A mathematical justification of the Isobe-Kakinuma model for water waves with and without bottom topography

Tatsuo Iguchi

Abstract

We consider the Isobe–Kakinuma model for water waves in both cases of the flat and the variable bottoms. The Isobe–Kakinuma model is a system of Euler–Lagrange equations for an approximate Lagrangian which is derived from Luke's Lagrangian for water waves by approximating the velocity potential in the Lagrangian appropriately. The Isobe–Kakinuma model consists of $(N+1)$ second order and a first order partial differential equations, where N is a nonnegative integer. We justify rigorously the Isobe–Kakinuma model as a higher order shallow water approximation in the strongly nonlinear regime by giving an error estimate between the solutions of the Isobe–Kakinuma model and of the full water wave problem in terms of the small nondimensional parameter δ, which is the ratio of the mean depth to the typical wavelength. It turns out that the error is of order $O(\delta^{4N+2})$ in the case of the flat bottom and of order $O(\delta^{4[N/2]+2})$ in the case of variable bottoms.

1 Introduction

In this paper we consider the motion of a water filled in $(n+1)$-dimensional Euclidean space together with the water surface. The water wave problem is mathematically formulated as a free boundary problem for an irrotational flow of an inviscid and incompressible fluid under the gravitational field. Let t be the time, $x = (x_1, \dots, x_n)$ the horizontal spatial coordinates, and z the vertical spatial coordinate. We assume that the water surface and the bottom are represented as $z = \eta(x,t)$ and $z = -h + b(x)$, respectively, where $\eta = \eta(x,t)$ is the surface elevation, h is the mean depth, and $b = b(x)$ represents the bottom topography. As was shown by J. C. Luke [17], the water wave problem has a variational structure. His Lagrangian density is of the form

$$\mathscr{L}_{\mathrm{Luke}}(\Phi,\eta) = \int_{-h+b(x)}^{\eta(x,t)} \left(\partial_t \Phi(x,z,t) + \frac{1}{2}|\nabla_X \Phi(x,z,t)|^2 + gz \right) \mathrm{d}z, \tag{1.1}$$

where $\Phi = \Phi(x,z,t)$ is the velocity potential, $\nabla_X = (\nabla, \partial_z) = (\partial_{x_1}, \dots, \partial_{x_n}, \partial_z)$, and g is the gravitational constant. He showed that the corresponding Euler–Lagrange equation is exactly the basic equations for water waves. Concerning the water wave problem, we refer to H. Lamb [13], J. J. Stoker [22], and D. Lannes [15].

M. Isobe [8, 9] and T. Kakinuma [10, 11, 12] approximated the velocity potential Φ in Luke's Lagrangian by

$$\Phi^{\mathrm{app}}(x,z,t) = \sum_{i=0}^{N} \Psi_i(z;b)\phi_i(x,t),$$

where $\{\Psi_i\}$ is an appropriate function system in the vertical coordinate z and may depend on the bottom topography b and $(\phi_0, \phi_1, \dots, \phi_N)$ are unknown variables, and derived an approximate Lagrangian density $\mathscr{L}^{\mathrm{app}}(\phi_0, \phi_1, \dots, \phi_N, \eta) = \mathscr{L}_{\mathrm{Luke}}(\Phi^{\mathrm{app}}, \eta)$. The Isobe–Kakinuma model is the corresponding Euler–Lagrange equation for the approximated Lagrangian. We have to choose the function system $\{\Psi_i\}$ carefully in order that the Isobe-Kakinuma model would be a good approximation for the full water wave problem. One of the choices is obtained by the bases of a Taylor series of the velocity potential $\Phi(x,z,t)$ with respect to the vertical spatial coordinate z around the bottom. Such an expansion has been already used by J. Boussinesq [2] in the case of the flat bottom. From this point of view, one of the natural choices of the function system is given by

$$\Psi_i(z;b) = \begin{cases} (z+h)^{2i} & \text{in the case of the flat bottom,} \\ (z+h-b(x))^i & \text{in the case of the variable bottom.} \end{cases}$$

Here we note that the later choice is valid also for the case of the flat bottom. However, it turns out that the terms of odd degree do not play any important role in such a case so that the former choice economizes the computational resources in the numerical computations. In this paper, to treat the both cases at the same time, we adopt the approximation

$$\Phi^{\mathrm{app}}(x,z,t) = \sum_{i=0}^{N} (z+h-b(x))^{p_i}\phi_i(x,t), \tag{1.2}$$

where $p_0, p_1, \dots, p_N$ are nonnegative integers satisfying $0 = p_0 < p_1 < \cdots < p_N$. Then, the corresponding Isobe–Kakinuma model has the form

$$\begin{cases} H^{p_i}\partial_t\eta + \displaystyle\sum_{j=0}^{N}\Big\{\nabla\cdot\Big(\frac{1}{p_i+p_j+1}H^{p_i+p_j+1}\nabla\phi_j - \frac{p_j}{p_i+p_j}H^{p_i+p_j}\phi_j\nabla b\Big) \\ \qquad\qquad + \dfrac{p_i}{p_i+p_j}H^{p_i+p_j}\nabla b\cdot\nabla\phi_j - \dfrac{p_ip_j}{p_i+p_j-1}H^{p_i+p_j-1}(1+|\nabla b|^2)\phi_j\Big\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i=0,1,\dots,N, \\ \displaystyle\sum_{j=0}^{N} H^{p_j}\partial_t\phi_j + g\eta + \frac{1}{2}\Big\{\Big|\sum_{j=0}^{N}(H^{p_j}\nabla\phi_j - p_jH^{p_j-1}\phi_j\nabla b)\Big|^2 + \Big(\sum_{j=0}^{N}p_jH^{p_j-1}\phi_j\Big)^2\Big\} = 0, \end{cases} \tag{1.3}$$

where $H = H(x,t) = h + \eta(x,t) - b(x)$ is the depth of the water. Here and in what follows we use the notational convention $0/0 = 0$. This is the Isobe–Kakinuma model that we are going to consider in this paper. This system consists of $(N+1)$ evolution equations for η and only one evolution equation for $(N+1)$ unknowns $(\phi_0, \phi_1, \dots, \phi_N)$, so that this is a overdetermined and underdetermined composite system. For more details to this model, we refer R. Nemoto and T. Iguchi [21].

One of the interesting features of the model is its linear dispersion relation. Let $c_{IK}(\xi)$ and $c_{WW}(\xi)$ be the phase speed of the plane wave solution related to the wave vector $\xi \in \mathbf{R}^n$ of the linearized Isobe–Kakinuma model and the linearized water wave problem around the rest state in the case of the flat bottom, respectively. Then, under the choice $p_i = 2i$, $(c_{IK}(\xi))^2$ becomes $[2N/2N]$ Padé approximant of $(c_{WW}(\xi))^2$. More precisely, it holds that

$$\left|\left(\frac{c_{WW}(\xi)}{\sqrt{gh}}\right)^2 - \left(\frac{c_{IK}(\xi)}{\sqrt{gh}}\right)^2\right| \le C(h|\xi|)^{4N+2}$$

with a positive constant C depending only on N. Under the choice $p_i = i$, we do not have such a beautiful result as above, but we still have following nice estimate

$$\left|\left(\frac{c_{WW}(\xi)}{\sqrt{gh}}\right)^2 - \left(\frac{c_{IK}(\xi)}{\sqrt{gh}}\right)^2\right| \le C(h|\xi|)^{4[N/2]+2}.$$

For the details, we refer to R. Nemoto and T. Iguchi [21]. Since $h|\xi|$ is essentially the ratio of the mean depth to the wave length, these estimates anticipate that the Isobe–Kakinuma model would be an approximation to the full water wave problem with an error of order $O(\delta^{4N+2})$ in the case of the flat bottom with the choice $p_i = 2i$, and of order $O(\delta^{4[N/2]+2})$ in the case of variable bottoms with the choice $p_i = i$, where δ is a nondimensional parameter defined as the aspect ratio. In this paper, we will show that this is correct even for the nonlinear problem with variable bottoms. In a particular case, that is, in the case $N = 1$ and $p_1 = 2$ with the flat bottom, this was shown by T. Iguchi [7]. Therefore, this paper is a generalization of his results.

In order to compare this Isobe–Kakinuma model with the full water wave problem more precisely in the shallow water regime, we need to rewrite (1.3) in an appropriate nondimensional form. Let λ be the typical wave length and introduce a nondimensional parameter δ by the aspect ratio $\delta = h/\lambda$, which measures the shallowness of the water. We rescale the independent and the dependent variables by

$$x = \lambda\tilde{x}, \quad z = h\tilde{z}, \quad t = \frac{\lambda}{\sqrt{gh}}\tilde{t}, \quad \eta = h\tilde{\eta}, \quad \phi_i = \frac{\lambda\sqrt{gh}}{\lambda^{p_i}}\tilde{\phi}_i.$$

Here we note that these rescaling of dependent variables are related to the strongly nonlinear regime of the wave. Plugging these into (1.3) and dropping the tilde sign in the notation we obtain the Isobe–Kakinuma model in the nondimensional form. It follows directly from the equations that if the solution and its derivatives are uniformly bounded with respect to the small parameter δ, then ϕ_i is of order $O(\delta)$ in the case where p_i is an odd integer. Therefore, it is more convenient to rescale ϕ_i again by

$$\phi_i = \delta^{p_i - 2[p_i/2]}\tilde{\phi}_i,$$

where $[p_i/2]$ is the integer part of $p_i/2$. Note that $p_i - 2[p_i/2] = 0$ if p_i is even, $= 1$ if p_i is odd. Then, the Isobe–Kakinuma model in nondimensional form has the form

$$(1.4)\quad \begin{cases} H^{p_i}\partial_t\eta + \displaystyle\sum_{j=0}^{N}\delta^{2(p_j-[p_j/2])}\biggl\{\nabla\cdot\Bigl(\frac{1}{p_i+p_j+1}H^{p_i+p_j+1}\nabla\phi_j - \frac{p_j}{p_i+p_j}H^{p_i+p_j}\phi_j\nabla b\Bigr) \\ \qquad + \displaystyle\frac{p_i}{p_i+p_j}H^{p_i+p_j}\nabla b\cdot\nabla\phi_j - \frac{p_ip_j}{p_i+p_j-1}H^{p_i+p_j-1}(\delta^{-2}+|\nabla b|^2)\phi_j\biggr\} = 0 \\ \qquad\qquad \text{for} \quad i = 0,1,\ldots,N, \\ \displaystyle\sum_{j=0}^{N}\delta^{2(p_j-[p_j/2])}H^{p_j}\partial_t\phi_j + \eta + \frac{1}{2}\biggl\{\Bigl|\sum_{j=0}^{N}\delta^{2(p_j-[p_j/2])}(H^{p_j}\nabla\phi_j - p_jH^{p_j-1}\phi_j\nabla b)\Bigr|^2 \\ \qquad + \delta^2\Bigl(\displaystyle\sum_{j=0}^{N}\delta^{2(p_j-[p_j/2]-1)}p_jH^{p_j-1}\phi_j\Bigr)^2\biggr\} = 0, \end{cases}$$

where $H = H(x,t) = 1 + \eta(x,t) + b(x)$. We consider the initial value problem to this Isobe–Kakinuma model (1.4) under the initial conditions

$$(1.5)\qquad (\eta, \phi_0, \ldots, \phi_N) = (\eta_{(0)}, \phi_{0(0)}, \ldots, \phi_{N(0)}) \qquad \text{at} \quad t = 0.$$

Solvability of the initial value problem (1.4)–(1.5) was first given by Y. Murakami and T. Iguchi [20] in a particular case and then by R. Nemoto and T. Iguchi [21] in the general case under physically reasonable conditions on the initial data.

On the other hand, the initial value problem to the full water wave problem in Zakharov–Craig–Sulem formulation in the nondimensional form is written as

$$\begin{cases} \partial_t \eta - \Lambda(\eta, b, \delta)\phi = 0, \\ \partial_t \phi + \eta + \dfrac{1}{2}|\nabla\phi|^2 - \delta^2 \dfrac{(\Lambda(\eta, b, \delta)\phi + \nabla\eta \cdot \nabla\phi)^2}{2(1 + \delta^2|\nabla\eta|^2)} = 0, \end{cases} \tag{1.6}$$

$$(\eta, \phi) = (\eta_{(0)}, \phi_{(0)}) \qquad \text{at} \quad t = 0, \tag{1.7}$$

where $\phi = \phi(x, t)$ is the trace of the velocity potential Φ on the water surface, that is, $\phi(x, t) = \Phi(x, \eta(x, t), t)$ and $\Lambda(\eta, b, \delta)$ is the Dirichlet-to-Neumann map for Laplace's equation. More precisely, the linear operator $\Lambda(\eta, b, \delta)$ depending nonlinearly on the surface elevation η, the bottom topography b, and the parameter δ is defined by

$$\Lambda(\eta, b, \delta)\phi = (\delta^{-2}\partial_z\Phi - \nabla\eta \cdot \nabla\Phi)|_{z=\eta(x,t)}, \tag{1.8}$$

where Φ is a unique solution to the boundary value problem for Laplace's equation

$$\begin{cases} \Delta\Phi + \delta^{-2}\partial_z^2\Phi = 0 & \text{in} \quad -1 + b(x) < z < \eta(x, t), \\ \Phi = \phi & \text{on} \quad z = \eta(x, t), \\ \delta^{-2}\partial_z\Phi - \nabla b \cdot \nabla\Phi = 0 & \text{on} \quad z = -1 + b(x). \end{cases} \tag{1.9}$$

For the details of this formulation to the water wave problem, we refer to V. E. Zakharov [23], W. Craig and C. Sulem [3], and D. Lannes [15]. In this paper we will give an error estimate between the solutions of the initial value problems to the Isobe–Kakinuma model (1.4)–(1.5) and to the full water wave problem (1.6)–(1.7) under appropriate conditions on the initial data.

There are huge literatures devoted to modelization for the full water wave problem and many approximate models were proposed and analyzed, especially, in weekly nonlinear regimes. Even in the strongly nonlinear regime, there are several model equations. Among them, the most famous model is the shallow water equations, which is also called Saint-Venant equations. The equations in the full water wave problem (1.6) can be expanded with respect to δ^2 and the shallow water equations are derived in the limit $\delta \to +0$, so that the shallow water equations are the approximation of the full water wave problem with an error $O(\delta^2)$. This approximation of the equations leads to the approximation of the solution in the same order of the error as

$$|\eta^{\mathrm{WW}}(x, t) - \eta^{\mathrm{SW}}(x, t)| \lesssim \delta^2$$

on some time interval independent of δ, where η^{WW} and η^{SW} are the solutions to the full water wave problem and to the shallow water equations, respectively. The other famous model in the strongly nonlinear regime is the Green–Naghdi equations, which are derived by introducing the vertically averaged horizontal velocity field and by retaining the terms of order δ^2 in the expansion of the equations. Therefore, the Green–Naghdi equations are the approximation of the full water wave problem with an error $O(\delta^4)$. This approximation of the equations leads again to the approximation of the solution in the same order of the error as

$$|\eta^{\mathrm{WW}}(x, t) - \eta^{\mathrm{GN}}(x, t)| \lesssim \delta^4$$

on some time interval independent of δ, where η^{GN} is a solution to the Green–Naghdi equations. Concerning these and related results, we refer to Y. A. Li [16], T. Iguchi [5, 6], B. Alvarez-Samaniego and D. Lannes [1], and H. Fujiwara and T. Iguchi [4].

