_{n+2}a_{n+3}\left(1+\frac{1}{a_{n+1}a_{n+2}}\right)\left(1+\frac{1}{a_{n+2}a_{n+3}}\left(1+\frac{1}{a_{n+1}a_{n+2}}\right)^{-1}\right)$$
$$= a_{n+1}a_{n+2}a_{n+3}\left(1+\frac{1}{a_{n+1}a_{n+2}}\right)\left(1+\frac{v_{n,2}}{a_{n+2}a_{n+3}}\right)$$

for some $v_{n,2}$ with $\frac{1}{2} < v_{n,2} < \frac{3}{2}$, in view of (3) and the following inequality

$$\frac{1}{2} < \frac{1}{1+xy} < \frac{3}{2}, \quad \text{if} \quad -\frac{2}{9} < x < \frac{2}{3} \quad \text{and} \quad \frac{1}{2} < y < \frac{3}{2}.$$

Repeating this we get

$$q_{n+k} = a_{n+1}a_{n+2}\cdots a_{n+k}\prod_{j=1}^{k-1}\left(1+\frac{v_{n,j}}{a_{n+j}a_{n+j+1}}\right)$$

for some $v_{n,j}$ with $\frac{1}{2} < v_{n,j} < 3/2$, and thus the ratio $q_{n,k}/(a_{n+1}a_{n+2}\cdots a_{n+k})$ is also converges to a non-zero limit because of (2). If we regard $q_{n,k}$ as a polinomial in k variables $a_{n+1}, a_{n+2}, \ldots, a_{n+k}$ and write it as $q_{n,k}=q_{n,k}(a_{n+1}, a_{n+2}, \ldots, a_{n+k})$, we have

$$p_{n,k} = q_{n+1,k-1}(a_{n+2}, a_{n+3}, \ldots, a_{n+k})$$

and hence $p_{n,k}/(a_{n+2}a_{n+3}\cdots a_{n+k})$ is also convergent.

We may assume $p_n q_n \neq 0$, since there are infinitely many such n's. Then from the recurrence relations

$$p_{n+k} = p_n q_{n,k} + p_{n-1}p_{n,k},$$
$$q_{n+k} = q_n q_{n,k} + q_{n-1}p_{n,k},$$

we have

$$\frac{p_{n+k}}{a_2a_3\cdots a_{n+k}} = \frac{p_n}{a_2a_3\cdots a_n}\,\frac{q_{n,k}}{a_{n+1}\cdots a_{n+k}}\left(1+\frac{p_{n-1}}{p_n}\,\frac{p_{n,k}}{q_{n,k}}\right),$$

$$\frac{q_{n+k}}{a_1a_2\cdots a_{n+k}} = \frac{q_n}{a_1a_2\cdots a_n}\,\frac{q_{n,k}}{a_{n+1}\cdots a_{n+k}}\left(1+\frac{q_{n-1}}{q_n}\,\frac{p_{n,k}}{q_{n,k}}\right).$$

The right-hand sides converge as $k\to\infty$ to finite limits different from zero, since under the assumption the continued fraction

$$\frac{1}{a_{n+1}}\;\frac{1}{+\,a_{n+2}}\;\frac{1}{+\,a_{n+3}}\;\cdots = \lim_{k\to\infty}\frac{p_{n,k}}{q_{n,k}}$$

converges to an irrational number.

As we have seen above

$$\begin{aligned} a_{n+1}\alpha_n &= a_{n+1}\lim_{k\to\infty}\frac{p_{n,k}}{q_{n,k}} \\ &= \prod_{k=1}^{\infty}\left(1+\frac{u_{n,k}}{a_{n+k}a_{n+k+1}}\right)\left(1+\frac{v_{n,k}}{a_{n+k}a_{n+k+1}}\right)^{-1} \end{aligned}$$

with $\frac{1}{2} < u_{n,k} < 3/2$, $\frac{1}{2} < v_{n,k} < 3/2$, which converges to 1 as $n\to\infty$.

