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Computing the Inertia from Sign Patterns

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Abstract

A symmetric matrix A is said to be sign-nonsingular if every symmetric matrix with the same sign pattern as A is nonsingular. Hall, Li and Wang (2001) showed that the inertia of a sign-nonsingular symmetric matrix is determined uniquely by its sign pattern. The purpose of this paper is to present an efficient algorithm for computing the inertia of such matrices. The algorithm runs in $\mathrm{O}(nm)$ time for a symmetric matrix of order n with m nonzero entries. The correctness of the algorithm provides an alternative proof of the result by Hall et al. In addition, for a symmetric matrix in general, it is shown to be NP-complete to decide whether the inertia of the matrix is not determined by the sign pattern.

1 Introduction

The inertia of a symmetric matrix indicates the number of the positive/negative eigenvalues. This is an important quantity invariant under the congruence transformation. Quadratic forms are classified by the inertia of the coefficient matrices. In this paper, we discuss computing the inertia of a given symmetric matrix from the sign pattern of its entries without numerical information.

Matrix analysis with sign pattern has been studied by many researchers (see [2]). The analysis is called *qualitative matrix theory*. It is useful because the sign pattern of a matrix can be easily inferred in a variety of situations, rather than the exact value of the matrix entries. For a matrix A, we denote by $\mathcal{Q}(A)$ the set of all matrices having the same sign pattern as A, which is called the *sign pattern class* of A. Klee, Ladner and Manber[5] proved that for a rectangular matrix A, it is NP-complete to discern whether there exists $\tilde{A} \in \mathcal{Q}(A)$ which is not row full-rank. However, Robertson, Seymour and Thomas[8] devised a polynomial time algorithm to discern whether $\tilde{A}$ is nonsingular for any square matrix $\tilde{A} \in \mathcal{Q}(A)$.

This paper deals with the sign pattern class of symmetric matrices, which is denoted by $\mathcal{Q}^*(A) := \{\tilde{A} \mid \tilde{A} \in \mathcal{Q}(A), \tilde{A}^\top = \tilde{A}\}$. A symmetric matrix A is said to be *sign-nonsingular* if $\tilde{A}$ is nonsingular for any symmetric matrix $\tilde{A} \in \mathcal{Q}^*(A)$. We first discuss computing the inertia of a sign-nonsingular symmetric matrix from the sign pattern. Hall, Li and Wang[3] gave several characterizations of the symmetric sign patterns that require unique inertia in 2001. Their result implies that the inertia of a sign-nonsingular symmetric matrix is determined uniquely by the sign pattern. We prove the uniqueness of the inertia of a sign-nonsingular symmetric matrix in a different way from Hall et al., that is, we find a nested sequence $\{A_k\}_{k=1}^n$ of the principal submatrices of order k such that at least one of A_{k-1} and A_k is sign-nonsingular for any $k = 2, \ldots, n$. Our proof is based on a characterization of the structure of a symmetric bipartite graph. Furthermore, our proof naturally provides an $\mathrm{O}(nm)$ time algorithm for computing the inertia of a sign-nonsingular symmetric matrix of order n with m nonzero entries.

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We also discuss the problem of deciding whether a symmetric matrix is sign-nonsingular or not. We prove that the decision problem whether a given matrix is not a sign-nonsingular symmetric matrix is NP-complete by the result of Klee et al. Hence, for a symmetric matrix A in general, it is NP-complete to discern whether the inertia of $\tilde{A}$ is not the same for all $\tilde{A} \in \mathcal{Q}^*(A)$.

The present paper is organized as follows. Section 2 contains some notations and preliminaries about matrices and bipartite graphs. In Section 3, we recapitulate the inertia of a symmetric matrix in terms of linear algebra. From Section 4 to 6, we discuss the inertia of a sign-nonsingular symmetric matrix. In Section 4, we give the structure of a symmetric bipartite graph with perfect matchings. In Section 5, we show that the inertia of a sign-nonsingular symmetric matrix A is determined uniquely be the sign pattern. In Section 6, we design an efficient algorithm for computing the inertia of a sign-nonsingular symmetric matrix. In Section 7, we describe NP-completeness of the decision problem whether any $\tilde{A} \in \mathcal{Q}^*(A)$ does not have the same inertia for a symmetric matrix A in general.

2 Matrices and Bipartite Graphs

For a matrix A, the *row index set* and the *column index set* are denoted by $\mathrm{Row}(A)$ and $\mathrm{Col}(A)$, i.e., $A = (a_{ij} \mid i \in \mathrm{Row}(A), j \in \mathrm{Col}(A))$, where a_{ij} is the (i,j)-entry of A. For $I \subseteq \mathrm{Row}(A)$ and $J \subseteq \mathrm{Col}(A)$, $A[I,J] = (a_{ij} \mid i \in I, j \in J)$ means the submatrix of A with row set I and column set J.