Compared to these approximations, the Isobe–Kakinuma model (1.4) is not the approximation of the equations so that it is not straight forward to derive a precise error estimate, even in the formal level. To analyze the shallow water approximation, we will further restrict ourselves to the following two cases:

(H1) $p_i = 2i$ $(i = 0, 1, \dots, N)$ with the flat bottom, that is, $b(x) \equiv 0$

(H2) $p_i = i$ $(i = 0, 1, \dots, N)$ with general bottom topographies

We rewrite the approximation (1.2) in the nondimensional form as

$$\Phi^{\mathrm{app}}(x,z,t) = \sum_{i=0}^{N} \delta^{2(p_i - [p_i/2])}(z+1-b(x))^{p_i}\phi_i(x,t), \tag{1.10}$$

while as was shown by J. Boussinesq [2], in the case of the flat bottom (which corresponds to the case (H1)) the velocity potential Φ which satisfies the boundary value problem (1.9) can be expanded in a Taylor series as

$$\Phi(x,z,t) = \sum_{i=0}^{\infty} \delta^{2i}(z+1)^{2i}\frac{(-\Delta)^i\phi_0(x,t)}{(2i)!},$$

where ϕ_0 is the trace of the velocity potential Φ on the bottom. Although ϕ_i is not equal to $\frac{1}{(2i)!}(-\Delta)^i\phi_0$, we may regard (1.10) to an approximation with an error of order $O(\delta^{2N+2})$ in the case (H1) and of order $O(\delta^{2[N/2]+2})$ in the case (H2). Therefore, one may expect that the Isobe–Kakinuma model would be an approximation with an error of these orders. However, surprisingly we shall see in this paper that the precise error is much smaller than these orders and is given by

$$|\eta^{\mathrm{WW}}(x,t) - \eta^{\mathrm{IK}}(x,t)| \lesssim \begin{cases} \delta^{4N+2} & \text{in the case (H1)}, \\ \delta^{4[N/2]+2} & \text{in the case (H2)}, \end{cases} \tag{1.11}$$

as was expected by the analysis of linear dispersion relations, where η^{IK} is the solution of the Isobe–Kakinuma model. As mentioned before, in the case $N=1$ and $p_1 = 2$ with flat bottom, this error estimate was shown by T. Iguchi [7].

As another higher order shallow water approximation in the strongly nonlinear regime, extended Green–Naghdi equations were proposed by Y. Matsuno [18, 19]. His δ^{2N} model is an approximation of the full water wave equations with an error of order δ^{2N+2} and contains $(2N+1)$th order derivative terms. As is well known, higher order derivative terms are troublesome in a numerical computation. Moreover, it is not so easy to write down explicitly the extended Green–Naghdi equations for large N. We remark also that until now there is no rigorous justification of his δ^{2N} model in the sense of approximation of the solutions as mentioned above. Compared to this model, the Isobe–Kakinuma model does not contain any higher order derivative terms. This is one of strong advantages of the Isobe–Kakinuma model.

The contents of this paper are as follows. In Section 2 we present our main results in this paper, that is, an existence of the solution of the initial value problem to the Isobe–Kakinuma model (1.4)–(1.5) on some time interval independent of the parameter $\delta \in (0,1]$, the consistency of the Isobe–Kakinuma model with the water wave equations (1.6), and the rigorous justification of the model by establishing an error estimate of the solutions such as (1.11). In Section 3 we derive estimates for the time derivatives of the solution to (1.4) and related partial differential operators of elliptic type with particular care on the dependence on the parameter δ. Since the hypersurface $t=0$ in the space-time $\mathbf{R}^n \times \mathbf{R}$ is characteristic for the Isobe–Kakinuma model, these estimations are not straightforward. In Section 4 we prove the existence of the solution to (1.4)–(1.5) on some time interval independent of δ. Here, we do not need the special choice of the indices p_i. In Section 5, under the additional conditions (H1) or (H2) we prove uniform boundedness of the solution to (1.4) and its derivatives. In Sections 6–8 we prove a consistency of the Isobe–Kakinuma model with the water wave equations. One of the key elements for obtaining the precise error estimates is to introduce a modified approximate velocity potential $\widetilde{\Phi}^{\rm app}$, which approximates the velocity potential Φ for the full water wave problem with an error of the order indicated in (1.11). The approximation $\Phi^{\rm app}$ does not possess such a nice property. In Section 9 we derive an error estimate between the solutions to the Isobe–Kakinuma model and to the water wave equations by using the stability of the water wave equations.

Acknowledgement This work was carried out when the author was visiting Université de Bordeaux on his sabbatical leave during the 2017 academic year. He is very grateful to the member of Institut de Mathḿatiques de Bordeaux, especially, David Lannes for their kind hospitalities and for fruitful discussions. This work was partially supported by JSPS KAKENHI Grant Number JP17K18742 and JP17H02856.

2 Main results

Notation. We denote by $W^{m,p}(\mathbf{R}^n)$ the L^p Sobolev space of order m on $\mathbf{R}^n$. The norms of the Sobolev space $H^m = W^{m,2}(\mathbf{R}^n)$ and of a Banach space B are denoted by $\|\cdot\|_m$ and by $\|\cdot\|_B$, respectively. The L^2-norm and the L^2-inner product are simply denoted by $\|\cdot\|$ and $(\cdot,\cdot)_{L^2}$, respectively. We put $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, and $\partial_z = \partial/\partial z$. $[P,Q] = PQ - QP$ denotes the commutator. We put $J = (1-\Delta)^{1/2}$ and $J_\delta = (1-\delta^2\Delta)^{1/2}$. $[a]$ denotes the integer part of the real number a and $a_1 \vee a_2 = \max\{a_1, a_2\}$. We denote the Kronecker delta by δ_{ij}, that is, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. We fix $t_0 > n/2$, which satisfies $t_0 - n/2 \ll 1$ if necessary. For a matrix A we denote by $A^{\rm T}$ the transpose of A. For a vector $\boldsymbol{\phi} = (\phi_0, \phi_1, \ldots, \phi_N)^{\rm T}$ we denote the last N components by $\boldsymbol{\phi}' = (\phi_1, \ldots, \phi_N)^{\rm T}$. We use the notational convention $0/0 = 0$. We denote by $C(a_1, a_2, \ldots)$ a positive constant depending on $a_1, a_2, \ldots$. $f \lesssim g$ means that there exists a non-essential positive constant C such that $f \leq Cg$ holds.

To state our main results it is convenient to introduce rescaled variables $\boldsymbol{\phi}^\delta = (\phi_0^\delta, \ldots, \phi_N^\delta)^{\rm T}$ by

$$\phi_i^\delta = \delta^{2(p_i - [p_i/2])}\phi_i \tag{2.1}$$

for $i = 0, 1, \ldots, N$. Then, the Isobe–Kakinuma model (1.4) is written in these rescaled variables as

$$\begin{cases} H^{p_i}\partial_t\eta + \displaystyle\sum_{j=0}^N \biggl\{ \nabla\cdot\biggl(\frac{1}{p_i+p_j+1}H^{p_i+p_j+1}\nabla\phi_j^\delta - \frac{p_j}{p_i+p_j}H^{p_i+p_j}\phi_j^\delta\nabla b \biggr) \\ \qquad + \displaystyle\frac{p_i}{p_i+p_j}H^{p_i+p_j}\nabla b\cdot\nabla\phi_j^\delta - \frac{p_ip_j}{p_i+p_j-1}H^{p_i+p_j-1}(\delta^{-2}+|\nabla b|^2)\phi_j^\delta \biggr\} = 0 \\ \qquad\qquad \text{for} \quad i = 0, 1, \ldots, N, \\ \displaystyle\sum_{j=0}^N H^{p_j}\partial_t\phi_j^\delta + \eta + \frac{1}{2}\biggl\{ \biggl| \sum_{j=0}^N (H^{p_j}\nabla\phi_j^\delta - p_jH^{p_j-1}\phi_j^\delta\nabla b) \biggr|^2 + \delta^{-2}\biggl(\sum_{j=0}^N p_jH^{p_j-1}\phi_j^\delta \biggr)^2 \biggr\} = 0. \end{cases} \tag{2.2}$$

We denote the corresponding initial data by $\boldsymbol{\phi}_{(0)}^\delta = (\phi_{0(0)}^\delta, \ldots, \phi_{N(0)}^\delta)^{\rm T}$ with

$$\phi_{i(0)}^\delta = \delta^{2(p_i - [p_i/2])}\phi_{i(0)} \tag{2.3}$$

for $i=0,1,\dots,N$ and put $\boldsymbol{\phi}_{(0)}^{\delta\prime} = (\phi_{1(0)}^{\delta},\dots,\phi_{N(0)}^{\delta})^{\mathrm{T}}$, where $\boldsymbol{\phi}_{(0)} = (\phi_{0(0)},\dots,\phi_{N(0)})^{\mathrm{T}}$ is the initial data for the original variables $\boldsymbol{\phi} = (\phi_0,\dots,\phi_N)^{\mathrm{T}}$ in (1.5).

As explained in the previous section, the Isobe–Kakinuma model (2.2) is a overdetermined and underdetermined composite system and we have $(N+1)$ evolution equations for only one unknown η, so that the initial value problem (1.4)–(1.5) is not solvable in general. In fact, if the problem has a solution $(\eta,\phi_0,\dots,\phi_N)$, then by eliminating the time derivative $\partial_t\eta$ from the evolution equations we see that the solution has to satisfy the relations

$$
\begin{aligned}
&H^{p_i}\sum_{j=0}^{N}\nabla\cdot\left(\frac{1}{p_j+1}H^{p_j+1}\nabla\phi_j^{\delta}-\frac{p_j}{p_j}H^{p_j}\phi_j^{\delta}\nabla b\right)\\
&=\sum_{j=0}^{N}\biggl\{\nabla\cdot\left(\frac{1}{p_i+p_j+1}H^{p_i+p_j+1}\nabla\phi_j^{\delta}-\frac{p_j}{p_i+p_j}H^{p_i+p_j}\phi_j^{\delta}\nabla b\right)\\
&\qquad+\frac{p_i}{p_i+p_j}H^{p_i+p_j}\nabla b\cdot\nabla\phi_j^{\delta}-\frac{p_ip_j}{p_i+p_j-1}H^{p_i+p_j-1}(\delta^{-2}+|\nabla b|^2)\phi_j^{\delta}\biggr\}
\end{aligned}
\tag{2.4}
$$

for $i=1,\dots,N$. Therefore, as a necessary condition the initial date $(\eta_{(0)},\boldsymbol{\phi}_{(0)})$ and the bottom topography b have to satisfy the relation (2.4) for the existence of the solution.

Another important condition on the well-posedness of the Isobe–Kakinuma model is related to a generalized Rayleigh–Taylor sign condition for water wave problem and states the positivity of the function a defined by

$$
\begin{aligned}
a&=1+\sum_{j=1}^{N}p_jH^{p_j-1}\partial_t\phi_j^{\delta}\\
&\quad+\boldsymbol{u}\cdot\sum_{j=1}^{N}\Bigl(p_jH^{p_j-1}\nabla\phi_j^{\delta}-p_j(p_j-1)H^{p_j-2}\phi_j^{\delta}\nabla b\Bigr)+w\sum_{j=1}^{N}p_j(p_j-1)H^{p_j-2}\phi_j^{\delta}.
\end{aligned}
\tag{2.5}
$$

For the details of this function a, we refer to R. Nemoto and T. Iguchi [21].

The following theorem is one of the main results in this paper and asserts the existence of the solution to the initial value problem (1.4)–(1.5) on a time interval independent of the small parameter δ with a uniform bounds of the solution $(\eta,\boldsymbol{\phi}^{\delta})$ in the rescaled variables.

Theorem 2.1 *Let c_0, M_0 be positive constants and m an integer such that $m>n/2+1$. There exist a time $T_1>0$ and a constant M such that for any $\delta\in(0,1]$ if the initial data $(\eta_{(0)},\boldsymbol{\phi}_{(0)}^{\delta})$ and b satisfy the relation (2.4) and*

$$
\begin{cases}
\|(\eta_{(0)},\nabla\boldsymbol{\phi}_{(0)}^{\delta})\|_m+\delta^{-1}\|\boldsymbol{\phi}_{(0)}^{\delta\prime}\|_m+\|b\|_{W^{m+1,\infty}}\le M_0,\\
1+\eta_{(0)}(x)-b(x)\ge c_0,\quad a(x,0)\ge c_0\qquad \text{for}\quad x\in\mathbf{R}^n,
\end{cases}
\tag{2.6}
$$

then the initial value problem (1.4)–(1.5) has a unique solution $(\eta,\phi_0,\dots,\phi_N)$ on the time interval $[0,T_1]$. Moreover, the solution satisfies the uniform bound:

$$
\begin{cases}
\|\eta(t)\|_m+\|\nabla\boldsymbol{\phi}^{\delta}(t)\|_m+\delta^{-1}\|\boldsymbol{\phi}^{\delta\prime}(t)\|_m+\delta^{-2}\|\boldsymbol{\phi}^{\delta\prime}(t)\|_{m-1}\\
\quad+\|\partial_t\eta(t)\|_{m-1}+\|\partial_t\boldsymbol{\phi}^{\delta}(t)\|_m+\delta^{-1}\|\partial_t\boldsymbol{\phi}^{\delta\prime}(t)\|_{m-1}+\delta^{-2}\|\partial_t\boldsymbol{\phi}^{\delta\prime}(t)\|_{m-2}\\
\quad+\|\partial_t^2\eta(t)\|_{m-2}+\|\partial_t^2\boldsymbol{\phi}^{\delta}(t)\|_{m-1}+\delta^{-1}\|\partial_t^2\boldsymbol{\phi}^{\delta\prime}(t)\|_{m-2}\le M,\\
1+\eta(x,t)-b(x)\ge c_0/2,\quad a(x,t)\ge c_0/2\qquad\text{for}\quad x\in\mathbf{R}^n, 0\le t\le T_1,
\end{cases}
\tag{2.7}
$$

where $\boldsymbol{\phi}^\delta = (\phi_0^\delta, \dots, \phi_N^\delta)^{\rm T}$ *are the rescaled variables defnied by* (2.1) *and* $\boldsymbol{\phi}^{\delta\prime} = (\phi_1^\delta, \dots, \phi_N^\delta)^{\rm T}$.

Furthermore, if we assume in addition that (H1) *or* (H2)*, then we have the uniform bound:*

$$\begin{cases} \|\phi_j(t)\|_{m-2j+1} \le M & \textit{in the case } \text{(H1)}, \\ \|\phi_j(t)\|_{m-2[(j+1)/2]+1} \le M & \textit{in the case } \text{(H2)} \end{cases}$$

for $i = 1, \dots, N$ *and* $0 \le t \le T_1$.

We proceed to show that the Isobe–Kakinuma model (1.4) is consistent with the water wave equations (1.6) at order $O(\delta^{4N+2})$ in the case (H1) and at order $O(\delta^{4[N/2]+2})$ in the case (H2). In view of the facts that the unknown ϕ in Zakharov–Craig–Sulem formulation is the trace of the velocity potential Φ on the water surface and that the unknowns $\boldsymbol{\phi} = (\phi_0, \dots, \phi_N)^{\rm T}$ for Isobe–Kakinuma model appear in the approximation (1.10) of Φ, these variables are related approximately by the formula

$$\phi = \sum_{i=0}^N H^{p_i}\phi_i^\delta = \sum_{i=0}^N \delta^{2(p_i - [p_i/2])} H^{p_i}\phi_i. \tag{2.8}$$

Theorem 2.2 *In addition to hypothesis of Theorem 2.1 we assume that* (H1) *or* (H2) *and that* $m \ge 4N+2$ *and* $m > n/2+2N+2$ *in the case* (H1)*,* $m \ge 4[N/2]+2+\delta_{N1}$ *and* $m > n/2+2[N/2]+2$ *in the case* (H2)*. Let* $(\eta, \phi_0, \dots, \phi_N)$ *be the solution obtained in Theorem 2.1 and define* ϕ *by* (2.8)*. Then,* (η, ϕ) *satisfy the water wave equations approximately as*

$$\begin{cases} \partial_t\eta - \Lambda(\eta, b, \delta)\phi = \mathfrak{r}_1, \\ \partial_t\phi + \eta + \dfrac{1}{2}|\nabla\phi|^2 - \delta^2\dfrac{(\Lambda(\eta, b, \delta)\phi + \nabla\eta\cdot\nabla\phi)^2}{2(1+\delta^2|\nabla\eta|^2)} = \mathfrak{r}_2. \end{cases} \tag{2.9}$$

Here, $(\mathfrak{r}_1, \mathfrak{r}_2)$ *satisfy*

$$\begin{cases} \|(\mathfrak{r}_1(t), \mathfrak{r}_2(t))\|_{m-4(N+1)} \le C\delta^{4N+2} & \textit{in the case } \text{(H1)}, \\ \|(\mathfrak{r}_1(t), \mathfrak{r}_2(t))\|_{m-4([N/2]+1)} \le C\delta^{4[N/2]+2} & \textit{in the case } \text{(H2)}, \end{cases} \tag{2.10}$$

where C *is a positive constant independent of* $\delta \in (0,1]$ *and* $t \in [0, T_1]$.