Proof of Theorem 1. The proof is based on the continued fraction

$$e^x = 1 + \frac{2x}{2-x}\;\frac{x^2}{+\,2\cdot 3}\;\frac{x^2}{+\,2\cdot 5}\;\frac{x^2}{+\,2\cdot 7}\;\frac{x^2}{+\,2\cdot 9}\;+\cdots .$$

This formula appears in [6;(2.4.30)]; however, it is stated there incorrectly. Using the equivalence transformation

$$c_0 + \frac{b_1}{c_1}\;\frac{b_2}{+\,c_2}\;\frac{b_3}{+\,c_3}\;+\cdots = c_0 + \frac{r_1b_1}{r_1c_1}\;\frac{r_1r_2b_2}{+\,r_2c_2}\;\frac{r_2r_3b_3}{+\,r_3c_3}\;\cdots,$$

we find the regular continued fraction

$$e^x = 1 + \frac{1}{a_1}\;\frac{1}{+\,a_2}\;\frac{1}{+\,a_3}\;+\cdots$$

with

$$a_1 = \frac{2-x}{2x},\quad a_{2n} = \frac{4(4n-1)}{x},\quad a_{2n+1} = \frac{4n+1}{x},\quad n\geqslant 1.$$

Now we put $x=a/b$. Then the continued fractions satisfies the conditions of Lemma 1. We denote its nth convergent by p_n/q_n. It can be shown by induction that d_np_n, d_nq_n are integers for all $n\geqslant 1$, where

(4) $$d_n = 2a^n,\quad n\geqslant 1.$$

applying now Lemma 1, we see

$$\lim_{n\to\infty}\frac{q_n}{q_{n+1}} = \lim_{n\to\infty}\frac{q_n d_{n+1}}{q_{n+1}d_n} = \lim_{n\to\infty}\alpha_n = 0,$$

so that we can choose a positive integer n_0 such that

(5) $\quad |q_n| < |q_{n+1}|, \quad |q_n/d_n| < |q_{n+1}/d_{n+1}|, \quad |\alpha_n| < \frac{1}{2}$

for all integers $n \geqslant n_0$, for some $n_0 = n_0(a,b)$.

Now let p and $q > 0$ be given integers. We may assume

$$|q_{n_0}/d_{n_0}| < 4q.$$

Then by (5) there is an integer $n=n(q) \geqslant n_0$ determined uniquely by the inequality

(6) $\quad |q_{n-1}/d_{n-1}| \leqslant 4q < |q_n/d_n|$.

By virture of the formula

$$p_n q_{n-1} - p_{n-1}q_n = \pm 1,$$

we can deduce

$$p_n q - q_n p \neq 0 \quad \text{or} \quad p_{n-1}q - q_{n-1}p \neq 0.$$

Assume first that $p_n q - q_n p$ is defferent from zero. Then we have

$$d_n q_n (e^{a/b} - \frac{p}{q}) = \frac{d_n(p_n q - q_n p)}{q} + d_n(q_n e^{a/b} - p_n),$$

where $d_n(p_n q - q_n p)$ is a non-zero integer, so that

$$|d_n(p_n q - q_n p)| \geqslant 1,$$

and

$$|d_n(q_n e^{a/b} - p_n)| < \frac{1}{2q},$$

because of the formula

$$q_n e^{a/b} - p_n = \frac{\pm 1}{q_{n+1} + \alpha_{n+1} q_n}$$

with (5) and (6). Hence we get

$$\left|d_n q_n (e^{a/b} - \frac{p}{q})\right| > \frac{1}{2q},$$

or equivalently

$$\left|e^{a/b} - \frac{p}{q}\right| > \tfrac{1}{2} q^{-1-\log|d_n q_n|/\log q} .$$

We will find the same inequality, if we start with another possibility $p_{n-1}q - q_{n-1}p \neq 0$. Therefore, using (4), (5), and (6), we obtain

(7) $\quad \left|e^{a/b} - \frac{p}{q}\right| > c_1 q^{-2-(2\log d_n + \log|a_n|)/\log q}$

(8) $\quad > c_2 q^{-2-((2\log a)n + \log n)/\log q}$.