For a square matrix A of order n, the *determinant* of A is defined by

$$\det A := \sum_{\pi \in \mathcal{S}_n} \mathrm{sgn}\pi \prod_{i=1}^{n} a_{i\pi(i)}, \tag{2.1}$$

where $\mathcal{S}_n$ denotes the set of all the permutations of order n, and $\mathrm{sgn}\pi \in \{1,-1\}$ is the signature of the permutation $\pi \in \mathcal{S}_n$. A square matrix is said to be *nonsingular* if its determinant is distinct from zero.

As a combinatorial counterpart of nonsingularity, we say that a matrix A is *term-nonsingular* if the expansion in (2.1) contains at least one nonvanishing term, that is, if $a_{i\pi(i)} \neq 0$ ($\forall i \in \mathrm{Row}(A)$) for some permutation $\pi \in \mathcal{S}_n$. Obviously, nonsingularity implies term-nonsingularity, since the right-hand side of (2.1) is distinct from zero only if the summation contains a nonzero term. The *term-rank* of A is the maximum size of a term-nonsingular submatrix of A. We denote term-rank of A by $\text{t-rank}A$. Hence we have $\text{t-rank}A \geq \mathrm{rank}A$.

Let $G = (U,V;E)$ be a bipartite graph with vertex sets U, V and an edge set $E \subseteq U \times V$. A *path* $P \subseteq E$ is a sequence of consecutive edges in a graph. A *circuit* $C \subseteq E$ is a path which ends at the vertex it begins. For $F \subseteq E$, we denote by ∂F the set of all the end-vertices of edges in F, i.e., $\partial F := \{u_i, v_j \mid (u_i, v_j) \in F\}$. An edge subset M in G is called a *matching* if $2|M| = |\partial M|$, and a matching M is said to be a *perfect matching* if $\partial M = U \cup V$.

With a matrix A, we associate a bipartite graph $G(A) = (U,V;E)$ with vertex sets $U := \{u_i \mid i \in \mathrm{Row}(A)\}$ and $V := \{v_j \mid j \in \mathrm{Col}(A)\}$. The edge set E is given by $E := \{(u_i, v_j) \mid a_{ij} \neq 0, u_i \in U, v_j \in V\}$, that is, an edge of $G(A)$ represents a nonvanishing entry of A. A perfect matching in $G(A)$ corresponds to one nonvanishing term of $\det A$. Therefore, A is term-nonsingular if and only if $G(A)$ has a perfect matching. Furthermore, the term-rank of A is equal to the maximum size of a matching in $G(A)$.

For a square matrix A, we say that A has an *equisignum* determinant if every nonvanishing term of the determinant of A has the same sign. If a square matrix has an equisignum determinant, the matrix is obviously nonsingular. It is known that $\tilde{A}$ is nonsingular for any $\tilde{A} \in \mathcal{Q}(A)$ if and only if A has an equisignum determinant[5]. Hence this is a sufficient condition for a matrix to be a sign-nonsingular symmetric matrix.

Consider a directed bipartite graph $D = (U,V;E)$. An edge subset F is said to be *central* if the subgraph obtained from D by deleting the vertices ∂F has a perfect matching. A circuit is said to be *oddly oriented* in D if the circuit contains an odd number of edges that are directed in the direction of each orientation of the circuit. We say that a directed graph D is *Pfaffian* if every central circuit in D is oddly oriented.

Let $D(A)$ be a directed bipartite graph associated with a matrix A defined as follows. A vertex set of $D(A)$ is the same as that of $G(A)$, and an edge $e = (u_i, v_j)$ is oriented from u_i to v_j if $a_{ij} > 0$, and from v_j to u_i if $a_{ij} < 0$. It is known that A has an equisignum determinant if and only if $D(A)$ is Pfaffian[6]. The complexity status of the decision problem whether a given matrix has an equisignum determinant or not had been an open problem for a long time. This problem is polynomial time equivalent to the decision problem whether a given undirected bipartite graph has a Pfaffian orientation[9]. In 1999, Robertson, Seymour and Thomas[8] gave a polynomial time algorithm to discern if a given bipartite graph has a Pfaffian orientation. This algorithm can be easily applied to discern if a given directed bipartite graph is Pfaffian or not. Thus it can be tested in polynomial time whether $\tilde{A}$ is nonsingular for any $\tilde{A} \in \mathcal{Q}(A)$ for a given matrix A.

3 The Inertia of Symmetric Matrices

In this section, we recapitulate the inertia of a symmetric matrix in terms of linear algebra. We consider a symmetric matrix A of order n, where $\mathrm{Row}(A)$ and $\mathrm{Col}(A)$ are both identical with a finite set N of cardinality n.