The above theorem concerns the approximation of the equations. Next, we will be concerned with the approximation of the solution to give a rigorous justification of the Isobe–Kakinuma model. Here we recall the existence theorem for the initial value problem to the water wave equations (1.6)–(1.7) obtained by T. Iguchi [5] and B. Alvarez-Samaniego and D. Lannes [1]. See also D. Lannes [15].

Theorem 2.3 *Let* $c_0, M_0 > 0$ *and* $m > n/2+1$*. There exist a time* $T_2 > 0$ *and constants* $C, \delta_* > 0$ *such that for any* $\delta \in (0, \delta_2]$ *if the initial data* $(\eta_{(0)}, \phi_{(0)})$ *satisfy*

$$\begin{cases} \|\eta_{(0)}\|_{m+3+1/2} + \|\nabla\phi_{(0)}\|_{m+3} \le M_0, \\ 1 + \eta_{(0)}(x) \ge c_0 \qquad \textit{for} \quad x \in \mathbf{R}^n, \end{cases}$$

then the initial value problem (1.6)–(1.7) *has a unique solution* (η, ϕ) *on the time interval* $[0, T_2]$*. Moreover, the solution satisfies the uniform bound:*

$$\begin{cases} \|\eta(t)\|_{m+3} + \|\nabla\phi(t)\|_{m+2} + \|\partial_t\eta(t)\|_{m+2} + \|\partial_t\phi(t)\|_{m+2} \le C, \\ 1 + \eta(x,t) \ge c_0/2, \qquad \textit{for} \quad x \in \mathbf{R}^n,\ 0 \le t \le T_2. \end{cases}$$

Remark 2.4 In the above theorem the constant δ_2 is small. As in the case of Theorem 2.1 we can reduce the restriction $0 < \delta \le \delta_1$ to $0 \le \delta \le 1$, if we impose the sign condition $a^{\mathrm{WW}}(x,0) \ge c_0$ on the initial data, where $a^{\mathrm{WW}} = 1 + \delta^2 \partial_t Z + \delta^2 \boldsymbol{v} \cdot \nabla Z$ with

$$\begin{cases} Z = (1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda(\eta,\delta)\phi + \nabla\eta\cdot\nabla\phi), \\ \boldsymbol{v} = \nabla\phi - \delta^2 Z\nabla\eta. \end{cases}$$

In order that the solution to the Isobe–Kakinuma model (1.3)–(1.5) approximates the solution to the water wave problem (1.6)–(1.7), we need to prepare the initial data $\boldsymbol{\phi}^\delta_{(0)}$ for the Isobe–Kakinuma model appropriately in terms of the initial data $\eta_{(0)}$ and $\phi_{(0)}$ for the water wave equations. As a matter of fact, the necessary conditions (2.4) and the approximate relation (2.8) between ϕ and $\boldsymbol{\phi}^\delta$ determine uniquely the initial data $\boldsymbol{\phi}^\delta_{(0)}$ from $(\eta_{(0)}, \phi_{(0)})$, and b as guaranteed by Lemma 3.4 in the Section 3. Moreover, it follows from Lemma 4.4 in Section 4 that $\|a(\cdot,0)-1\|_{m-1} \le C\delta$ with a constant independent of δ. Therefore, by taking δ_* sufficiently small if necessary, we have $a(x,0) \ge 1/2$, so that the conditions (2.6) in Theorem 2.1 will be satisfied and we can construct the solution to the Isobe–Kakinuma model. The next theorem gives a rigorous justification of the Isobe–Kakinuma model for the water wave problem as a higher order shallow water approximation.

Theorem 2.5 *Let c_0, M_0 be positive constants and m an integer such that $m > n/2+1$, suppose that* (H1) *or* (H2) *holds, and put $T_* = \min\{T_1, T_2\}$, where T_1 and T_2 are those in Theorems* 2.1 *and* 2.3. *Suppose also that $0 < \delta \le \delta_*$ and the initial data $(\eta_{(0)}, \phi_{(0)})$ and b satisfy*

$$(2.11) \qquad \begin{cases} \|\eta_{(0)}\|_{m+4N+8} + \|\nabla\phi_{(0)}\|_{m+4N+7} \le M_0 & \textit{in the case } \text{(H1)}, \\ \|\eta_{(0)}\|_{m+4[N/2]+8} + \|\nabla\phi_{(0)}\|_{m+4[N/2]+7} + \|b\|_{W^{m+4[N/2]+8,\infty}} \le M_0 & \textit{in the case } \text{(H2)}, \\ 1 + \eta_{(0)}(x) - b(x) \ge c_0 \qquad \textit{for} \quad x \in \mathbf{R}^n. \end{cases}$$

Then, (2.4) *and* (2.8) *determine uniquely the initial data $\boldsymbol{\phi}^\delta_{(0)}$ to the Isobe–Kakinuma model. Let $(\eta^{\mathrm{WW}}, \phi^{\mathrm{WW}})$ be the solution to the initial value problem* (1.6)–(1.7) *obtained in Theorem 2.3 and $(\eta^{\mathrm{IK}}, \boldsymbol{\phi}^\delta)$ the solution to the initial value problem* (2.2)–(2.3) *obtained in Theorem 2.1, and define ϕ^{IK} by* (2.8). *Then, for any $\delta \in (0, \delta_*]$ and $t \in [0, T_*]$ we have*

$$(2.12) \qquad \|\eta^{\mathrm{WW}}(t) - \eta^{\mathrm{IK}}(t)\|_{m+2} + \|\phi^{\mathrm{WW}}(t) - \phi^{\mathrm{IK}}(t)\|_{m+2} \le \begin{cases} C\delta^{4N+2} & \textit{in the case } \text{(H1)}, \\ C\delta^{4[N/2]+2} & \textit{in the case } \text{(H2)}, \end{cases}$$

where C is a positive constant independent of δ and t.

Remark 2.6 The error estimate (2.12) together with the Sobolev imbedding theorem implies the pointwise error estimate (1.11).

We will give the proof of Theorems 2.1, 2.2, and 2.5 in Sections 4–5, 6–8, and 9, respectively.

3 Estimate of the time derivate and related operators

One of the difficulties for the analysis of the Isobe–Kakinuma model (1.4) (equivalently (2.2)) lies in the fact that the hypersurface $t = 0$ in the space-time $\mathbf{R}^n \times \mathbf{R}$ is characteristic for the model. In fact, the evolution equation for $\boldsymbol{\phi}^\delta = (\phi^\delta_0, \ldots, \phi^\delta_N)^{\mathrm{T}}$ is underdetermined so that we cannot express the time derivative $\partial_t \boldsymbol{\phi}^\delta$ in terms of the spatial derivatives directly from the equation. Nevertheless, we can express it implicitly along with the calculations in R. Nemoto and T. Iguchi [21]. Since we need to trace carefully the dependence of the small parameter δ, we outline them.

We introduce second order differential operators $L_{ij} = L_{ij}(H, b, \delta)$ $(i, j = 0, 1, \dots)$ depending on the water depth H, the bottom topography b, and the parameter δ by

$$L_{ij}\psi_j = -\nabla \cdot \left(\frac{1}{p_i + p_j + 1} H^{p_i+p_j+1} \nabla \psi_j - \frac{p_j}{p_i + p_j} H^{p_i+p_j} \psi_j \nabla b \right) - \frac{p_i}{p_i + p_j} H^{p_i+p_j} \nabla b \cdot \nabla \psi_j + \frac{p_i p_j}{p_i + p_j - 1} H^{p_i+p_j-1} (\delta^{-2} + |\nabla b|^2) \psi_j. \tag{3.1}$$

Then, we have $L_{ij}^* = L_{ji}$, where L_{ij}^* is the adjoint operator of L_{ij} in $L^2(\mathbf{R}^n)$. We introduce also the functions $\boldsymbol{u}$ and w by

$$\boldsymbol{u} = \sum_{i=0}^{N} (H^{p_i} \nabla \phi_i^\delta - p_i H^{p_i-1} \phi_i^\delta \nabla b), \quad w = \delta^{-2} \sum_{i=1}^{N} p_i H^{p_i-1} \phi_i^\delta. \tag{3.2}$$

Since $\boldsymbol{u} = (\nabla \Phi^{\mathrm{app}})|_{z=\eta}$ and $\delta^2 w = (\partial_z \Phi^{\mathrm{app}})|_{z=\eta}$, where Φ^{app} is the approximate velocity potential defined by (1.10), $\boldsymbol{u}$ and $\delta^2 w$ represent approximately the horizontal and the vertical components of the velocity field on the water surface, respectively. We note that both $\boldsymbol{u}$ and w would be expected of order $O(1)$. Then, the Isobe–Kakinuma model (1.4) and the necessary conditions (2.4) can be written simply as

$$\begin{cases} H^{p_i} \partial_t \eta - \displaystyle\sum_{j=0}^{N} L_{ij} \phi_j^\delta = 0 \qquad \text{for} \quad i = 0, 1, \dots, N, \\ \displaystyle\sum_{j=0}^{N} H^{p_j} \partial_t \phi_j^\delta + \eta + \frac{1}{2} (|\boldsymbol{u}|^2 + \delta^2 w^2) = 0 \end{cases} \tag{3.3}$$

and

$$\sum_{j=0}^{N} (L_{ij} - H^{p_i} L_{0j}) \phi_j^\delta = 0 \qquad \text{for} \quad i = 1, \dots, N, \tag{3.4}$$

respectively. In view of these equations we introduce also linear operators $\mathscr{L}_i = \mathscr{L}_i(H, b, \delta)$ $(i = 0, 1, \dots, N)$ depending on the water depth H, the bottom topography b, and the parameter δ, and acting on $\boldsymbol{\varphi} = (\varphi_0, \dots, \varphi_N)^{\mathrm{T}}$ by

$$\mathscr{L}_0 \boldsymbol{\varphi} = \sum_{j=0}^{N} H^{p_j} \varphi_j, \qquad \mathscr{L}_i \boldsymbol{\varphi} = \sum_{j=0}^{N} (L_{ij} - H^{p_i} L_{0j}) \varphi_j \quad \text{for} \quad i = 1, \dots, N, \tag{3.5}$$

and put

$$\mathscr{L} \boldsymbol{\varphi} = (\mathscr{L}_0 \boldsymbol{\varphi}, \dots, \mathscr{L}_N \boldsymbol{\varphi})^{\mathrm{T}}. \tag{3.6}$$

Then, the necessary conditions (3.4) have the simple form

$$\mathscr{L}_i \boldsymbol{\phi}^\delta = 0 \quad \text{for} \quad i = 1, \dots, N. \tag{3.7}$$

We note that the operators $\mathscr{L}_i$ for $i = 1, \ldots, N$ can be written explicitly as

$$\begin{aligned}(3.8) \qquad \mathscr{L}_i \boldsymbol{\varphi} = \sum_{j=0}^{N} \biggl\{ & -\left(\frac{1}{p_i + p_j + 1} - \frac{1}{p_j + 1} \right) H^{p_i + p_j + 1} \Delta \varphi_j \\ & + \left(\frac{p_j}{p_i + p_j} - \frac{p_j}{p_j} \right) H^{p_i + p_j} \nabla \cdot (\varphi_j \nabla b) \\ & - \frac{p_i}{p_i + p_j} H^{p_i + p_j} \nabla b \cdot \nabla \varphi_j + \frac{p_i p_j}{p_i + p_j - 1} H^{p_i + p_j - 1} (\delta^{-2} + |\nabla b|^2) \varphi_j \biggr\},\end{aligned}$$

and that they do not include the term ∇H. Therefore, differentiating (3.7) with respect to the time t and using the first equation in (3.3) with $i = 0$ to eliminate $\partial_t \eta$, we obtain

$$(3.9) \qquad \mathscr{L}_i \partial_t \boldsymbol{\phi}_j^\delta = F_i \qquad \text{for} \quad i = 1, \ldots, N,$$

where

$$(3.10) \qquad F_i = -\left(\left(\frac{\partial}{\partial H} \mathscr{L}_i \right) \boldsymbol{\phi}^\delta \right) \sum_{j=0}^{N} L_{0j} \phi_j^\delta$$

for $i = 1, \ldots, N$. We note that F_i does not contain any time derivatives. Then, it follows from (3.9) and the second equation in (3.3) that

$$(3.11) \qquad \mathscr{L} \partial_t \boldsymbol{\phi}^\delta = \boldsymbol{F},$$

where $\boldsymbol{F} = (F_0, \ldots, F_N)^{\mathrm{T}}$ and

$$(3.12) \qquad F_0 = -\eta - \frac{1}{2}(|\boldsymbol{u}|^2 + \delta^2 w^2).$$

Therefore, the time derivative of $\boldsymbol{\phi}^\delta$ can be represented implicitly in terms of the spatial derivatives by $\partial_t \boldsymbol{\phi}^\delta = \mathscr{L}^{-1} \boldsymbol{F}$.

To investigate the operator $\mathscr{L}^{-1}$, assuming $\boldsymbol{F}$ to be a given function we consider the equation

$$(3.13) \qquad \mathscr{L} \boldsymbol{\varphi} = \boldsymbol{F}.$$

Let $\boldsymbol{\varphi}$ be a solution of this equation. It follows from the first component of (3.13) that

$$(3.14) \qquad \varphi_0 = F_0 - \sum_{j=1}^{N} H^{p_j} \varphi_j.$$

Plugging this into the other components of (3.13) we obtain

$$(3.15) \qquad P_i \boldsymbol{\varphi}' = F_i - (L_{i0} - H^{p_i} L_{00}) F_0 \quad \text{for} \quad i = 1, \ldots, N,$$

where $\boldsymbol{\varphi}' = (\varphi_1, \ldots, \varphi_N)^{\mathrm{T}}$ and $P_j = P_j(H, b, \delta)$ $(j = 1, \ldots, N)$ are second order differential operators defined by

$$(3.16) \qquad P_i \boldsymbol{\varphi}' = \sum_{j=1}^{N} \{ (L_{ij} - H^{p_i} L_{0j}) \varphi_j - (L_{i0} - H^{p_i} L_{00})(H^{p_j} \varphi_j) \}.$$

We further introduce the operator $P \boldsymbol{\varphi}' = (P_1 \boldsymbol{\varphi}', \ldots, P_N \boldsymbol{\varphi}')^{\mathrm{T}}$. Since $L_{ij}^* = L_{ji}$, we see easily that P is symmetric in $L^2(\mathbf{R}^n)$. Moreover, we have the following lemma.