The constants c_1, $c_2 > 0$, and in the sequel those implied in O-symbols depend possibly on a and b. It remains to replace n and log n by functions of q.

By Lemma 1 and (6) we have

(9) $$\log q = \log|a_1 \dots a_n| - \log d_n + O(1).$$

Here we see

$$a_1 a_2 \dots a_{2n+1} = \frac{2-x}{16x}\left(\frac{8}{x}\right)^{2n+1}(n!)^2 \prod_{k=1}^{\infty}\left(1-\frac{1}{16k^2}\right),$$

so that

$$\log|a_1 \dots a_{2n+1}| = 2\log n! + (2n+1)\log\left(\frac{8}{x}\right) + O(1)$$

$$= \log|a_1 \dots a_{2n}| + \log n + O(1).$$

Using Stirlig's formula

$$\log \Gamma(x) = x\log x - x - \tfrac{1}{2}\log x + O(1), \quad x \to \infty,$$

we get

$$\log|a_1 \dots a_n| = n\log n + n\log\frac{4}{e|x|} + O(1),$$

which together with (4) and (9) yields

(10) $$\log q = n\log n + \gamma n + O(\log n), \quad \gamma = \log\frac{4}{ea|x|}.$$

Hence we have

(11) $$\log n = \log\log q - \log\log\log q + O(1),$$

and so

$$F_\gamma(n) = n\log n + \gamma n = \log q + O(\log\log q).$$

Therefore we obtain

$$n = G_\gamma(\log q + O(\log\log q))$$

$$= G_\gamma(\log q) + O(1).$$

From this, (8), and (11) Theorem 1 follows.

Corollary 1 is an immeadiate consequence of the following

Lemma 2. The function $G_\gamma(Y)$ can be developed in the series

$$G_\gamma(Y) = \frac{Y}{\log Y}\left\{1 + \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} A_{n,k}\frac{(\log\log Y - \gamma)^{n-k}}{(\log Y)^n}\right\},$$

provided Y is sufficiently large, where the coefficients $A_{n,k}$ are given by the following relations;

$$A_{n+1,K+1} = A_{n,k+1} - \sum_{\substack{n_1+n_2=n \\ k_1+k_2=k \\ 0\le k_1\le n_1 \\ 0\le k_2<n_2}} A_{n_1,k_1}B_{n_2,k_2}, \quad 0 \le k < n, \quad n \ge 1,$$

where

$$B_{n,k} = \sum_{m=1}^{n-k} \frac{(-1)^{m-1}}{m} \sum_{\substack{n_1+\cdots n_m=n \\ k_1+\cdots k_m=k \\ 0\leqslant k_i< n_i}} A_{n_1,k_1}\cdots A_{n_m,k_m},$$

with $A_{n,0}=1,\ n\geqslant 0$, and $A_{n,n}=0,\ n\geqslant 1$.

The first few terms of the series are

$$G_\gamma(Y) = \frac{Y}{\log Y}\Big\{1+\frac{\log\log Y-\gamma}{\log Y}+\frac{(\log\log Y-\gamma)^2}{(\log Y)^2}+\frac{\log\log Y-\gamma}{(\log Y)^2}$$

$$+\frac{(\log\log Y-\gamma)^3}{(\log Y)^3}-\frac{5}{2}\frac{(\log\log Y-\gamma)^2}{(\log Y)^3}+\frac{\log\log Y-\gamma}{(\log Y)^3}+\cdots\Big\}.$$

The proof of Lemma 2 will be found in the last part of this paper.