Let $p(A)$ be the number of positive eigenvalues of A, $q(A)$ the number of negative eigenvalues, and $z(A)$ the number of zero eigenvalues, all counting multiplicity. The ordered triple $\mathrm{In}(A) := (p(A), q(A), z(A))$ is called the *inertia* of A. By the definition of the inertia, we have

$$\begin{aligned} \mathrm{rank} A &= p(A) + q(A), && (3.1) \\ z(A) &= n - \mathrm{rank} A. && (3.2) \end{aligned}$$

Consider transforming A to $S^\top AS$ by a nonsingular matrix S. This transformation is called a *congruence transformation*. It is known that the inertia of a symmetric matrix is invariant under congruence transformations (Sylvester's law of inertia[4]). Hence, a symmetric matrix A is congruent to a diagonal matrix as follows:

$$S^\top AS = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1, 0, \ldots, 0), \tag{3.3}$$

where the indicated matrix has exactly $p(A)$ "1", $q(A)$ "-1", and $z(A)$ "0" in its diagonal entries.

Let us consider the determinant of $S^\top AS$. If A is nonsingular, it follows from (3.3) that

$$\det(S^\top AS) = (\det S)^2 \det A = (-1)^{q(A)}. \tag{3.4}$$

The sign of $\det A$ is thus determined by the parity of $q(A)$.

Let A_k be a principal submatrix of order k for $1 \leq k \leq n$. Then there exists a nonsingular matrix S such that

$$S^\top AS = \begin{pmatrix} I_{p(A_k)} & O & O \\ O & -I_{q(A_k)} & O \\ O & O & H \end{pmatrix}, \tag{3.5}$$

where H is a symmetric matrix. This implies

$$p(A_k) \leq p(A), \quad q(A_k) \leq q(A).$$

Let $\{A_k\}_{k=1}^n$ be a nested sequence of principal submatrices of order k ($A_n := A$). In a similar way, we have

$$p(A_k) \leq p(A_l), \quad q(A_k) \leq q(A_l), \tag{3.6}$$

for each $1 \le k \le l \le n$.

A nonsingular symmetric matrix A has a nested sequence $\{A_k\}_{k=1}^n$ of principal submatrices such that no two consecutive submatrices are singular. We call such a sequence *admissible*. If we find an admissible nested sequence $\{A_k\}_{k=1}^n$ of principal submatrices, then the inertia of A can be obtained by comparing the signs of $\det A_k$ as follows.

If both A_k and A_{k-1} are nonsingular ($k \ge 2$), then (3.4) implies that the signs of the principal minors $\det A_k$ and $\det A_{k-1}$ are equal to $(-1)^{q(A_k)}$ and $(-1)^{q(A_{k-1})}$, respectively. Moreover, the difference between $q(A_k)$ and $q(A_{k-1})$ is at most one by (3.1) and (3.6). Therefore, we have

$$\mathrm{In}(A_k) = \begin{cases} \mathrm{In}(A_{k-1}) + (1,0,0) & (\text{if } \operatorname{sgn}\det A_k = \operatorname{sgn}\det A_{k-1}) \\ \mathrm{In}(A_{k-1}) + (0,1,0) & (\text{if } \operatorname{sgn}\det A_k \neq \operatorname{sgn}\det A_{k-1}). \end{cases} \tag{3.7}$$

In an admissible sequence $\{A_k\}_{k=1}^n$, if A_k is nonsingular and A_{k-1} is singular ($k \ge 3$), then A_{k-2} is nonsingular. Consider transforming A_k by a nonsingular matrix S to diagonalize A_{k-2} as (3.5). Since A_{k-1} is singular and A_k is nonsingular, this implies the determinant of the submatrix H in the right-hand side of (3.5) is nonpositive. Hence $\operatorname{sgn}\det A_k \neq \operatorname{sgn}\det A_{k-2}$. Then, by (3.4), the parities of $q(A_k)$ and $q(A_{k-2})$ are different. Moreover, the difference between $q(A_k)$ and $q(A_{k-2})$ is at most two by (3.1) and (3.6). Therefore, we have

$$\mathrm{In}(A_k) = \mathrm{In}(A_{k-2}) + (1,1,0). \tag{3.8}$$

4 Symmetric Bipartite Graphs

Let $G = (U, V; E)$ be a bipartite graph with vertex sets $U := \{u_1, \dots, u_n\}$ and $V := \{v_1, \dots, v_n\}$. Both of the vertex sets are identified with $N := \{1, \dots, n\}$. A bipartite graph $G = (U, V; E)$ is said to be *symmetric* if $e = (u_i, v_j) \in E$ implies $(u_j, v_i) \in E$ for all $e \in E$. The bipartite graph $G(A)$ associated with a symmetric matrix A is symmetric.