Lemma 3.1 *Let c_0, c_1 be positive constants. There exists a positive constant $C = C(c_0, c_1)$ depending only on c_0 and c_1 such that if $H, \nabla b \in L^\infty(\mathbf{R}^n)$ satisfy $H(x) \geq c_0$ and $|\nabla b(x)| \leq c_1$, then for any $\delta \in (0,1]$ we have*

$$(P\boldsymbol{\varphi}', \boldsymbol{\varphi}')_{L^2} \geq C^{-1}(\|\nabla\boldsymbol{\varphi}'\|^2 + \delta^{-2}\|\boldsymbol{\varphi}'\|^2).$$

Proof. Introducing $\varphi_0 = -\sum_{j=1}^N H^{p_j}\varphi_j$, we have

$$\begin{aligned}(P\boldsymbol{\varphi}', \boldsymbol{\varphi}')_{L^2} &= \sum_{i,j=0}^N (L_{ij}\varphi_j, \varphi_i)_{L^2} \\ &= \int_{\mathbf{R}^n} \mathrm{d}x \int_0^H \left\{ \left| \sum_{i=0}^N (z^{p_i}\nabla\varphi_i - p_i z^{p_i-1}\varphi_i\nabla b) \right|^2 + \delta^{-2}\left(\sum_{i=0}^N p_i z^{p_i-1}\varphi_i \right)^2 \right\} \mathrm{d}z,\end{aligned}$$

which gives the desired estimate. For the details, we refer to R. Nemoto and T. Iguchi [21]. □

Once we obtain such a coercive estimate, by the standard theory of elliptic partial differential equations, we can obtain the following lemma.

Lemma 3.2 *Let c_0, M be positive constants and m an integer such that $m > n/2 + 1$. There exists a positive constant $C = C(c_0, M, m)$ such that if η and b satisfy*

$$\begin{cases} \|\eta\|_m + \|b\|_{W^{m,\infty}} \leq M, \\ c_0 \leq H(x) = 1 + \eta(x) - b(x) \quad \text{for} \quad x \in \mathbf{R}^n, \end{cases} \tag{3.17}$$

then for $k = 0, \pm 1, \ldots, \pm(m-1)$ and $\delta \in (0,1]$ we have

$$\|J_\delta P^{-1}\boldsymbol{G}'\|_k \leq C\delta^2\|J_\delta^{-1}\boldsymbol{G}'\|_k. \tag{3.18}$$

Remark 3.3 For the estimation to the time derivate $\partial_t\boldsymbol{\phi}^\delta$, it is sufficient to show the above estimate (3.18) in the Sobolev space with nonnegative indices. However, the estimate with negative indices plays an important role in deriving an error estimate between the solutions to the Isobe–Kakinuma model and to the full water wave problem.

Proof. Put $\boldsymbol{\varphi}' = P^{-1}\boldsymbol{G}'$. Noting that $\|\nabla\boldsymbol{\varphi}'\|^2 + \delta^{-2}\|\boldsymbol{\varphi}'\|^2$ is equivalent to $\delta^{-2}\|J_\delta\boldsymbol{\varphi}'\|^2$ uniformly with respect to $\delta \in (0,1]$, we see by Lemma 3.1 that

$$\|J_\delta\boldsymbol{\varphi}'\|^2 \lesssim \delta^2(P\boldsymbol{\varphi}', \boldsymbol{\varphi}')_{L^2} = \delta^2(\boldsymbol{G}', \boldsymbol{\varphi}')_{L^2} \leq \delta^2\|J_\delta^{-1}\boldsymbol{G}'\|\|J_\delta\boldsymbol{\varphi}'\|,$$

which yields the estimate (3.18) in the case $k = 0$.

Let $1 \leq k \leq m-1$ and α be a multi-index such that $|\alpha| \leq k$. Applying the differential operator ∂^α to the equation $P\boldsymbol{\varphi}' = \boldsymbol{G}'$, we have $P\partial^\alpha\boldsymbol{\varphi}' = \partial^\alpha\boldsymbol{G}' - [\partial^\alpha, P]\boldsymbol{\varphi}'$, so that

$$\|J_\delta\partial^\alpha\boldsymbol{\varphi}'\| \lesssim \delta^2(\|J_\delta^{-1}\partial^\alpha\boldsymbol{G}'\| + \|J_\delta^{-1}[\partial^\alpha, P]\boldsymbol{\varphi}'\|).$$

We evaluate the commutator $[\partial^\alpha, P]$ by writing down explicitly the operator P. Let $t_0 > n/2$ and remember the standard commutator estimate

$$\|[\partial^\alpha, u]v\| \lesssim \begin{cases} \|u\|_{W^{|\alpha|,\infty}}\|v\|_{|\alpha|-1}, \\ \|u\|_{|\alpha|\vee t_0+1}\|v\|_{|\alpha|-1}. \end{cases}$$

By expanding the commutator $[\partial^\alpha, u]v = \partial^\alpha(uv) - u\partial^\alpha v$, evaluating each terms separately, and using the calculus inequalities $\|uv\|_k \lesssim \|u\|_{|k|\vee t_0}\|u\|_k$ and $\|uv\|_k \lesssim \|u\|_{W^{|k|,\infty}}\|v\|_k$ for any integers k, we also have $\|[\partial^\alpha, u]v\| \lesssim \|u\|_{|\alpha|\vee t_0}\|v\|_{|\alpha|}$ and

$$\|[\partial^\alpha, u]v\|_{-1} \lesssim \begin{cases} \|u\|_{W^{|\alpha|-1\vee 1,\infty}}\|v\|_{|\alpha|-1}, \\ \|u\|_{|\alpha|-1\vee 1\vee t_0}\|v\|_{|\alpha|-1}. \end{cases}$$

In the following, we use these calculus inequalities without any comment. We also note that we need to handle a smooth function $f(H)$ of $H = 1 + \eta - b$. Under the conditions in (3.17), $f(H)$ does not belong to H^m nor $W^{m,\infty}$, in general. However, we can decompose it as $f(H) = f(1-b) + f_1(\eta, b)\eta$ with a smooth function f_1, and the first term belongs to $W^{m,\infty}$ and the second one to H^m. We will also use this fact without any comment. Noting

$$\delta\|J_\delta^{-1}\nabla u\| \le \|u\|, \quad \delta\|J_\delta^{-1}u\| \le \|u\|_{-1}, \quad \|J_\delta^{-1}u\| \le \|u\|, \tag{3.19}$$

and using the above calculus inequalities, we see that

$$\delta^2\|J_\delta^{-1}[\partial^\alpha, P]\boldsymbol{\varphi}'\| \lesssim \delta\|\nabla\boldsymbol{\varphi}'\|_{k-1} + \|\boldsymbol{\varphi}'\|_{k-1} \lesssim \|J_\delta\boldsymbol{\varphi}'\|_{k-1},$$

so that $\|J_\delta\boldsymbol{\varphi}'\|_k \lesssim \delta^2\|J_\delta^{-1}\boldsymbol{G}'\|_k + \|J_\delta\boldsymbol{\varphi}'\|_{k-1}$, which yields the estimate (3.18) for positive k by induction on k.

The estimate for negative k comes from the standard duality argument. We note that the operator P is symmetric in $L^2(\mathbf{R}^n)$ so is P^{-1}. Let $1 \le k' \le m$. Then, we see that

$$\begin{aligned} |(J_\delta P^{-1}\boldsymbol{G}', \boldsymbol{F}')_{L^2}| &= |(J_\delta^{-1}\boldsymbol{G}', J_\delta P^{-1}J_\delta\boldsymbol{F}')_{L^2}| \le \|J_\delta^{-1}\boldsymbol{G}'\|_{-k'}\|J_\delta P^{-1}J_\delta\boldsymbol{F}'\|_{k'} \\ &\lesssim \delta^2\|J_\delta^{-1}\boldsymbol{G}'\|_{-k'}\|J_\delta^{-1}(J_\delta\boldsymbol{F}')\|_{k'} = \delta^2\|J_\delta^{-1}\boldsymbol{G}'\|_{-k'}\|\boldsymbol{F}'\|_{k'}, \end{aligned}$$

which gives $\|J_\delta P^{-1}\boldsymbol{G}'\|_{-k'} \lesssim \delta^2\|J_\delta^{-1}\boldsymbol{G}'\|_{-k'}$. This gives the estimate (3.18) for negative k. □

Thanks of Lemma 3.2, a unique existence of the solution $\boldsymbol{\varphi}$ to (3.13) is guaranteed in appropriate function spaces. Concerning estimates of the solution, we have the following lemma.

Lemma 3.4 *Let c_0, M be positive constants and m an integer such that $m > n/2 + 1$. There exists a positive constant $C = C(c_0, M, m)$ such that if η and b satisfy the conditions in* (3.17) *and if $\boldsymbol{\varphi}$ is a solution of* (3.13)*, then for $k = 0, \pm1, \ldots, \pm(m-1)$ and $\delta \in (0,1]$ we have*

$$\begin{cases} \|\nabla\varphi_0\|_k + \delta^{-1}\|J_\delta\boldsymbol{\varphi}'\|_k \le C(\|\nabla F_0\|_k + \delta\|J_\delta^{-1}\boldsymbol{F}'\|_k), \\ \|\boldsymbol{\varphi}\|_{k+1} \le C(\|F_0\|_{k+1} + \delta\|J_\delta^{-1}\boldsymbol{F}'\|_k), \end{cases} \tag{3.20}$$

where $\boldsymbol{F}' = (F_1, \ldots, F_N)^{\mathrm{T}}$.

If, in addition, $F_0 = 0$, then we have

$$\|\boldsymbol{\varphi}\|_k \le C\delta^2\|\boldsymbol{F}'\|_k. \tag{3.21}$$

Proof. Since $\boldsymbol{\varphi}'$ satisfy (3.15), it follows from Lemma 3.2 that

$$\|J_\delta\boldsymbol{\varphi}'\|_k \lesssim \delta^2\|J_\delta^{-1}\boldsymbol{F}'\|_k + \sum_{i=1}^N \delta^2\|J_\delta^{-1}(L_{i0} - H^{p_i}L_{00})F_0\|_k.$$

Here, by writing down the operator $L_{i0} - H^{p_i}L_{00}$ explicitly and noting (3.19), we have

$$\delta^2\|J_\delta^{-1}(L_{i0} - H^{p_i}L_{00})F_0\|_k \lesssim \delta\|\nabla F_0\|_k,$$

so that $\|J_\delta \boldsymbol{\varphi}'\|_k \lesssim \delta^2 \|J_\delta^{-1} \boldsymbol{F}'\|_k + \delta \|\nabla F_0\|_k$. Now, we estimate φ_0 by using (3.14) and obtain $\|\nabla \varphi_0\|_k \lesssim \|\nabla F_0\|_k + \|\boldsymbol{\varphi}'\|_{k+1} \lesssim \|\nabla F_0\|_k + \delta^{-1} \|J_\delta \boldsymbol{\varphi}'\|_k$. Therefore, we obtain the first estimate in (3.20). Similarly, by (3.14) we also have $\|\varphi_0\|_{k+1} \lesssim \|F_0\|_{k+1} + \delta^{-1} \|J_\delta \boldsymbol{\varphi}'\|_k$. In view of $\|u\|_{k+1} \le \delta^{-1} \|J_\delta u\|_k$, we obtain the second estimate in (3.20).

If $F_0 = 0$, then it follows from the first estimate in (3.20) that $\|\boldsymbol{\varphi}'\|_k \le \|J_\delta \boldsymbol{\varphi}'\|_k \lesssim \delta^2 \|J_\delta^{-1} \boldsymbol{F}'\|_k \lesssim \delta^2 \|\boldsymbol{F}'\|_k$. This together with (3.14) gives the estimate for φ_0. □

Now, we are ready to give an estimate for the time derivative $\partial_t \boldsymbol{\phi}^\delta$. We introduce a mathematical energy $E_m(t)$ by

$$E_m(t) = \|\eta(t)\|_m^2 + \|\nabla \boldsymbol{\phi}^\delta(t)\|_m^2 + \delta^{-2} \|\boldsymbol{\phi}^{\delta\prime}(t)\|_m^2, \tag{3.22}$$

where $\boldsymbol{\phi}^{\delta\prime} = (\phi_1^\delta, \dots, \phi_N^\delta)^{\mathrm{T}}$.

Lemma 3.5 *Let c_0, M be positive constants and m an integer such that $m > n/2+1$. There exists a positive constant $C = C(c_0, M, m)$ such that if $(\eta, \boldsymbol{\phi}^\delta)$ is a solution to the Isobe–Kakinuma model* (2.2) *satisfying*

$$\begin{cases} E_m(t) + \|b\|_{W^{m+1,\infty}} \le M, \\ c_0 \le H(x,t) = 1 + \eta(x,t) - b(x) \quad \text{for} \quad x \in \mathbf{R}^n, \ 0 \le t \le T, \end{cases} \tag{3.23}$$

then we have $\|\partial_t \eta(t)\|_{m-1}^2 + \|\partial_t \boldsymbol{\phi}^\delta(t)\|_m^2 + \delta^{-2} \|\partial_t \boldsymbol{\phi}^{\delta\prime}(t)\|_{m-1}^2 \le C E_m(t)$ for $0 \le t \le T$.

Proof. We remind that $\boldsymbol{u}$ and w were defined by (3.2), so that we easily have $\|\boldsymbol{u}\|_m^2 + \delta^2 \|w\|_m^2 \lesssim E_m(t)$. We remind also that $\partial_t \boldsymbol{\phi}^\delta$ satisfies (3.11) where F_0 and $\boldsymbol{F}' = (F_1, \dots, F_N)^{\mathrm{T}}$ are defined by (3.12) and (3.10), respectively. By using the explicit expressions, we see that $\|F_0\|_m^2 + \delta^2 \|\boldsymbol{F}'\|_m^2 \lesssim E_m(t)$. Therefore, applying the second estimate in Lemma 3.4 and noting $\|J_\delta^{-1} u\|_k \le \|u\|_k$ we obtain $\|\partial_t \boldsymbol{\phi}^\delta\|_m^2 \lesssim \|F_0\|_m^2 + \delta^2 \|\boldsymbol{F}'\|_{m-1}^2 \lesssim E_m(t)$. On the other hand, applying the first estimate in Lemma 3.4 and noting $\|u\|_k \le \|J_\delta u\|_k$ we obtain $\delta^{-2} \|\partial_t \boldsymbol{\phi}^{\delta\prime}\|_{m-1}^2 \lesssim \|\nabla F_0\|_{m-1}^2 + \delta^2 \|\boldsymbol{F}'\|_{m-1}^2 \lesssim E_m(t)$. The estimate for $\partial_t \eta$ follows directly from the first equation in (2.2) with $i = 0$. □

4 Uniform estimate of the solution I

In this section we will prove the first half of Theorem 2.1, that is, the existence of the solution on a time interval independent of $\delta \in (0,1]$ and a uniform bound (2.6) to the rescaled variables $(\eta, \boldsymbol{\phi}^\delta)$ by using an energy method.