Our method can be available for the continued fractions of Gauss which represent some confluent hypergeometric functions. The following are such examples.

$$\tan x = \frac{x}{1-}\frac{x^2}{3-}\frac{x^2}{5-}\frac{x^2}{7-}\cdots.$$

$$\tanh x = \frac{x}{1+}\frac{x^2}{3+}\frac{x^2}{5+}\frac{x^2}{7+}\cdots.$$

$$\frac{f_\lambda(x)}{f'_\lambda(x)} = \lambda+\frac{x}{\lambda+1+}\frac{x}{\lambda+2+}\frac{x}{\lambda+3+}\cdots,$$

where $-\lambda$ is not a positive integer and $f_\lambda(x)$ is defined by

$$f_\lambda(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!(\lambda+1)(\lambda+2)\cdots(\lambda+n)}.$$

$$\frac{J_{\lambda+1}(x)}{J_\lambda(x)} = \frac{x}{2(\lambda+1)-}\frac{x^2}{2(\lambda+2)-}\frac{x^2}{2(\lambda+3)-}\cdots,$$

where $J_\lambda(x)$ is the Bessel function of the first kind of order λ defined by

$$J_\lambda(x) = \frac{1}{\Gamma(\lambda+1)}\left(\frac{x}{2}\right)^\lambda f_\lambda\left(-\frac{x^2}{4}\right).$$

(cf. [6; § 6.1.3], [8; Chap.II].)

Theorem 2. Let θ be one of the numbers given below. Then there is a positive constant C depending only on θ such that

$$\left|\theta-\frac{p}{q}\right| > Cq^{-2-\beta G_\gamma(\log q)/\log q\ \frac{\log\log q}{\log q}}$$

for all integers p, q with $q \geqslant 3$, where θ and the correspondig constants β and γ are as follows: let a, b be positive integers and r, s be integers with $\lambda = r/s \neq -1, -2. -3, \ldots, s > 0$.

θ	β	γ
$\tan\frac{a}{b}$, $\tanh\frac{a}{b}$	$2\log a$	$\log\frac{2b}{ea^2}$
$\frac{\tan\sqrt{a/b}}{\sqrt{a/b}}$, $\frac{\tanh\sqrt{a/b}}{\sqrt{a/b}}$	$\log a$	$\log\frac{2\sqrt{b}}{ea}$
$\frac{f_\lambda(a/b)}{f'_\lambda(a/b)}$	$2\log(\sqrt{a}s)$	$\log\frac{\sqrt{b}}{eas}$
$\frac{J_{\lambda+1}(a/b)}{J_\lambda(a/b)}$	$2\log(as)$	$\log\frac{2b}{ea^2s}$
$\frac{J_{\lambda+1}(\sqrt{a}/b)}{a/b\ J_\lambda(\sqrt{a/b})}$	$\log(as)$	$\log\frac{2\sqrt{b}}{eas}$

Corollary 3. Let θ, β, and γ be as above. Then for any positive ε, there is a positive constant C depending at most on θ and ε such that

$$\left|\theta - \frac{p}{q}\right| > Cq^{-2-(\beta+\varepsilon)/\log\log q}$$

for all integersp, q with $q \geqslant 3$.

Corollary 4. If a=s=1, then there is a positive constant C depending only on θ such that

$$\left|\theta - \frac{p}{q}\right| > Cq^{-2\,\frac{\log\log q}{\log q}}$$

for all integers p, q with $q \geqslant 3$.