Let $G = (U, V; E)$ be a symmetric bipartite graph with perfect matchings. An edge $(u_i, v_i) \in E$ is called a *diagonal* edge for $i \in N$. For an edge subset F of G, we call $F^\top := \{(u_j, v_i) \mid (u_i, v_j) \in F\}$ the *transpose* of F. If $F^\top$ coincides with F, it is called *symmetric*, otherwise *asymmetric*. For edge subsets F_1 and F_2 of G, we denote by $F_1 \triangle F_2$ the symmetric difference between F_1 and F_2.

Let M be a perfect matching in G. We denote by π a permutation corresponding to M. Since G is symmetric, $M^\top$ is also a perfect matching in G. Then $M^\top$ corresponds to the inverse permutation π^{-1}. A permutation π is expressed uniquely as a product of some cyclic permutations. We suppose $\pi = \prod_{k=1}^h \sigma_k$, where h is a positive integer and σ_k is a cyclic permutation. Let $M_k \subseteq M$ be a matching in G corresponding to σ_k. Then M is the disjoint union of M_k, that is, $M = \bigcup_{k=1}^h M_k$.

Let C be a circuit in G. Since G is symmetric, the transpose $C^\top$ is also a circuit in G. A *chord* of a circuit is an edge in $E \setminus C$ having both ends on the circuit. We say that a circuit C is *chordless* if C has no chord.

Theorem 4.1 *Let G be a symmetric bipartite graph with perfect matchings. Then G satisfies the following* (a), (b) or (c).

(a) There exists a perfect matching M in G such that $(u_i, v_i) \in M$ for some $i \in N$.

(b) There exists a perfect matching M in G such that $(u_i, v_j) \in M$ and $(u_j, v_i) \in M$ for some distinct $i, j \in N$.

(c) The symmetric bipartite graph G is a disjoint union of some chordless symmetric circuits.

Proof. By the assumption, the symmetric bipartite graph G has a perfect matching M and its transposed perfect matching $M^\top$. Furthermore, M is the disjoint union of matchings M_k for $k = 1, \dots, h$ corresponding to cyclic permutations.

We will show that if G does not satisfy the condition (a) or (c), then G satisfies the condition (b).

First, we claim that $M \cup M^\top$ consists of $C_k \cup C_k^\top$ for $k = 1, \ldots, h$, where C_k is a circuit (C_k may coincide with $C_k^\top$). Indeed, $M \cup M^\top$ is the disjoint union of $M_k \cup M_k^\top$ for $k = 1, \ldots, h$. Since M does not have either (u_i, v_j) or (u_j, v_i) for some $i, j \in N$ (otherwise, G satisfies (a) or (b)), $M_k \cup M_k^\top$ consists of circuits. If $|M_k|(\geq 3)$ is odd, then $M_k \cup M_k^\top$ is a symmetric circuit of length $2|M_k|$ (see Fig.1 for the case of $|M_k| = 5$). If $|M_k|(\geq 4)$ is even, then $M_k \cup M_k^\top$ consists of an asymmetric circuit of length $|M_k|$ and its transpose (see Fig.2 for the case of $|M_k| = 4$).

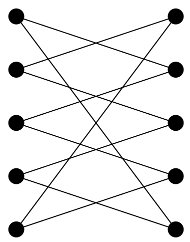

Figure 1: The union $M_k \cup M_k^\top$ ($|M_k| = 5$).

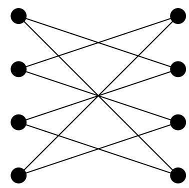

Figure 2: The union $M_k \cup M_k^\top$ ($|M_k| = 4$).

Suppose that there exists an asymmetric circuit C_k and its transposed circuit $C_k^\top (\neq C_k)$ in $M \cup M^\top$. Then $M' := M \triangle C_k^\top$ is also a perfect matching in G, and M' has both (u_j, v_i) and (u_i, v_j) for each $(u_i, v_j) \in C_k \cap M$. This implies G satisfies the condition (b).

Hence we may suppose there exist only symmetric circuits in $M \cup M^\top$. Since G is not a disjoint union of some chordless circuits, (i) there exists $(u_i, v_j) \in E \setminus (M \cup M^\top), u_i \in \partial C_k, v_j \in \partial C_l$ for some distinct $1 \leq k, l \leq h$, or (ii) there exists $(u_i, v_j) \in E \setminus (M \cup M^\top), u_i \in \partial C_k, v_j \in \partial C_k$ for some $1 \leq k \leq h$.

Suppose G satisfies (i). Then there also exists $(u_j, v_i) \in E \setminus (M \cup M^\top), u_j \in \partial C_l, v_i \in \partial C_k$, because of symmetry of $G(A)$. Let $P_k \subseteq C_k$ be a path from $u_i \in U \cap \partial C_k$ to $v_i \in V \cap \partial C_k$ such that both end-vertices of P_k are incident with an edge of $M \cap P_k$. Let $P_l \subseteq C_l$ be a path in a similar way. Then $C' := (P_k \cup P_l) \cup \{(u_i, v_j), (u_j, v_i)\}$ is a circuit in G. Furthermore, $M' := M \triangle C'$ is also a perfect matching in G, and M' has both (u_i, v_j) and (u_j, v_i). Moreover, it shows that there exists $(u_{j'}, v_{i'}) \in (P_k^\top \cup P_l^\top) \cap M'$ for each $(u_{i'}, v_{j'}) \in (P_k \cup P_l) \cap M'$ because C_k and C_l are symmetric. Thus G satisfies the condition (b).