Let α be a multi-index satisfying $1 \le |\alpha| \le m$. Applying ∂^α to the Isobe–Kakinuma model (2.2), after a tedious but straightforward calculation, we obtain

$$\begin{cases} H^{p_i}((\partial_t + \boldsymbol{u} \cdot \nabla) \partial^\alpha \eta) - \displaystyle\sum_{j=0}^N L_{ij}(\partial^\alpha \phi_j^\delta) = -f_{i,\alpha} \quad \text{for} \quad i = 0, 1, \dots, N, \\ \displaystyle\sum_{j=0}^N H^{p_j}((\partial_t + \boldsymbol{u} \cdot \nabla) \partial^\alpha \phi_j^\delta) + a \partial^\alpha \eta = f_{N+1,\alpha}, \end{cases} \tag{4.1}$$

where

$$f_{i,\alpha} = [\partial^\alpha, H^{p_i}]\partial_t \eta + ((\nabla \cdot (H^{p_i}\boldsymbol{u}))\partial^\alpha \eta \\
+ \sum_{j=0}^{N} \biggl\{ \nabla \cdot \biggl\{ \Bigl([\partial^\alpha, \frac{1}{p_i+p_j+1} H^{p_i+p_j+1}] - H^{p_i+p_j}(\partial^\alpha \eta) \Bigr) \nabla \phi_j^\delta \\
+ \Bigl([\partial^\alpha, \frac{p_j}{p_i+p_j} H^{p_i+p_j}(\nabla b)] - p_j H^{p_i+p_j-1}(\nabla b)(\partial^\alpha \eta) \Bigr) \phi_j^\delta \biggr\} \\
+ \frac{p_j}{p_i+p_j} [\partial^\alpha, H^{p_i+p_j}\nabla b] \cdot \nabla \phi_j^\delta - \frac{p_i p_j}{p_i+p_j-1} [\partial^\alpha, H^{p_i+p_j-1}(\delta^{-2} + |\nabla b|^2)] \phi_j^\delta \biggr\}, \tag{4.2}$$

$$f_{N+1,\alpha} = -\sum_{j=1}^{N} \Bigl([\partial^\alpha, H^{p_j}] - p_j H^{p_j-1}(\partial^\alpha \eta) \Bigr) \partial_t \phi_j^\delta \\
- \frac{1}{2}(\partial^\alpha(|\boldsymbol{u}|^2) - 2\boldsymbol{u} \cdot \partial^\alpha \boldsymbol{u}) - \frac{1}{2}\delta^2(\partial^\alpha(w^2) - 2w\partial^\alpha w) \\
- \boldsymbol{u} \cdot \sum_{j=1}^{N} \biggl\{ \Bigl([\partial^\alpha, H^{p_j}] - p_j H^{p_j-1}(\partial^\alpha \eta) \Bigr) \nabla \phi_j^\delta \\
- p_j \Bigl([\partial^\alpha, H^{p_j-1}(\nabla b)] - (p_j-1) H^{p_j-2}(\nabla b)(\partial^\alpha \eta) \Bigr) \phi_j^\delta \biggr\} \\
- w \cdot \sum_{j=1}^{N} p_j \Bigl([\partial^\alpha, H^{p_j-1}] - (p_j-1) H^{p_j-2}(\partial^\alpha \eta) \Bigr) \phi_j^\delta, \tag{4.3}$$

and a is related to a generalized Rayleigh–Taylor sign condition and is given by

$$a = 1 + \sum_{j=1}^{N} p_j H^{p_j-1} \partial_t \phi_j^\delta \\
+ \boldsymbol{u} \cdot \sum_{j=1}^{N} \Bigl(p_j H^{p_j-1} \nabla \phi_j^\delta - p_j(p_j-1) H^{p_j-2} \phi_j^\delta \nabla b \Bigr) + w \sum_{j=1}^{N} p_j(p_j-1) H^{p_j-2} \phi_j^\delta. \tag{4.4}$$

We can rewrite (4.1) in a matrix form as

$$\begin{pmatrix} 0 & \boldsymbol{l}^{\mathrm{T}} \\ -\boldsymbol{l} & O \end{pmatrix} (\partial_t + \boldsymbol{u} \cdot \nabla) \partial^\alpha \begin{pmatrix} \eta \\ \boldsymbol{\phi}^\delta \end{pmatrix} + \begin{pmatrix} a & \boldsymbol{0}^{\mathrm{T}} \\ \boldsymbol{0} & L \end{pmatrix} \partial^\alpha \begin{pmatrix} \eta \\ \boldsymbol{\phi}^\delta \end{pmatrix} = \begin{pmatrix} f_{N+1,\alpha} \\ \boldsymbol{f}_\alpha \end{pmatrix}, \tag{4.5}$$

where $\boldsymbol{f}_\alpha = (f_{0,\alpha}, \dots, f_{N,\alpha})^{\mathrm{T}}$, $L = (L_{ij})_{0 \le i,j \le N}$, and

$$\boldsymbol{l} = \boldsymbol{l}(H) = (H^{p_0}, \dots, H^{p_N})^{\mathrm{T}}. \tag{4.6}$$

Since $L_{ij}^* = L_{ji}$, the matrix operator L acting on $\boldsymbol{\varphi} = (\varphi_0, \dots, \varphi_N)^{\mathrm{T}}$ is symmetric in $L^2(\mathbf{R}^n)$. Moreover, we have already shown the positivity of L in the proof of Lemma 3.1, that is, we have the following lemma.

Lemma 4.1 *Let c_1, C_0 be positive constants. There exists a positive constant $C = C(c_1, C_0)$ such that if $H, \nabla b \in L^\infty(\mathbf{R}^n)$ satisfy $C_0^{-1} \le H(x) \le C_0$ and $|\nabla b(x)| \le c_1$, then for any $\delta \in (0,1]$ we have*

$$C^{-1}(\|\nabla \boldsymbol{\varphi}\|^2 + \delta^{-2}\|\boldsymbol{\varphi}'\|^2) \le (L\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L^2} \le C(\|\nabla \boldsymbol{\varphi}\|^2 + \delta^{-2}\|\boldsymbol{\varphi}'\|^2)$$

where $\boldsymbol{\varphi}' = (\varphi_1, \dots, \varphi_N)^{\mathrm{T}}$.

By making use of this nice structure of the equations, we derive an energy estimate which leads uniform bound of the solution in the rescaled variables. Before carrying out the estimate, we need to show that an appropriate norm of the right-hand side of (4.5) would be evaluated by our energy function $E_m(t)$ uniformly with respect to $\delta \in (0,1]$. However, it contains a term $\delta^{-2}[\partial^\alpha, H^{p_i+p_j-1}]\phi_j^\delta$, which cannot be estimated directly because of the coefficient δ^{-2}. Nevertheless, thanks of the commutator we can gain a regularity of order one. Using this fact and the necessary conditions (3.4), we can handle such a term. We remind that the necessary conditions can be written simply as $\mathscr{L}_i\boldsymbol{\phi}^\delta = 0$ for $i = 1, \ldots, N$.

Lemma 4.2 *Let c_0, M be positive constants and m an integer such that $m > n/2 + 1$. There exists a positive constant $C = C(c_0, M, m)$ such that if η and b satisfy*

$$(4.7) \qquad \begin{cases} \|\eta\|_{m-1} + \|b\|_{W^{m+1,\infty}} \le M, \\ c_0 \le H(x) = 1 + \eta(x) - b(x) \quad \text{for} \quad x \in \mathbf{R}^n, \end{cases}$$

and if $\boldsymbol{\varphi}$ satisfies $\mathscr{L}_i\boldsymbol{\varphi} = F_i$ for $i = 1, \ldots, N$, then for $k = 0, \pm 1, \ldots, \pm(m-1)$ and $\delta \in (0,1]$ we have

$$\delta^{-2}\|\boldsymbol{\varphi}'\|_k \le C(\|\nabla\boldsymbol{\varphi}\|_{k+1} + \|\boldsymbol{\varphi}'\|_{k+1} + \|\boldsymbol{F}'\|_k).$$

Proof. In view of (3.8), we see that the equation $\mathscr{L}_i\boldsymbol{\varphi} = F_i$ is equivalent to

$$(4.8) \qquad \sum_{j=1}^N \frac{p_i p_j}{p_i + p_j - 1} H^{p_j}\varphi_j = \frac{\delta^2}{1 + \delta^2|\nabla b|} \sum_{j=0}^N H^{p_j+1}\biggl\{ -\frac{p_i}{(p_i + p_j + 1)(p_j + 1)} H\Delta\varphi_j + \frac{p_i p_j}{(p_i + p_j)p_j}\nabla\cdot(\varphi_j\nabla b) + \frac{p_i}{(p_i + p_j)}\nabla b\cdot\nabla\varphi_j \biggr\} + \frac{\delta^2}{1 + \delta^2|\nabla b|} H^{1-p_i}F_i$$

for $i = 1, \ldots, N$. Since $N \times N$ matrix $A_1' = \bigl(\frac{p_i p_j}{p_i + p_j - 1}\bigr)_{1 \le i,j \le N}$ is nonsingular, the desired estimate comes from standard calculus inequalities. $\square$

Lemma 4.3 *Let c_0, M be positive constants and m an integer such that $m > n/2+1$. There exists a positive constant $C = C(c_0, M, m)$ such that if $(\eta, \boldsymbol{\phi}^\delta)$ is a solution to the Isobe–Kakinuma model* (2.2) *satisfying the conditions in* (3.23)*, then we have*

$$\begin{cases} \delta^{-4}(\|\boldsymbol{\phi}^{\delta\prime}(t)\|_{m-1}^2 + \|\partial_t\boldsymbol{\phi}^{\delta\prime}(t)\|_{m-2}^2) \le CE_m(t), \\ \|\boldsymbol{f}_\alpha\|^2 + \|f_{N+1,\alpha}\|_1^2 \le CE_m(t). \end{cases}$$

Proof. By Lemma 4.2, we have $\delta^{-2}\|\boldsymbol{\phi}^{\delta\prime}\|_{m-1} \lesssim \|\nabla\boldsymbol{\phi}^\delta\|_m + \|\boldsymbol{\phi}^{\delta\prime}\|_m \lesssim E_m(t)^{1/2}$. Let $\boldsymbol{F}' = (F_1, \ldots, F_N)^{\mathrm{T}}$ be defined by (3.10). Then, we have $\|\boldsymbol{F}'\|_{m-1} \lesssim \|\nabla\boldsymbol{\phi}^\delta\|_m + \delta^{-2}\|\boldsymbol{\phi}^{\delta\prime}\|_{m-1} \lesssim E_m(t)^{1/2}$. Since $\partial_t\boldsymbol{\phi}^\delta$ satisfies (3.11), by Lemmas 4.2 and 3.5, we get $\delta^{-2}\|\partial_t\boldsymbol{\phi}^{\delta\prime}\|_{m-2} \lesssim \|\partial_t\boldsymbol{\phi}^\delta\|_m + \|\boldsymbol{F}'\|_{m-2} \lesssim E_m(t)^{1/2}$. Therefore, we obtain the first estimate of the lemma. Note that we also have $\|\partial_t\eta\|_{m-1} + \|\boldsymbol{u}\|_m + \delta\|w\|_m \lesssim E_m(t)^{1/2}$. Thus, by using the standard commutator estimate and an estimate for a symmetric commutator $\|\partial^\alpha(uv) - (\partial^\alpha u)v - u(\partial^\alpha v)\|_1 \lesssim \|u\|_{|\alpha|\vee t_0+1}\|v\|_{|\alpha|\vee t_0+1}$, we obtain the second estimate of the lemma. $\square$

In our energy estimate, we need to handle the time derivative $\partial_t a$. Since the coefficient a contains $\partial_t\boldsymbol{\phi}^{\delta\prime}$, we need to estimate the second order time derivative $\partial_t^2\boldsymbol{\phi}^{\delta\prime}$.

Lemma 4.4 *Let c_0, M be positive constants and m an integer such that $m > n/2+1$. There exists a positive constant $C = C(c_0, M, m)$ such that if $(\eta, \boldsymbol{\phi}^\delta)$ is a solution to the Isobe–Kakinuma model* (2.2) *satisfying the conditions in* (3.23)*, then we have*

$$\begin{cases} \|\partial_t^2 \eta(t)\|_{m-2}^2 + \|\partial_t^2 \boldsymbol{\phi}^\delta(t)\|_{m-1}^2 + \delta^{-2}\|\partial_t^2 \boldsymbol{\phi}^\delta(t)\|_{m-2}^2 \le CE_m(t), \\ \|a-1\|_m^2 + \delta^{-2}\|a-1\|_{m-1}^2 + \|\partial_t a\|_{m-1}^2 \le CE_m(t). \end{cases}$$

Proof. Differentiating the first equation in (3.3) (equivalently (2.2)) with $i=0$ with respect to t, we have

$$\partial_t^2 \eta = \sum_{j=0}^N L_{0j}\partial_t \phi_j^\delta - \nabla \cdot ((\partial_t \eta)\boldsymbol{u}),$$

which together with Lemma 3.5 yields $\|\partial_t^2 \eta\|_{m-2}^2 \lesssim E_m(t)$. Note that the operator $\mathscr{L}$ depends on H but not on ∇H. Therefore, differentiating the second equation in (3.3) and the necessary conditions $\mathscr{L}_i \boldsymbol{\phi}^\delta = 0$ for $i = 1, \ldots, N$ twice with respect to t, we have

$$\mathscr{L}\partial_t^2 \boldsymbol{\phi}^\delta = \boldsymbol{F}_1,$$

where $\boldsymbol{F}_1 = (F_{0,1}, \ldots, F_{N,1})^{\mathrm{T}}$, and

$$\begin{cases} F_{0,1} = -\partial_t \eta - (\partial_t \eta)\sum_{j=1}^N p_j H^{p_j-1}\partial_t \phi_j^\delta - \boldsymbol{u}\cdot\partial_t \boldsymbol{u} - \delta^2 w \partial_t w, \\ F_{i,1} = -(\partial_t^2 \eta)\Big(\frac{\partial}{\partial H}\mathscr{L}_i\Big)\boldsymbol{\phi}^\delta - (\partial_t \eta)^2\Big(\frac{\partial^2}{\partial H^2}\mathscr{L}_i\Big)\boldsymbol{\phi}^\delta - 2(\partial_t \eta)\Big(\frac{\partial}{\partial H}\mathscr{L}_i\Big)\partial_t \boldsymbol{\phi}^\delta \end{cases}$$

for $i = 1, \ldots, N$. Here, we note that $\big(\frac{\partial}{\partial H}\big)^j \mathscr{L}_i$ is also a second order differential operators like $\mathscr{L}_i$. Therefore, by Lemmas 3.5 and 4.3 we have $\|\partial_t \boldsymbol{u}\|_{m-1}^2 + \delta^2\|\partial_t w\|_{m-1}^2 \lesssim E_m(t)$ and $\|F_{0,1}\|_{m-1}^2 + \|\boldsymbol{F}_1'\|_{m-2}^2 \lesssim E_m(t)$, where $\boldsymbol{F}_1' = (F_{1,1}, \ldots, F_{N,1})^{\mathrm{T}}$. Applying the both estimates in Lemma 3.4, we obtain $\|\partial_t^2 \boldsymbol{\phi}^\delta\|_{m-1} + \delta^{-1}\|\partial_t^2 \boldsymbol{\phi}^\delta\|_{m-2} \lesssim \|F_{0,1}\|_{m-1} + \delta\|J_\delta^{-1}\boldsymbol{F}_1'\|_{m-2} \lesssim E_m(t)$, so that the first estimate of the lemma is proved. In view of (4.4), the above estimates together with Lemmas 3.5 and 4.3 yield the second estimate of the lemma. □

Now, we are ready to give a proof of the first half of Theorem 2.1. Since the existence theorem has already been established by R. Nemoto and T. Iguchi [21] in the function spaces, it is sufficient to show (2.6) for some time interval independent of $\delta \in (0,1]$. Moreover, in view of Lemmas 3.5, 4.3, and 4.3, it is sufficient to show that

$$E_m(t) \le M_1, \quad c_0/2 \le H(x,t) \le 2C_0, \quad c_0/2 \le a(x,t) \le 2C_0 \tag{4.9}$$

for any $x \in \mathbf{R}^n$, $0 \le t \le T$, and $0 < \delta \le 1$, where C_0 is chosen so that $H(x,0) \le C_0$ and $a(x,0) \le C_0$ and the constant M_1 and the time T will be determined later. Note that by Lemma 4.4 such a constant C_0 exists under our assumption on the initial data and the bottom topography. In the following we simply write the constants depending only on (c_0, C_0, M_0, m) by C_1 and the constants depending also on M_1 by C_2, which may change from line to line.