Proof. Let $\theta = \tanh(a/b)$. By the equivalence transformatin, the continued fraction is transformed into the regular one

$$\tanh x = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots, \quad a_n = \frac{2n-1}{x}, \ n \geqslant 1,$$

which satisfies the conditions of Lemma 1. Let x=a/b. Then $d_n p_n$ and $d_n q_n$, $n \geqslant 1$, are integers, where $d_n = a^n$, $n \geqslant 1$. Choose n_0 such that (5) holds for all $n \geqslant n_0$. For any given p and q with $q \geqslant c_3$, there is an integer $n = n(q) \geqslant n_0$ satisfying (6). Then we have (7) and (9). We see

$$a_1 a_2 \cdots a_n = \frac{(2n-1)!}{(n-1)!} \frac{1}{2^n x^n},$$

so that

$$\log |a_1 a_2 \cdots a_n| = n\log n + n\log \frac{2}{ex} + O(1).$$

Hence we obtain (10) with $\gamma = \log(2b/(ea^2))$. The rest of the proof is the same as that of Theorem 1.

For the number $\tanh\sqrt{a/b}\,/\sqrt{a/b}$, we use the continued fraction

$$\frac{\tanh\sqrt{x}}{\sqrt{x}} = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots,$$

where

$$a_1 = 1, \quad a_{2n+1} = 4n+1, \quad a_{2n} = \frac{4n-1}{x}, \quad n \geqslant 1.$$

This with $x=a/b$ also satisfies the conditions of Lemma 1. We have

$$\log |a_1 a_2 \cdots a_n| = n\log n + n\log \frac{2}{e\sqrt{x}} + O(1)$$

and

$$d_{2n} = d_{2n+1} = a^n, \quad n \geqslant 0;$$

and the proof will be carried out as above.

The remaining cases can be proved in the same way.

Remark. We have assumed a, b, and s to be positive integers only for brevity. All the results stated above are valid even for integers in a given imaginary quadratic field.

Proof of Lemma 2. Put $x=\log X$ and $y=\log Y$. Then

$$Y = X\log X + \gamma X, \quad X > e^{-\gamma}$$

implies

(13) $$e^y = e^x(x+\gamma), \quad x > -\gamma,$$

so that

$$y = x + \log(x+\gamma) = \phi(x), \quad \text{say}.$$

We denote by $x=\phi^{-1}(y)$ the inverse function of $y=\phi(x)$, and difine

$$f(y) = ye^{x-y}, \quad x = \phi^{-1}(y),$$

then we have, using (2),

(14) $$f(y) = \frac{y}{x+\gamma} = \frac{y}{y-\log y + \gamma + \log f(y)}.$$

We have to show

(15) $$f(y) = 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} A_{n,k} \frac{(\log y - \gamma)^{n-k}}{y^n}.$$

In what follows, we always assume that x and y are sufficiently large. Noticing that $\log f(y) = O(1)$, we have by (14)

$$f(y) = \sum_{n=0}^{\infty} \frac{(\log y - \gamma - \log f(y))^n}{y^n}$$

$$(16) \qquad = \sum_{k=0}^{\infty} (-\log f(y))^k \sum_{n=k}^{\infty} \binom{n}{k} \frac{(\log y - \gamma)^{n-k}}{y^n} .$$

Also

$$\log f(y) = \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\log y - \gamma - \log f(y)}{y}\right)^m$$

$$= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=0}^{m} \binom{m}{n} \frac{(\log y - \gamma)^{m-n}}{y^m} (-\log f(y))^n$$

$$= \sum_{m=1}^{\infty} \frac{1}{m} \frac{(\log y - \gamma)^m}{y^m} + \sum_{n=1}^{\infty} (-\log f(y))^n \sum_{m=n}^{\infty} \frac{1}{m} \binom{m}{n} \frac{(\log y - \gamma)^{m-n}}{y^m},$$

so that

$$-\sum_{m=1}^{\infty} \frac{1}{m} \frac{(\log y - \gamma)^m}{y^m} = (-\log f(y))\left(1 + \sum_{m=1}^{\infty} \frac{(\log y - \gamma)^{m-1}}{y^m}\right),$$

$$+ \sum_{n=2}^{\infty} (-\log f(y))^n \sum_{m=n}^{\infty} \frac{1}{m} \binom{m}{n} \frac{(\log y - \gamma)^{m-n}}{y^m} .$$