Suppose G satisfies (ii). Then let $P_k \subseteq C_k$ be a path from $u_i \in U \cap \partial C_k$ to $v_j \in V \cap \partial C_k$ such that both end-vertices of P_k are incident with an edge of $M \cap P_k$. Then $C' := P_k \cup \{(u_i, v_j)\}$ is a circuit in G. Furthermore, $M' := M \triangle C'$ is also a perfect matching in G, and there exists $(u_{j'}, v_{i'}) \in C_k \cap M'$ for each $(u_{i'}, v_{j'}) \in (P_k \triangle P_k^\top) \cap M'$. Thus G satisfies the condition (b).

Consequently, if G does not satisfy the condition (a) or (c), G satisfies the condition (b). ■

5 Uniqueness of the Inertia

In this section, we discuss the uniqueness of the inertia of a sign-nonsingular symmetric matrix A. For convenience, we denote by $A[J]$ the principal submatrix $A[J, J]$ for $J \subseteq N$. Since sign-nonsingularity implies nonsingularity, it is clear that $z(A) = 0$.

A nested sequence $\{A_k\}_{k=1}^n$ of principal submatrices is called *sign-admissible* if at least one of A_{k-1} and A_k is sign-nonsingular for any $k = 2, \ldots, n$. It follows from (3.7) and (3.8) that if A has a sign-admissible sequence, the inertia of A is determined uniquely by the sign pattern.

However, it is not easy to find such a sequence. Even if a symmetric matrix A has an equisignum determinant, a principal submatrix of A may not be sign-nonsingular. For example,

$$A = \begin{pmatrix} +1 & +1 & +1 & -1 \\ +1 & +1 & +1 & +1 \\ +1 & +1 & -1 & 0 \\ -1 & +1 & 0 & 0 \end{pmatrix}$$

has an equisignum determinant, and hence A is sign-nonsingular, while a nested sequence of the leading principal submatrices is not sign-admissible ($A[\{1,2\}]$ and $A[\{1,2,3\}]$ are not sign-nonsingular). However, the inertia is determined independently of the magnitude of the entries, $\mathrm{In}(A) = (2,1,0)$. This is because each of $A[\{2\}]$, $A[\{2,3\}]$ and $A[\{1,2,3,4\}]$ has an equisignum determinant.

Thus, it is sufficient to find one sign-admissible nested sequence of principal submatrices for determining the inertia uniquely from the sign pattern. To find a sign-admissible nested sequence of principal submatrices, we will use the characterization of a symmetric bipartite graph described in Section 4.

The main result in this section is the following theorem.

Theorem 5.1 *Let A be a sign-nonsingular symmetric matrix. Then there exists a sign-admissible nested sequence of principal submatrices. Hence $\tilde{A}$ has the same inertia for any $\tilde{A} \in \mathcal{Q}^*(A)$.*

We first provide the following three lemmas to prove Theorem 5.1.

Lemma 5.2 *Let A be a sign-nonsingular symmetric matrix such that $G(A)$ satisfies the condition* (a) of Theorem 4.1, that is, there exists a perfect matching M in $G(A)$ containing (u_i, v_i) for some $i \in N$. Then the principal submatrix $A' := A[N \setminus \{i\}]$ is a sign-nonsingular symmetric matrix, and the inertia of A is obtained by

$$\mathrm{In}(A) = \begin{cases} \mathrm{In}(A') + (1,0,0) & (\text{if } a_{ii} > 0) \\ \mathrm{In}(A') + (0,1,0) & (\text{if } a_{ii} < 0). \end{cases} \tag{5.1}$$

Proof. The determinant of A is expanded as

$$\det A = a_{ii} \det A' + \sum_{j \neq i} (-1)^{i+j} a_{ij} \det A[N \setminus \{i\}, N \setminus \{j\}].$$

Since $G(A')$ has a perfect matching $M \setminus \{(u_i, v_i)\}$, A' is term-nonsingular. It then follows from the sign-nonsingularity of A that A' is a sign-nonsingular symmetric matrix (otherwise, the sign of the determinant of A can change by the magnitude of a_{ii}). Therefore, by (3.4), we have

$$\mathrm{sgn} \det A = (-1)^{q(A)} = \mathrm{sgn}(a_{ii})(-1)^{q(A')},$$

which implies that the parity of $q(A)$ depends on that of $q(A')$ and the sign of a_{ii}. Therefore, by (3.7), we have

$$(p(A), q(A)) = \begin{cases} (p(A') + 1, q(A')) & (\text{if } a_{ii} > 0) \\ (p(A'), q(A') + 1) & (\text{if } a_{ii} < 0). \end{cases}$$