We remind that the solution satisfies (4.5). In view of this symmetric form of the equations, we introduce an energy function $\mathscr{E}_m(t)$ by

$$\mathscr{E}_m(t) = \sum_{|\alpha| \le m}\{(a\partial^\alpha \eta(t), \partial^\alpha \eta(t))_{L^2} + (L\partial^\alpha \boldsymbol{\phi}^\delta(t), \partial^\alpha \boldsymbol{\phi}^\delta(t))_{L^2}\}.$$

Now, suppose that the solution satisfies (4.9). Then, by Lemma 4.1 we see that

$$C_1^{-1}E_m(t) \le \mathscr{E}_m(t) \le C_1 E_m(t) \tag{4.10}$$

for $0 \le t \le T$. For $1 \le |\alpha| \le m$ we take the L^2-inner product of (4.5) with $(\partial_t + \boldsymbol{u}\cdot\nabla)\partial^\alpha(\eta, \boldsymbol{\phi}^\delta)^{\mathrm{T}}$ and use the symmetry of the operator L and integration by parts. For $|\alpha| = 0$ we evaluate it directly. Then, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_m(t) &= \sum_{|\alpha|\le m}\{((\partial_t a)\partial^\alpha\eta, \partial^\alpha\eta)_{L^2} + ([\partial_t, L]\partial^\alpha\boldsymbol{\phi}^\delta, \partial^\alpha\boldsymbol{\phi}^\delta)_{L^2}\} \\ &\quad + \sum_{1\le|\alpha|\le m}\{((\nabla\cdot(a\boldsymbol{u}))\partial^\alpha\eta, \partial^\alpha\eta)_{L^2} - 2(L\partial^\alpha\boldsymbol{\phi}^\delta, (\boldsymbol{u}\cdot\nabla)\partial^\alpha\boldsymbol{\phi}^\delta)_{L^2} \\ &\qquad\qquad + 2(f_{N+1,\alpha}, (\partial_t + \boldsymbol{u}\cdot\nabla)\partial^\alpha\eta)_{L^2} + 2(\boldsymbol{f}_\alpha, (\partial_t + \boldsymbol{u}\cdot\nabla)\partial^\alpha\boldsymbol{\phi}^\delta)_{L^2}\} \\ &\quad + 2(a\eta, \partial_t\eta)_{L^2} + 2(L\boldsymbol{\phi}^\delta, \partial_t\boldsymbol{\phi}^\delta)_{L^2}.\end{aligned} \tag{4.11}$$

To evaluate the term with the commutator $[\partial_t, L]$, it is sufficient to see that

$$\begin{aligned}([\partial_t, L]\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L^2} = \sum_{i,j=0}^N \int_{\mathbf{R}^n} (\partial_t\eta)\{&H^{p_i+p_j}\nabla\varphi_j\cdot\nabla\varphi_i - p_j H^{p_i+p_j-1}\varphi_j\nabla b\cdot\nabla\varphi_i \\ &- p_i H^{p_i+p_j-1}\varphi_i\nabla b\cdot\nabla\varphi_j + p_i p_j H^{p_i+p_j-2}(\delta^{-2} + |\nabla b|^2)\varphi_j\varphi_i\}\mathrm{d}x,\end{aligned}$$

which yields $|([\partial_t, L]\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L^2}| \lesssim \|\nabla\boldsymbol{\varphi}\|^2 + \delta^{-2}\|\boldsymbol{\varphi}\|^2$. To evaluate the term $(L\partial^\alpha\boldsymbol{\phi}^\delta, (\boldsymbol{u}\cdot\nabla)\partial^\alpha\boldsymbol{\phi}^\delta)_{L^2}$, we decompose the operator L into its principal term $L^{\mathrm{pr}} = (L_{ij}^{\mathrm{pr}})_{0\le i,j\le N}$ and the remainder part $L^{\mathrm{low}} = (L_{ij}^{\mathrm{low}})_{0\le i,j\le N}$, where

$$\begin{cases} L_{ij}^{\mathrm{pr}}\varphi_j = -\nabla\cdot\left(\dfrac{1}{p_i+p_j+1}H^{p_i+p_j+1}\nabla\varphi_j\right) + \delta^{-2}\dfrac{p_i p_j}{p_i+p_j-1}H^{p_i+p_j-1}\varphi_j, \\ L_{ij}^{\mathrm{low}}\varphi_j = \nabla\cdot\left(\dfrac{p_j}{p_i+p_j}H^{p_i+p_j}\varphi_j\nabla b\right) - \dfrac{p_i}{p_i+p_j}H^{p_i+p_j}\nabla b\cdot\nabla\varphi_j \\ \qquad\qquad + \dfrac{p_i p_j}{p_i+p_j-1}H^{p_i+p_j-1}|\nabla b|^2\varphi_j. \end{cases}$$

We can evaluate the term $(L^{\mathrm{low}}\partial^\alpha\boldsymbol{\phi}^\delta, (\boldsymbol{u}\cdot\nabla)\partial^\alpha\boldsymbol{\phi}^\delta)_{L^2}$ directly by the Cauchy–Schwarz inequality, whereas the term $(L^{\mathrm{pr}}\partial^\alpha\boldsymbol{\phi}^\delta, (\boldsymbol{u}\cdot\nabla)\partial^\alpha\boldsymbol{\phi}^\delta)_{L^2}$ is evaluated by the expression

$$\begin{aligned}&(L^{\mathrm{pr}}\boldsymbol{\varphi}, (\boldsymbol{u}\cdot\nabla)\boldsymbol{\varphi})_{L^2} \\ &= \sum_{i,j=0}^N \int_{\mathbf{R}^n}\left\{\frac{1}{p_i+p_j+1}\left(H^{p_i+p_j+1}\nabla\varphi_j\cdot[\nabla, \boldsymbol{u}\cdot\nabla]\varphi_i - \frac{1}{2}(\nabla\cdot(H^{p_i+p_j+1}\boldsymbol{u}))\nabla\varphi_j\cdot\nabla\varphi_i\right)\right. \\ &\qquad\qquad \left. - \frac{1}{2}\delta^{-2}\frac{p_i p_j}{p_i+p_j-1}(\nabla\cdot(H^{p_i+p_j-1}\boldsymbol{u}))\varphi_j\varphi_i\right\}\mathrm{d}x,\end{aligned}$$

where we used integration by parts. This yields $|(L^{\mathrm{pr}}\boldsymbol{\varphi}, (\boldsymbol{u}\cdot\nabla)\boldsymbol{\varphi})_{L^2}| \lesssim \|\nabla\boldsymbol{\varphi}\|^2 + \delta^{-2}\|\boldsymbol{\varphi}\|^2$. Concerning the terms with $f_{N+1,\alpha}$, by Lemma 4.3 and $\|\boldsymbol{u}\|_m \lesssim E_m(t)$ we evaluate it as

$$|(f_{N+1,\alpha}, (\partial_t + \boldsymbol{u}\cdot\nabla)\partial^\alpha\eta)_{L^2}| \le \|f_{N+1,\alpha}\|_1\|(\partial_t + \boldsymbol{u}\cdot\nabla)\partial^\alpha\eta\|_{-1} \le C_2 E_m(t).$$

The term with $\boldsymbol{f}_\alpha$ and the last two terms in the right-hand side of (4.11) can be evaluated directly by the Cauchy–Schwarz inequality. Therefore, in view of Lemmas 4.3 and 4.4 we obtain $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_m(t) \leq C_2\mathscr{E}_m(t)$, which together with Gronwall's inequality and the equivalence (4.10) implies

$$E_m(t) \leq C_1 E_m(0) e^{C_2 t} \leq C_1 M_0^2 e^{C_2 t}.$$

On the other hand, by the fundamental theorem of calculus, the Sobolev imbedding theorem, and Lemmas 3.5 and 4.4 we have

$$|H(x,t) - H(x,0)| + |a(x,t) - a(x,0)| \leq C_2 t.$$

By taking into account these two inequalities, we define the positive constant M_1 and the time T so that $M_1 = 2C_1M_0^2$ and then $T = C_2^{-1}\min\{\log 2, C_0, c_0/2\}$. Then, the above arguments show that the solution in fact satisfy (4.9) for $0 \leq t \leq T$ uniformly in $\delta \in (0,1]$.

5 Uniform estimate of the solution II

In this section we will prove the second half of Theorem 2.1, that is, the uniform bound (2.7) to the original variables $(\eta, \boldsymbol{\phi})$ by using the uniform bound (2.6) obtained in the previous section and the necessary conditions (2.4). To this end we have to use the advantage of our specific choice of the indices p_i, that is, $p_i = 2i$ in the case of the flat bottom and $p_i = i$ in the case with general bottom topographies.

5.1 The case $p_i = 2i$ with the flat bottom

Lemma 5.1 *Choose $p_i = 2i$ $(i = 0, 1, \ldots, N)$ and suppose that the bottom is flat. If $\boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_N)^{\mathrm{T}}$ satisfies $\mathscr{L}_i\boldsymbol{\varphi} = 0$ for $i = 1, \ldots, N$, then we have*

$$\varphi_j = \delta^2\left\{-\frac{1}{2j(2j-1)}\Delta\varphi_{j-1} + \beta_{j,N}H^{2(N-j)}\Delta\varphi_N\right\}$$

for $j = 1, \ldots, N$, where the constant $\beta_{j,N}$ is defined by (5.1) *below.*

Proof. It follows from (4.8) with $F_i = 0$ that

$$\sum_{j=1}^N \frac{4ij}{2(i+j)-1}H^{2j}\varphi_j = \sum_{j=1}^N \frac{4ij}{2(i+j)-1}\left(-\frac{\delta^2}{2j(2j-1)}H^{2j}\Delta\varphi_{j-1}\right) - \frac{2i}{(2(N+i)+1)(2N+1)}\delta^2H^{2N}\Delta\varphi_N.$$

In view of this, we define constants $\beta_{j,N}$ for $j = 1, 2, \ldots, N$ by

$$\sum_{j=1}^N \frac{4ij}{2(i+j)-1}\beta_{j,N} = -\frac{2i}{(2(N+i)+1)(2N+1)} \tag{5.1}$$

for $i = 1, 2, \ldots, N$. Since the matrix $\left(\frac{4ij}{2(i+j)-1}\right)_{1\leq i,j\leq N}$ is nonsingular, the constants $\beta_{j,N}$ $(j = 1, 2, \ldots, N)$ are uniquely determined. Then, we obtain the desired identity. □

Lemma 5.2 *Under the same hypothesis of Lemma 5.1, for any integer k we have*

$$\|(\varphi_j,\dots,\varphi_N)\|_k \le \delta^{2j} C(\|\eta\|_{|k|\vee|k+2(j-1)|\vee t_0})\|\nabla\boldsymbol{\varphi}\|_{k+2j-1}$$

for $j=1,\dots,N$.

Proof. It follows from Lemma (5.1) that $\|\varphi_j\|_k \le \delta^2\|\nabla\varphi_{j-1}\|_{k+1} + \delta^2 C(\|\eta\|_{|k|\vee t_0})\|\nabla\varphi_N\|_{k+1}$, so that

$$\begin{aligned}\|(\varphi_j,\dots,\varphi_N)\|_k &\le \delta^2 C(\|\eta\|_{|k|\vee t_0})\|\nabla(\varphi_{j-1},\dots,\varphi_N)\|_{k+1}\\ &\le \delta^2 C(\|\eta\|_{|k|\vee t_0})\|(\varphi_{j-1},\dots,\varphi_N)\|_{k+2}\end{aligned}$$

for $j=1,\dots,N$. Using this inductively, we obtain the desired estimate. □

Now, we will show the second half of Theorem 2.1 in the case (H1). Since $\boldsymbol{\phi}^\delta$ satisfies the necessary conditions (2.4), we can apply Lemma 5.2 with $\boldsymbol{\varphi}=\boldsymbol{\phi}^\delta$ and $k=m-2j+1$. Then, under our hypothesis we have $m-1>n/2$ and $m\ge j\ge 1$, so that $|k|\vee|k+2(j-1)|\vee t_0 = m-1$. Therefore, we obtain

$$\|\phi_j^\delta(t)\|_{m-2j+1} \le \delta^{2j} C(\|\eta(t)\|_{m-1})\|\nabla\boldsymbol{\phi}^\delta(t)\|_m,$$

which together with (2.6) yields the desired estimate (2.7) in the case (H1).

5.2 The case $p_i = i$ with general bottom topographies

For simplify the description, we introduce a differential operator $Q=Q(b)$ depending on the bottom topography b by

$$Q\psi = \nabla\cdot(\psi\nabla b) + \nabla b\cdot\nabla\psi. \tag{5.2}$$

Lemma 5.3 *Choose $p_i = i$ ($i=0,1,\dots,N$). If $\boldsymbol{\varphi}=(\varphi_0,\dots,\varphi_N)^{\mathrm{T}}$ satisfies $\mathscr{L}_i\boldsymbol{\varphi}=0$ for $i=1,\dots,N$, then we have*

$$\begin{cases}\varphi_1 = \dfrac{\delta^2}{1+\delta^2|\nabla b|^2}\Big\{\nabla b\cdot\nabla\varphi_0 + \gamma_{1,N-1}H^N(\Delta\varphi_{N-1} - NQ(b)\varphi_N) + \gamma_{1,N}H^{N+1}\Delta\varphi_N\Big\},\\ \varphi_j = \dfrac{\delta^2}{1+\delta^2|\nabla b|^2}\Big\{-\dfrac{1}{j(j-1)}\Delta\varphi_{j-2} + \dfrac{1}{j}Q(b)\varphi_{j-1}\\ \qquad + \gamma_{j,N-1}H^{N-j+1}(\Delta\varphi_{N-1} - NQ(b)\varphi_N) + \gamma_{j,N}H^{N-j+2}\Delta\varphi_N\Big\} \quad \text{for} \quad j=2,\dots,N,\end{cases}$$

where the constant $\gamma_{j,k}$ is defined by (5.3) *below.*

Proof. It follows from (4.8) with $F_i=0$ that

$$\begin{aligned}\sum_{j=1}^N \frac{ij}{i+j-1}H^j\varphi_j &= \frac{\delta^2}{1+\delta^2|\nabla b|^2}\Big\{\sum_{j=2}^N \frac{ij}{i+j-1}H^j\Big(-\frac{1}{j(j-1)}\Delta\phi_{j-2} + \frac{1}{j}\nabla\cdot(\varphi_{j-1}\nabla b)\Big)\\ &\quad + \sum_{j=1}^N \frac{ij}{i+j-1}H^j\Big(\frac{1}{j}\nabla b\cdot\nabla\varphi_{j-1}\Big)\\ &\quad - \frac{i}{(N+i)N}H^{N+1}(\Delta\varphi_{N-1} - NQ(b)\varphi_N) - \frac{i}{(N+i+1)(N+1)}H^{N+2}\Delta\phi_N^\delta\Big\}.\end{aligned}$$

In view of this, we define constants $\gamma_{j,k}$ for $j = 1, 2, \dots, N$ and $k \geq 0$ by

$$\sum_{j=1}^{N} \frac{ij}{i+j-1}\gamma_{j,k} = -\frac{i}{(k+i+1)(k+1)} \tag{5.3}$$

for $i = 1, 2, \dots, N$. Since the matrix $(\frac{ij}{i+j-1})_{1\leq i,j\leq N}$ is nonsingular, the constants $\gamma_{j,k}$ ($j = 1, 2, \dots, N,\ k \geq 0$) are uniquely determined. Then, we obtain the desired identity. □

Lemma 5.4 *Under the same hypothesis of Lemma 5.3, for any integer k we have*

$$\begin{aligned}&\|(\varphi_{2j-1}, \varphi_{2j}, \dots, \varphi_N)\|_k \\ &\leq \delta^{2j} C(\|\eta\|_{|k|\vee|k+2(j-1)|\vee t_0}, \|b\|_{W^{|k|+1\vee|k+2j-1|+1,\infty}})(\|\nabla\varphi_0\|_{k+2j-1} + \|\boldsymbol{\varphi}'\|_{k+2j})\end{aligned}$$

for $j = 1, \dots, [(N+1)/2]$, where $\boldsymbol{\varphi}' = (\varphi_1, \dots, \varphi_N)^{\mathrm{T}}$.