Multipling the both sides of the equation above by the series

$$\left(1 + \sum_{m=1}^{\infty} \frac{(\log y - \gamma)^{m-1}}{y^m}\right)^{-1} = \sum_{k=0}^{\infty} \left(- \sum_{m=1}^{\infty} \frac{(\log y - \gamma)^{m-1}}{y^m}\right)^k ,$$

we find

$$A_0 = -\log f(y) + A_2(-\log f(y))^2 + A_3(-\log f(y))^3 + \dots,$$

where A_n are of the forms

$$A_n = \sum_{m=n}^{\infty} \sum_{k=0}^{m} a_{m,k}^{(n)} \frac{(\log y - \gamma)^{m-k}}{y^m}$$

with some constant coefficients $a_{m,k}^{(n)}$.

We consider now the power series

$$Y = S(X) = X + A_2X^2 + A_3X^3 + \dots,$$

which is convergent in a neighbourhood of the origin, and $S(0) = 0$, $S'(0) = 1$. Then there is the power series

$$X = T(Y) = Y + B_2Y^2 + B_3Y^3 + \dots$$

having positive radius of convergence such that

$$S(T(Y)) = Y,$$

where the coefficients B_n are given inductively by

$$B_n = \sum_{k=2}^{n} A_k \sum_{n_1+\dots+n_k=n} B_{n_1}\cdots B_{n_k}.$$

(cf. Cartan [3].)

Putting $X = -\log f(y)$ and $Y=A_0$, we get

$$-\log f(y) = A_0 + B_2A_0^2 + B_3A_0^3 + \dots$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{n} C_{n,k} \frac{(\log y - \gamma)^{n-k}}{y^n},$$

wher $C_{n,k}$ are constants; which together with (16) yields (15).

It remains to prove the relations satisfied by the coefficients $A_{n,k}$. We have from (14)

(17) $$f(y)\log f(y) = (\log y - \gamma) f(y) - y(f(y)-1).$$

Here by (15)

$$\log f(y) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}(f(y)-1)^m$$

and

$$(f(y)-1)^m = \sum_{n=m}^{\infty} \sum_{k=0}^{n-m} A_{n,k}^{(m)} \frac{(\log y-\gamma)^{n-k}}{y^n},$$

where

$$A_{n,k}^{(m)} = \sum_{\substack{n_1+\dots+n_m=n \\ k_1+\dots+k_m=k \\ 0 \leq k_i < n_i}} A_{n_1,k_1}\cdots A_{n_m,k_m},$$

so that

$$\log f(y) = \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{k=0}^{n-m} \frac{(-1)^{m-1}}{m} A_{n,k}^{(m)} \frac{(\log y - \gamma)^{n-k}}{y^n}$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} B_{n,k}\frac{(\log y - \gamma)^{n-k}}{y^n}.$$

This together with with (15) leads to

$$f(y)\log f(y) = \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\Big(B_{n,k} + \sum_{\substack{n_1+n_2=n\\ k_1+k_2=k\\ 0\leq k_i<n_i}} A_{n_1,k_1}B_{n_2,k_2}\Big)\frac{(\log y-\gamma)^{n-k}}{y^n}.$$

On the other hand , we have from (17)

$$(\log y-\gamma)f(y) - y(f(y)-1)$$

$$= A_{1,0}(\log y-\gamma) + \sum_{n=1}^{\infty}\sum_{k=-1}^{n-1}(A_{n,k+1}-A_{n+1,k+1})\frac{(\log y-\gamma)^{n-k}}{y^n},$$

where $A_{n,n}=0$, $n \geqslant 1$. Comparing both sides of (17) with these equalities, we obtain the relations for $A_{n,k}$ and $B_{n,k}$ in Lemma 2.

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