■

Lemma 5.3 *Let A be a sign-nonsingular symmetric matrix such that $G(A)$ does not satisfy the condition* (a) of Theorem 4.1. Suppose $G(A)$ satisfies the condition (b), that is, there exists a perfect matching M in $G(A)$ such that $(u_i, v_j) \in M$ and $(u_j, v_i) \in M$ for some distinct $i, j \in N$. Then the principal submatrix $A' = A[N \setminus \{i, j\}]$ is a sign-nonsingular symmetric matrix, and the inertia of A is obtained by

$$\mathrm{In}(A) = \mathrm{In}(A') + (1,1,0). \tag{5.2}$$

Proof. In the same way as Lemma 5.2, the determinant of A is expanded as

$$\det A = -(a_{ij})^2 \det A' + \sum_{\substack{(k,l)\neq(i,j),\\ k\neq l}} \pm \det A[\{k,l\},\{i,j\}] \det A[N \setminus \{k,l\}, N \setminus \{i,j\}].$$

Notice that $\det A[\{i,j\}] = -(a_{ij})^2$, because $G(A)$ has no perfect matching containing any diagonal edge by the assumption. Since A is a sign-nonsingular symmetric matrix and $G(A')$ has a perfect matching $M \setminus \{(u_i, v_j), (u_j, v_i)\}$, the principal submatrix A' is sign-nonsingular. Then, by (3.4), we have

$$\operatorname{sgn} \det A = (-1)^{q(A)} = (-1)^{q(A')+1},$$

which implies that the parity of $q(A)$ is different from that of $q(A')$. Therefore, by (3.8), we have

$$(p(A), q(A)) = (p(A') + 1, q(A') + 1).$$

■

Notice that it is not necessary that the principal submatrix $A[N \setminus \{i\}]$ or $A[N \setminus \{j\}]$ is sign-nonsingular.

Lemma 5.4 *Let A be a sign-nonsingular symmetric matrix of order $n = 2s + 1$ such that $G(A)$ is a chordless symmetric circuit. Let $2t$ be the number of the negative entries of A. Then the inertia of A is obtained by*

$$\operatorname{In}(A) = \begin{cases} (s+1, s, 0) & (\text{if } t \equiv s \bmod 2) \\ (s, s+1, 0) & (\text{if } t \not\equiv s \bmod 2). \end{cases} \tag{5.3}$$

Proof. Since $G(A)$ is a chordless symmetric circuit, the matrix size n is odd. Since $\det A$ has only two nonvanishing terms, we have

$$\operatorname{sgn} \det A = (-1)^t. \tag{5.4}$$

Let A' be the principal submatrix obtained by deleting the ith row and the ith column from A for any $i \in N$. Then $G(A')$ consists of two paths. Since $G(A')$ has only one perfect matching, A' has an equisignum determinant. Hence A' is a sign-nonsingular symmetric matrix. Then, by (3.4), we have

$$\operatorname{sgn} \det A' = (-1)^{q(A')}. \tag{5.5}$$

Since $G(A')$ consists of two paths and $G(A')$ has no diagonal edge, $G(A')$ satisfies the condition (b) of Theorem 4.1. By applying (5.2) repeatedly, we have $p(A') = q(A') = s$. Hence, by (3.7), we have

$$(p(A), q(A)) = \begin{cases} (s+1, s) & (\text{if } \operatorname{sign} \det A = \operatorname{sign} \det A') \\ (s, s+1) & (\text{if } \operatorname{sign} \det A \neq \operatorname{sign} \det A'), \end{cases}$$

which together with (5.4) and (5.5) implies (5.3). ■

A sign-nonsingular symmetric matrix A such that $G(A)$ is a chordless circuit has a sign-admissible sequence. Indeed, by row and column permutations, A is represented as

$$A = \begin{pmatrix} 0 & * & 0 & \cdots & 0 & * \\ * & 0 & * & \ddots & & 0 \\ 0 & * & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * & 0 \\ 0 & & \ddots & * & 0 & * \\ * & 0 & \cdots & 0 & * & 0 \end{pmatrix},$$

where $*$ designates a nonzero entry of A. Hence any nested sequence of principal submatrices in A is sign-admissible.

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. By Lemmas 5.2–5.4 and Theorem 4.1, if A is a sign-nonsingular symmetric matrix, then there is a sign-admissible nested sequence of principal submatrices in A. Therefore, the inertia of A is uniquely determined by the sign pattern. ∎

6 Algorithm for Sign-Nonsingular Symmetric Matrices

In this section, we present a polynomial time algorithm for computing the inertia of a given sign-nonsingular symmetric matrix.