Proof. We note that in view of (5.2) we have $\|Q\psi\|_k \lesssim \|\nabla b\|_{W^{|k|\vee|k+1|,\infty}}\|\psi\|_{k+1}$. It follows from Lemma 5.3 that

$$\begin{cases}\|\varphi_1\|_k \leq \delta^2 C(\|\eta\|_{|k|\vee t_0}, \|b\|_{W^{|k|+1\vee|k+1|+1,\infty}})(\|\nabla\varphi_0\|_{k+1} + \|\boldsymbol{\varphi}'\|_{k+2}), \\ \|\varphi_j\|_k \leq \delta^2 C(\|\eta\|_{|k|\vee t_0}, \|b\|_{W^{|k|+1\vee|k+1|+1,\infty}})(\|\nabla\varphi_{j-2}\|_{k+1} + \|(\varphi_{j-1}, \dots, \varphi_N)\|_{k+2})\end{cases} \tag{5.4}$$

for $j = 2, \dots, N$, so that

$$\|(\varphi_{2j-1}, \varphi_{2j}, \dots, \varphi_N)\|_k \leq \delta^2 C(\|\eta\|_{|k|\vee t_0}, \|b\|_{W^{|k|+1\vee|k+1|+1,\infty}})\|(\varphi_{2j-3}, \varphi_{2j-2}, \dots, \varphi_N)\|_k$$

for $j = 2, \dots, N$. Using this inductively, we obtain

$$\begin{aligned}&\|(\varphi_{2j-1}, \varphi_{2j}, \dots, \varphi_N)\|_k \\ &\leq \delta^{2(j-1)} C(\|\eta\|_{|k|\vee|k+2(j-2)|\vee t_0}, \|b\|_{W^{|k|+1\vee|k+2j-3|+1,\infty}})\|(\varphi_1, \varphi_2, \dots, \varphi_N)\|_{k+2(j-1)}\end{aligned}$$

for $j = 2, \dots, N$. Applying (5.4) with k replaced by $k + 2(j-1)$ to the last term in the above inequality, we obtain the desired estimate. □

Now, we can show the second half of Theorem 2.1 in the case (H2). We apply Lemma 5.4 with $\boldsymbol{\varphi} = \boldsymbol{\phi}^\delta$ and $k = m-2j+1$. Then, $|k|\vee|k+2(j-1)|\vee t_0 = m-1$ and $|k|+1\vee|k+2j-1|+1 = m+1$ hold if and only if the integer j satisfies $1 \leq j \leq m$. We remind our hypothesis $m \geq [(N+1)/2]$. Therefore, in the case of even N, we obtain

$$\|(\phi_{2j-1}^\delta, \phi_{2j}^\delta)\|_{m-2j-1} \leq \delta^{2j} C_m(\|\nabla\phi_0^\delta\|_m + \|\boldsymbol{\phi}^{\delta\prime}\|_{m+1}) \quad \text{for} \quad j = 1, \dots, N/2,$$

where $C_m = C(\|\eta\|_{m-1}, \|b\|_{W^{m+1,\infty}})$. In the case of odd N, we have

$$\begin{cases}\|\phi_{2j-1}^\delta\|_{m-2j-1} \leq \delta^{2j} C_m(\|\nabla\phi_0^\delta\|_m + \|\boldsymbol{\phi}^{\delta\prime}\|_{m+1}) & \text{for} \quad j = 1, \dots, (N+1)/2, \\ \|\phi_{2j}^\delta\|_{m-2j-1} \leq \delta^{2j} C_m(\|\nabla\phi_0^\delta\|_m + \|\boldsymbol{\phi}^{\delta\prime}\|_{m+1}) & \text{for} \quad j = 1, \dots, (N+1)/2 - 1.\end{cases}$$

These estimates together with (2.6) yield the desired estimate (2.7) in the case (H2).

The proof of Theorem 2.1 is complete.

6 Consistency of the Isobe–Kakinuma model I

In this and the following two sections, we will prove Theorem 2.2. Suppose that $(\eta, \boldsymbol{\phi}^\delta)$ is a solution of the Isobe–Kakinuma model (2.2) and define ϕ by (2.8), which is an approximation of the trace of the velocity potential on the water surface. We will show that (η, ϕ) satisfies the water wave equations in Zakharov–Craig–Sulem formulation (1.6) with an error of order $O(\delta^{4N+2})$ in the case (H1) and of order $O(\delta^{4[N/2]+2})$ in the case (H2). Here, we remind that the water wave equations in terms of the surface elevation η and the velocity potential Φ have the form

$$\Delta\Phi + \delta^{-2}\partial_z^2\Phi = 0 \quad \text{in} \quad \Omega(t), \tag{6.1}$$

$$\begin{cases} \partial_t\Phi + \frac{1}{2}\Big(|\nabla\Phi|^2 + \delta^{-2}(\partial_z\Phi)^2\Big) + \eta = 0 & \text{on} \quad \Gamma(t), \\ \partial_t\eta + \nabla\eta\cdot\nabla\Phi - \delta^{-2}\partial_z\Phi = 0 & \text{on} \quad \Gamma(t), \end{cases} \tag{6.2}$$

$$\delta^{-2}\partial_z\Phi - \nabla\eta\cdot\nabla\Phi = 0 \quad \text{on} \quad \Sigma, \tag{6.3}$$

where $\Omega(t)$, $\Gamma(t)$, and Σ denote the water region, the water surface, and the bottom, respectively. Our strategy to show the desired consistency is to use an approximate velocity potential which satisfy (6.1)–(6.3) approximately.

We define an approximate velocity potential $\Phi^{\rm app}$ in the water region by (1.10). Then, we see that the second equation in (2.2) is equivalent to

$$\partial_t\Phi^{\rm app} + \frac{1}{2}\Big(|\nabla\Phi^{\rm app}|^2 + \delta^{-2}(\partial_z\Phi^{\rm app})^2\Big) + \eta = 0 \quad \text{on} \quad z = \eta(x,t), \tag{6.4}$$

which is exactly the first equation in (6.2), that is, Bernoulli's law restricted on the water surface. However, $\Phi^{\rm app}$ satisfies the other equations approximately with an error of order $O(\delta^{2N})$ in the case (H1) and of order $O(\delta^{2[N/2]})$ in the case (H2). These orders of the error are not sufficient to show the desired result, so that we have to modify $\Phi^{\rm app}$ appropriately.

In the following arguments, the time t is arbitrarily fixed so that we omit it in the notation. In (3.5)–(3.6) we defined operators $\mathscr{L}\boldsymbol{\varphi} = (\mathscr{L}_0\boldsymbol{\varphi}, \dots, \mathscr{L}_N\boldsymbol{\varphi})^{\rm T}$,which act on $(N+1)$ vector-valued functions $\boldsymbol{\varphi} = (\varphi_0, \dots, \varphi_N)^{\rm T}$. We denote these operators by $\mathscr{L}^{(N)}$ and $\mathscr{L}_i^{(N)}$ for $i = 0, 1, \dots, N$. We assume that η, ϕ, $\boldsymbol{\phi}^\delta = (\phi_0^\delta, \dots, \phi_N^\delta)$, and b are given so that

$$\mathscr{L}_0^{(N)}\boldsymbol{\phi}^\delta = \phi, \qquad \mathscr{L}_i^{(N)}\boldsymbol{\phi}^\delta = 0 \quad \text{for} \quad i = 1, \dots, N, \tag{6.5}$$

and that

$$\begin{cases} \|\eta\|_m + \|\nabla\phi\|_{m-1} + \|b\|_{W^{m+1,\infty}} \le M, \\ H(x) = 1 + \eta(x) - b(x) \ge c_0 \quad \text{for} \quad x \in \mathbf{R}^n, \end{cases} \tag{6.6}$$

where m is an integer satisfying $m \ge n/2 + 1$. Now, we define $\widetilde{\boldsymbol{\phi}}^\delta = (\widetilde{\phi}_0^\delta, \widetilde{\phi}_1^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)^{\rm T}$ by

$$\mathscr{L}_0^{(2N+2)}\widetilde{\boldsymbol{\phi}}^\delta = \phi, \qquad \mathscr{L}_i^{(2N+2)}\widetilde{\boldsymbol{\phi}}^\delta = 0 \quad \text{for} \quad i = 1, \dots, 2N+2, \tag{6.7}$$

and then a modified approximate velocity potential $\widetilde{\Phi}^{\rm app}$ by

$$\widetilde{\Phi}^{\rm app}(x,z,t) = \sum_{i=0}^{2N+2} (z + 1 - b(x))^{p_i}\widetilde{\phi}_i^\delta(x,t). \tag{6.8}$$

We will show that η and $\widetilde{\Phi}^{\rm app}$ satisfy the water wave equations (6.1)–(6.3) with an error of desirable order.

To compare $\widetilde{\phi}_j^\delta$ with ϕ_j^δ for $j=0,1,\ldots,N$, we introduce a new function $\boldsymbol{\varphi}^\delta$ by

$$\boldsymbol{\varphi}^\delta = (\varphi_0^\delta, \varphi_1^\delta, \ldots, \varphi_N^\delta)^{\rm T}, \qquad \varphi_j^\delta = \phi_j^\delta - \widetilde{\phi}_j^\delta \quad \text{for} \quad j=0,1,\ldots,N. \tag{6.9}$$

Then, we see that $\boldsymbol{\varphi}^\delta$ satisfies

$$\mathscr{L}^{(N)}\boldsymbol{\varphi}^\delta = \boldsymbol{R} = (R_0, R_1, \ldots, R_N)^{\rm T}, \tag{6.10}$$

where

$$R_0 = \sum_{j=N+1}^{2N+2} H^{p_j}\widetilde{\phi}_j^\delta, \qquad R_i = \sum_{j=N+1}^{2N+2}(L_{ij} - H^{p_i}L_{0j})\widetilde{\phi}_j^\delta \quad \text{for} \quad i=1,2,\ldots,N. \tag{6.11}$$

We decompose $R_i = R_{1,i} + \delta^{-2}R_{2,i}$, where

$$\begin{aligned} R_{1,i} = \sum_{j=N+1}^{2N+2}\biggl\{ &-\left(\frac{1}{p_i+p_j+1} - \frac{1}{p_j+1}\right)H^{p_i+p_j+1}\Delta\varphi_j \\ &+ \left(\frac{p_j}{p_i+p_j} - \frac{p_j}{p_j}\right)H^{p_i+p_j}\nabla\cdot(\varphi_j\nabla b) \\ &- \frac{p_i}{p_i+p_j}H^{p_i+p_j}\nabla b\cdot\nabla\varphi_j + \frac{p_ip_j}{p_i+p_j-1}H^{p_i+p_j-1}|\nabla b|^2\varphi_j \biggr\}, \end{aligned} \tag{6.12}$$

$$R_{2,i} = \sum_{j=N+1}^{2N+2}\frac{p_ip_j}{p_i+p_j-1}H^{p_i+p_j-1}\widetilde{\phi}_j^\delta, \tag{6.13}$$

for $i=1,2,\ldots,N$. These decompositions lead a decomposition $\boldsymbol{R} = \boldsymbol{R}_1 + \delta^{-2}\boldsymbol{R}_2$, where $\boldsymbol{R}_1 = (R_0, R_{1,1}, \ldots, R_{1,N})^{\rm T}$ and $\boldsymbol{R}_2 = (0, R_{2,1}, \ldots, R_{2,N})^{\rm T}$. Then, we have

$$\mathscr{L}^{(N)}\boldsymbol{\varphi}^\delta = \boldsymbol{R}_1 + \delta^{-2}\boldsymbol{R}_2. \tag{6.14}$$

By using equations (6.7) and (6.14), we will evaluate $\widetilde{\boldsymbol{\phi}}^\delta$ and $\boldsymbol{\varphi}^\delta$. We also note that the difference between the two approximate velocity potentials $\widetilde{\Phi}^{\rm app}$ and $\Phi^{\rm app}$ is represented as

$$\widetilde{\Phi}^{\rm app} - \Phi^{\rm app} = \sum_{j=0}^{N}(z+1-b)^{p_j}\varphi_j^\delta + \sum_{j=N+1}^{2N+2}(z+1-b)^{p_j}\widetilde{\phi}_j^\delta. \tag{6.15}$$

6.1 The case $p_i = 2i$ with the flat bottom

Lemma 6.1 *Choose $p_i = 2i$ $(i=0,1,\ldots,N)$ and suppose that $b=0$ and that (η,ϕ) satisfy (6.6). For any $j=1,2,\ldots,2N+2$, if an integer k satisfies $|k|\le m$ and $|k+2j-1|\le m-1$, then we have*

$$\|(\widetilde{\phi}_j^\delta, \widetilde{\phi}_{j+1}^\delta, \ldots, \widetilde{\phi}_{2N+2}^\delta)\|_k \le C\delta^{2j},$$

where $C = C(M, c_0, m, j, k, N)$ is a positive constant independent of $\delta\in(0,1]$.

Proof. By Lemma 3.4, particularly, the first estimate in (3.20), we have $\|\nabla\widetilde{\phi}_0^\delta\|_k + \|\widetilde{\phi}^{\delta\prime}\|_{k+1} \lesssim \|\nabla\phi\|_k \lesssim 1$ if $|k| \leq m-1$, so that $\|\nabla\widetilde{\boldsymbol{\phi}}^\delta\|_{k+2j-1} \lesssim 1$ if $|k+2j-1| \leq m-1$. On the other hand, it follows from Lemma 5.2 that $\|(\widetilde{\phi}_j^\delta, \widetilde{\phi}_{j+1}^\delta, \ldots, \widetilde{\phi}_N^\delta)\|_k \lesssim \delta^{2j}\|\nabla\widetilde{\boldsymbol{\phi}}^\delta\|_{k+2j-1}$ if $|k| \vee |k+2(j-1)| \leq m$. In view of $|k+2(j-1)| \leq |k+2j-1|+1$, these two estimates give the desired one. □

Lemma 6.2 *Choose $p_i = 2i$ $(i = 0, 1, \ldots, N)$ and suppose that $b = 0$ and that (η, ϕ) satisfy (6.6). For any $j = 0, 1, \ldots, N+1$, if an integer k satisfies $|k-1| \vee |k| \vee |k+2j-1| \leq m-1$, then we have*

$$\|\boldsymbol{\varphi}^\delta\|_k + \|(\widetilde{\phi}_{N+1}^\delta, \ldots, \widetilde{\phi}_{2N+2}^\delta)\|_k \leq C\delta^{2j},$$

where $C = C(M, c_0, m, j, k, N)$ is a positive constant independent of $\delta \in (0, 1]$.