Let A be a sign-nonsingular symmetric matrix of order n. Then $z(A) = 0$. It is easy to see from Lemmas 5.2–5.4 that the inertia is obtained by computing the inertia of a sign-nonsingular symmetric matrix smaller than A, if we find a perfect matching M in $G(A)$ which satisfies the condition (a) or (b) of Theorem 4.1. Thus we can compute the inertia of A recursively. However, we present a practically more efficient algorithm that computes the inertia by finding circuits in $M \cup M^\top$ in the same way as the proof of Theorem 4.1.

The algorithm starts with finding a perfect matching M_d in $G(A)$ with maximum number of diagonal edges. The index set N of the row and column set is partitioned into $J_+ := \{j \mid (u_j, v_j) \in M_d, a_{jj} > 0\}$, $J_- := \{j \mid (u_j, v_j) \in M_d, a_{jj} < 0\}$ and $J_* := \{j \mid (u_j, v_j) \notin M_d\}$. We can find M_d efficiently by the maximum weight perfect matching algorithm. Indeed, define the weight function $w : E \to \{0, 1\}$ on the bipartite graph $G(A)$ by $w_e = 1$ for each diagonal edge e and $w_e = 0$ for the other edges. Then a maximum weight perfect matching with respect to w corresponds to M_d. The maximum weight perfect matching algorithm runs in $\mathrm{O}(n(m + nW))$ time with Dial's implementation of Dijkstra's shortest path algorithm[1], where W is the largest weight in the graph and m is the number of edges. In this case, W is equal to 1. Hence it requires $\mathrm{O}(nm)$ time.

Let $\mathcal{C}$ be the set of all symmetric circuits in $M_d \cup M_d^\top$. If there exist a pair of symmetric circuits $C, C' \in \mathcal{C}$ connected by an edge in $E \setminus (M_d \cup M_d^\top)$, then delete C and C' from $\mathcal{C}$. We repeat this until there are no such pair of symmetric circuits.

We denote by $C_1, \ldots, C_h$ the remaining symmetric circuits in $\mathcal{C}$. Let J_k be the indices corresponding to C_k for each $k = 1, \ldots, h$, that is, $J_k := \{i \mid (u_i, v_j) \in C_k\}$. Let J_0 be the remaining indices, that is, $J_0 := J_* \setminus \bigcup_{k=1}^h J_k$. Then it follows from Lemmas 5.2 and 5.3 that the inertia of A is obtained by

$$\begin{aligned} p(A) &= |J_+| + \frac{1}{2}|J_0| + \sum_{k=1}^{h} p(A[J_k]), \\ q(A) &= |J_-| + \frac{1}{2}|J_0| + \sum_{k=1}^{h} q(A[J_k]). \end{aligned} \tag{6.1}$$

The first term in the right-hand side of (6.1) is obtained by applying (5.1) repeatedly to each $i \in J_+ \cup J_-$, and the second term by applying (5.2) repeatedly to each $i, j \in J_0$.

The inertia of $A[J_k]$ for $k = 1, \ldots, h$ is obtained by Lemmas 5.3 and 5.4. Indeed, if C_k has a chord, then find a chordless symmetric circuit C'_k in $G(A[J_k])$ (otherwise $C'_k := C_k$). Let J'_k be the indices corresponding to C'_k. Let $2t'_k$ be the number of the negative entries of $A[J'_k]$. Comparing $s'_k = (|J'_k| - 1)/2$ with t'_k, we have, by (5.2) and (5.3), that

$$(p(A[J_k]), q(A[J_k])) = \begin{cases} (s_k + 1, s_k) & (\text{if } t'_k \equiv s'_k \bmod 2) \\ (s_k, s_k + 1) & (\text{if } t'_k \not\equiv s'_k \bmod 2), \end{cases} \tag{6.2}$$

where $s_k := (|J_k| - 1)/2$. Thus the inertia of A can be computed from (6.1) and (6.2).

1. Find a perfect matching M_d in $G(A)$ with maximum number of diagonal edges.
2. Find the set of all symmetric circuits $\mathcal{C}$ in $M_d \cup M_d^\top$.
3. Repeat the following.
   - If there exists a pair of symmetric circuits C and C' in $\mathcal{C}$ connected by an edge in $E \setminus (M_d \cup M_d^\top)$, then delete C and C' from $\mathcal{C}$.
4. Denote the remaining symmetric circuits in $\mathcal{C}$ by $C_1, \ldots, C_h$. Let J_k be the indices corresponding to C_k. For each connected component $G(A[J_k])$ for $k = 1, \ldots, h$, do the following.
   - If C_k has a chord, then find a chordless symmetric circuit C'_k in $G(A[J_k])$.
   - Compute the inertia of $A[J_k]$ by (6.2).
5. Return the inertia of A obtained by (6.1).

Figure 3: Algorithm for computing the inertia of a sign-nonsingular symmetric matrix

The algorithm is now summarized in Fig. 3.

The algorithm requires $\mathrm{O}(mn)$ time in total. Indeed, it requires $\mathrm{O}(nm)$ time to find a perfect matching M_d. In addition, it requires $\mathrm{O}(m)$ time to find symmetric circuits $C_1, \ldots, C_h$ in $M_d \cup M_d^\top$, and $\mathrm{O}(m)$ time to find a chordless symmetric circuit in $G(A[J_k])$ for $k = 1, \ldots, h$.

Theorem 6.1 *For a sign-nonsingular symmetric matrix A of order n, the inertia of A can be computed in $\mathrm{O}(nm)$ time, where m is the number of nonzero entries of A.*

7 Complexity of Testing Sign-Nonsingularity

In this section, we discuss the complexity status of the decision problem whether the inertia of a given symmetric matrix is uniquely determined by the sign pattern of the matrix entries. Hall, Li and Wang[3] proved the following theorem.

Theorem 7.1 (Hall, Li and Wang [3]) *For a given symmetric matrix A, $\tilde{A}$ has the same inertia for any $\tilde{A} \in \mathcal{Q}^*(A)$ if and only if*

$$\max\{\mathrm{rank}\tilde{A} \mid \tilde{A} \in \mathcal{Q}^*(A)\} = \min\{\mathrm{rank}\tilde{A} \mid \tilde{A} \in \mathcal{Q}^*(A)\}.$$

Furthermore, it is known that t-rank$A = \max\{\mathrm{rank}\tilde{A} \mid \tilde{A} \in \mathcal{Q}^*(A)\}$ [7]. Hence the inertia of A is uniquely determined by the sign pattern if and only if A satisfies

$$\text{t-rank}A = \min\{\mathrm{rank}\tilde{A} \mid \tilde{A} \in \mathcal{Q}^*(A)\}. \tag{7.1}$$

However, it is not clear how to discern efficiently if a given matrix satisfies (7.1) or not.

First, we suppose A is a term-nonsingular matrix of order n. Then A satisfies (7.1) if and only if A is a sign-nonsingular symmetric matrix. As already mentioned in Section 2, it can be done in polynomial time to discern whether a given symmetric matrix has an equisignum determinant. However, having an equisignum determinant is not a necessary condition for a symmetric matrix to be sign-nonsingular. For example,

$$A = \begin{pmatrix} +1 & 0 & +1 & +1 \\ 0 & +1 & +1 & +1 \\ +1 & +1 & -1 & 0 \\ +1 & +1 & 0 & -1 \end{pmatrix}$$

is a sign-nonsingular symmetric matrix, while it does not have an equisignum determinant[3].

We prove the following theorem.

Theorem 7.2 *For a symmetric matrix A, the problem of deciding whether A is not a sign-nonsingular symmetric matrix is NP-complete.*

To prove Theorem 7.2, we use the following result by Klee, Ladner and Manber[5].

Theorem 7.3 (Klee, Ladner and Manber [5]) *For each positive integer k, the problem of deciding whether an $n \times (n + \lfloor n^{1/k} \rfloor)$ matrix is not row full-rank only by the sign pattern is NP-complete.*

Proof of Theorem 7.2. It is clear that the decision problem whether a given matrix A is not a sign-nonsingular symmetric matrix is in NP, as it suffices to exhibit a singular symmetric matrix with the same sign pattern as A.

Suppose we can discern in polynomial time whether A is not a sign-nonsingular symmetric matrix. Consider

$$A = \begin{pmatrix} O & B \\ B^\top & I \end{pmatrix},$$

where B is a rectangular matrix such that $|\mathrm{Col}(B)| = |\mathrm{Row}(B)|$ and t-rank$B = |\mathrm{Row}(B)|$, and I is an identity matrix. Then $\det A = -\det B^\top B$ holds. Since $B^\top B$ is a positive semidefinite matrix, this implies $\det A \leq 0$. By the assumption, we can discern in polynomial time whether there exists $\tilde{A} = \begin{pmatrix} O & \tilde{B} \\ \tilde{B}^\top & \tilde{I} \end{pmatrix} \in \mathcal{Q}^*(A)$ such that $\det \tilde{A} = -\det \tilde{B}^\top \tilde{B} = 0$. This implies that we can test in polynomial time whether there exists $\tilde{B} \in \mathcal{Q}(B)$ which is not row full-rank. Hence it is NP-complete to discern whether A is not a sign-nonsingular symmetric matrix by Theorem 7.3. ■

It follows from Theorem 7.2 that the decision problem whether the inertia of A is not determined by the sign pattern is also NP-complete.

Corollary 7.4 *For a symmetric matrix A, the problem of deciding whether the inertia of A is not determined by the sign pattern is NP-complete.*

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