Proof. It follows from Lemma 3.4, particularly, the second estimate in (3.20) with k replaced by $k-1$ that $\|(\mathscr{L}^{(N)})^{-1}\boldsymbol{F}\|_k \lesssim \|F_0\|_k + \|\boldsymbol{F}'\|_{k-2}$ if $|k-1| \leq m-1$. Moreover, if $F_0 = 0$, then we can apply (3.21) and obtain $\|(\mathscr{L}^{(N)})^{-1}\boldsymbol{F}\|_k \lesssim \delta^2\|\boldsymbol{F}'\|_k$ if $|k| \leq m-1$. Therefore, in view of (6.14) we obtain

$$\begin{aligned}\|\boldsymbol{\varphi}^\delta\|_k &\leq \|(\mathscr{L}^{(N)})^{-1}\boldsymbol{R}_1\|_k + \delta^{-2}\|(\mathscr{L}^{(N)})^{-1}\boldsymbol{R}_2\|_k \\ &\lesssim \|R_0\|_k + \|\boldsymbol{R}_1'\|_{k-2} + \|\boldsymbol{R}_2'\|_k \quad \text{if} \quad |k-1| \vee |k| \leq m-1.\end{aligned}$$

Here, by the explicit form (6.11)–(6.13) of R_0, $\boldsymbol{R}_1'$, and $\boldsymbol{R}_2'$, we see that

$$\begin{cases}\|R_0\|_k + \|\boldsymbol{R}_2'\|_k \leq C(\|\eta\|_{|k|\vee t_0})\|(\widetilde{\phi}_{N+1}^\delta, \ldots, \widetilde{\phi}_{2N+2}^\delta)\|_k, \\ \|\boldsymbol{R}_1'\|_{k-2} \leq C(\|\eta\|_{|k-2|\vee|k-1|\vee t_0})\|(\widetilde{\phi}_{N+1}^\delta, \ldots, \widetilde{\phi}_{2N+2}^\delta)\|_k,\end{cases}$$

so that

$$\|R_0\|_k + \|\boldsymbol{R}_1'\|_{k-2} + \|\boldsymbol{R}_2'\|_k \lesssim \|(\widetilde{\phi}_{N+1}^\delta, \ldots, \widetilde{\phi}_{2N+2}^\delta)\|_k \quad \text{if} \quad |k-2| \vee |k| \leq m.$$

On the other hand, if $0 \leq j \leq N+1$, then by Lemma 6.1 we have

$$\|(\widetilde{\phi}_{N+1}^\delta, \ldots, \widetilde{\phi}_{2N+2}^\delta)\|_k \leq \|(\widetilde{\phi}_j^\delta, \ldots, \widetilde{\phi}_{2N+2}^\delta)\|_k \lesssim \delta^{2j}$$

if $|k| \leq m$ and $|k+2j-1| \leq m-1$. These three estimates yield $\|\boldsymbol{\varphi}^\delta\|_k \leq C\delta^{2j}$ if $|k-1| \vee |k| \leq m-1$, $|k-2| \vee |k| \leq m$, $|k| \leq m$, and $|k+2j-1| \leq m-1$. Since these last conditions on k are equivalent to $|k-1| \vee |k| \vee |k+2j-1| \leq m-1$, we obtain the desired result. □

Remark 6.3 Lemmas 6.1 and 6.2 imply that $(\widetilde{\phi}_{N+1}^\delta, \ldots, \widetilde{\phi}_{2N+2}^\delta)$ and $(\varphi_0^\delta, \ldots, \varphi_N^\delta)$ are both of order $O(\delta^{2N+2})$ if m is sufficiently large. In view of (6.15), the difference between the two approximate velocity potentials $\widetilde{\Phi}^{\rm app}$ and $\Phi^{\rm app}$ is of order $O(\delta^{2N+2})$.

We remind that $\widetilde{\Phi}^{\rm app}$ was defined by (6.8). In the case (H1), by direct calculation and Lemma 5.1, we see that

$$\begin{cases}\Delta\widetilde{\Phi}^{\rm app} + \delta^{-2}\partial_z^2\widetilde{\Phi}^{\rm app} = R & \text{in} \quad \Omega, \\ \widetilde{\Phi}^{\rm app} = \phi & \text{on} \quad \Gamma, \\ \delta^{-2}\partial_z\widetilde{\Phi}^{\rm app} = 0 & \text{on} \quad \Sigma,\end{cases} \tag{6.16}$$

where

$$R(x,z) = \sum_{j=0}^{2N+2}(z+1)^{2j}r_j(x) \tag{6.17}$$

and

$$r_j(x) = \begin{cases} (2j+2)(2j+1)\beta_{j+1,2N+2}H^{4N+2-2j}\Delta\widetilde{\phi}^\delta_{2N+2} & \text{for} \quad j=0,1,\ldots,2N+1, \\ \Delta\widetilde{\phi}^\delta_{2N+2} & \text{for} \quad j=2N+2. \end{cases}$$

Concerning the remainder term R, we have the following lemma.

Lemma 6.4 *Choose $p_i = 2i$ $(i=0,1,\ldots,N)$ and suppose that $b=0$ and that (η,ϕ) satisfy (6.6). For any $j=0,1,\ldots,2N+2$, if an integer k satisfies $|k|\vee|k+2|\leq m$ and $|k+2j+1|\leq m-1$, then we have*

$$\|(r_0,r_1,\ldots,r_{2N+2})\|_k \leq C\delta^{2j},$$

where $C=C(M,c_0,m,j,k,N)$ is a positive constant independent of $\delta\in(0,1]$.

Proof. It is easy to see that $\|(r_0,r_1,\ldots,r_{2N+2})\|_k \lesssim \|\widetilde{\phi}_{2N+2}\|_{k+2}$ if $|k|\leq m$. By Lemma 6.1 with k replaced by $k+2$, we have $\|\widetilde{\phi}_{2N+2}\|_{k+2}\lesssim \delta^{2j}$ if $|k+2|\vee|k+2j+1|+1\leq m$ for $j=0,1,\ldots,2N+2$. Combining these estimates we obtain the desired one. □

Remark 6.5 Lemma 6.4 implies that the remainder term R is of order $O(\delta^{4N+2})$ if m is sufficiently large, so that the approximate velocity potential $\widetilde{\Phi}^{\rm app}$ satisfies the continuity equation (6.1) with an error of order $O(\delta^{4N+2})$ while it satisfies the boundary condition (6.3) on the bottom precisely in the case of the flat bottom.

6.2 The case $p_i = i$ with general bottom topographies

Lemma 6.6 *Choose $p_i=i$ $(i=0,1,\ldots,N)$ and suppose that (η,ϕ) and b satisfy (6.6). For any $j=1,2,\ldots,N+1$, if an integer k satisfies $|k|\leq m$ and $|k+2j-1|\leq m-1$, then we have*

$$\|(\widetilde{\phi}^\delta_{2j-1},\widetilde{\phi}^\delta_{2j},\ldots,\widetilde{\phi}^\delta_{2N+1},\widetilde{\phi}^\delta_{2N+2})\|_k \leq C\delta^{2j},$$

where $C=C(M,c_0,m,j,k,N)$ is a positive constant independent of $\delta\in(0,1]$.

Proof. As in the proof of Lemma 6.1, by Lemma 3.4 we have $\|\nabla\widetilde{\phi}^\delta_0\|_{k+2j-1}+\|\widetilde{\phi}^{\delta\prime}\|_{k+2j}\lesssim 1$ if $|k+2j-1|\leq m-1$. On the other hand, it follows from Lemma 5.4 that $\|(\widetilde{\phi}^\delta_{2j-1},\widetilde{\phi}^\delta_{2j},\ldots,\widetilde{\phi}^\delta_{2N+2})\|_k \lesssim \delta^{2j}(\|\nabla\widetilde{\phi}^\delta_0\|_{k+2j-1}+\|\widetilde{\phi}^{\delta\prime}\|_{k+2j})$ if $|k|\vee|k+2j-1|\leq m$. These two estimates give the desired one. □

Lemma 6.7 *Choose $p_i=i$ $(i=0,1,\ldots,N)$ and suppose that (η,ϕ) and b satisfy (6.6). For any $j=0,1,\ldots,[N/2]+1$, if an integer k satisfies $|k-1|\vee|k|\vee|k+2j-1|\leq m-1$, then we have*

$$\|\boldsymbol{\varphi}^\delta\|_k + \|(\widetilde{\phi}^\delta_{N+1},\ldots,\widetilde{\phi}^\delta_{2N+2})\|_k \leq C\delta^{2j},$$

where $C=C(M,c_0,m,j,k,N)$ is a positive constant independent of $\delta\in(0,1]$.

Proof. As in the proof of Lemma 6.2, we have

$$\|\boldsymbol{\varphi}^\delta\|_k \lesssim \|R_0\|_k + \|\boldsymbol{R}_1'\|_{k-2} + \|\boldsymbol{R}_2'\|_k \quad \text{if} \quad |k-1|\vee|k|\leq m-1.$$

Here, by the explicit form (6.11)–(6.13) of R_0, $\boldsymbol{R}_1'$, and $\boldsymbol{R}_2'$, we see that

$$\begin{cases} \|R_0\|_k + \|\boldsymbol{R}_2'\|_k \leq C(\|\eta\|_{|k|\vee t_0}, \|b\|_{W^{|k|,\infty}})\|(\widetilde{\phi}^\delta_{N+1},\ldots,\widetilde{\phi}^\delta_{2N+2})\|_k, \\ \|\boldsymbol{R}_1'\|_{k-2} \leq C(\|\eta\|_{|k-2|\vee|k-1|\vee t_0}, \|b\|_{W^{|k-2|+1\vee|k-1|+1,\infty}})\|(\widetilde{\phi}^\delta_{N+1},\ldots,\widetilde{\phi}^\delta_{2N+2})\|_k, \end{cases}$$

so that

$$\|R_0\|_k + \|\boldsymbol{R}_1'\|_{k-2} + \|\boldsymbol{R}_2'\|_k \lesssim \|(\widetilde{\phi}_{N+1}^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_k \quad \text{if} \quad |k-2| \vee |k| \le m.$$

(i) The case of even $N = 2N_1$: if $0 \le j \le N_1 + 1 = [N/2]+1$, then by Lemma 6.1 we have

$$\begin{aligned}\|(\widetilde{\phi}_{N+1}^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_k &= \|(\widetilde{\phi}_{2(N_1+1)-1}^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_k \le \|(\widetilde{\phi}_{2j-1}^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_k \\ &\lesssim \delta^{2j} \quad \text{if} \quad |k| \vee |k+2j-1| + 1 \le m.\end{aligned}$$

(ii) The case of odd $N = 2N_1 - 1$: if $0 \le j \le N_1 = [N/2]+1$, then by Lemma 6.1 we have

$$\begin{aligned}\|(\widetilde{\phi}_{N+1}^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_k &= \|(\widetilde{\phi}_{2N_1}^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_k \le \|(\widetilde{\phi}_{2j-1}^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_k \\ &\lesssim \delta^{2j} \quad \text{if} \quad |k| \vee |k+2j-1| + 1 \le m.\end{aligned}$$

Combining the above estimates, we obtain the desired result. □

Remark 6.8 Lemma 6.7 imply that $(\widetilde{\phi}_{N+1}^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)$ and $(\varphi_0^\delta, \dots, \varphi_N^\delta)$ are both of order $O(\delta^{2[N/2]+2})$ if m is sufficiently large. In view of (6.15), the difference between the two approximate velocity potentials $\widetilde{\Phi}^{\mathrm{app}}$ and Φ^{app} is of order $O(\delta^{2[N/2]+2})$.

In the case (H2), by Lemma 5.3 we see that the approximate velocity potential $\widetilde{\Phi}^{\mathrm{app}}$ defined by (6.8) satisfies

$$\begin{cases} \Delta\widetilde{\Phi}^{\mathrm{app}} + \delta^{-2}\partial_z^2\widetilde{\Phi}^{\mathrm{app}} = R & \text{in} \quad \Omega, \\ \widetilde{\Phi}^{\mathrm{app}} = \phi & \text{on} \quad \Gamma, \\ \delta^{-2}\partial_z\widetilde{\Phi}^{\mathrm{app}} - \nabla b \cdot \nabla\widetilde{\Phi}^{\mathrm{app}} = r_B & \text{on} \quad \Sigma, \end{cases} \tag{6.18}$$

where

$$R(x,z) = \sum_{j=0}^{2N+2} (z+1-b(x))^j r_j(x) \tag{6.19}$$

and

$$r_j(x) = \begin{cases} (j+2)(j+1)\{\gamma_{j+2,2N+1}H^{2N+1-j}(\Delta\widetilde{\phi}_{2N+1}^\delta - (2N+2)Q(b)\widetilde{\phi}_{2N+2}^\delta) \\ \qquad\qquad + \gamma_{j+2,2N+2}H^{2N+2-j}\Delta\widetilde{\phi}_{2N+2}^\delta\} & \text{for} \quad j = 0,1,\dots,2N, \\ \Delta\widetilde{\phi}_{2N+1}^\delta - (2N+2)Q(b)\widetilde{\phi}_{2N+2}^\delta & \text{for} \quad j = 2N+1, \\ \Delta\widetilde{\phi}_{2N+2}^\delta & \text{for} \quad j = 2N+2, \end{cases}$$

$$r_B(x) = \gamma_{1,2N+1}H^{2N+2}(\Delta\widetilde{\phi}_{2N+1}^\delta - (2N+2)Q(b)\widetilde{\phi}_{2N+2}^\delta) + \gamma_{1,2N+2}H^{2N+3}\Delta\widetilde{\phi}_{2N+2}^\delta. \tag{6.20}$$

Concerning the remainder term R and r_B, we have the following lemma.

Lemma 6.9 *Choose $p_i = i$ $(i = 0,1,\dots,N)$ and suppose that (η,ϕ) and b satisfy (6.6). For any $j = 0,1,\dots,N+1$, if an integer k satisfies $|k| \vee |k+2| \le m$ and $|k+2j+1| \le m-1$, then we have*

$$\|(r_0, r_1, \dots, r_{2N+2})\|_k + \|r_B\|_k \le C\delta^{2j},$$

where $C = C(M, c_0, m, j, k, N)$ is a positive constant independent of $\delta \in (0,1]$.

Proof. In view of $\|Q(b)\psi\|_k \leq C(\|\nabla b\|_{W^{|k|\vee|k+1|,\infty}})\|\psi\|_k$, we see that $\|(r_0, r_1, \dots, r_{2N+2})\|_k + \|r_B\|_k \lesssim \|(\widetilde{\phi}^\delta_{2N+1}, \widetilde{\phi}^\delta_{2N+2})\|_{k+2}$ if $|k| \vee |k+1| \leq m$. By Lemma 6.6 with k replaced by $k+2$, we have $\|(\widetilde{\phi}^\delta_{2N+1}, \widetilde{\phi}^\delta_{2N+2})\|_{k+2} \lesssim \delta^{2j}$ if $|k+2| \vee |k+2j+1| + 1 \leq m$ for $j = 0, 1, \dots, N+1$. Combining these estimates we obtain the desired one. □

Remark 6.10 Lemma 6.9 implies that the remainder term R and r_B are of order $O(\delta^{2N+2})$ if m is sufficiently large, so that the approximate velocity potential $\widetilde{\Phi}^{\mathrm{app}}$ satisfies the continuity equation (6.1) and the boundary condition (6.3) on the bottom with an error of order $O(\delta^{2N+2})$. We also note that $4[N/2] + 2 \leq 2N + 2$.

7 Consistency of the Isobe–Kakinuma model II

We proceed to show that η and $\widetilde{\Phi}^{\mathrm{app}}$ satisfy approximately the second condition in (6.2), that is, the kinematic boundary condition on the water surface. Since the solution $(\eta, \boldsymbol{\phi}^\delta)$ to the Isobe–Kakinuma model satisfies

$$\partial_t \eta - \sum_{j=0}^{N} L_{0j}\phi_j^\delta = 0, \tag{7.1}$$

we need to compare $(\delta^{-2}\partial_z \widetilde{\Phi}^{\mathrm{app}} - \nabla\eta\cdot\nabla\widetilde{\Phi}^{\mathrm{app}})|_{z=\eta}$ with $\sum_{j=0}^{N} L_{0j}\phi_j^\delta$. To estimate the difference between them in Sobolev spaces, we will utilize fully the duality $(H^k)^* = H^{-k}$, so that we will evaluate the quantity

$$I = ((\delta^{-2}\partial_z \widetilde{\Phi}^{\mathrm{app}} - \nabla\eta\cdot\nabla\widetilde{\Phi}^{\mathrm{app}})|_{z=\eta} - \sum_{j=0}^{N} L_{0j}\phi_j^\delta, \psi)_{L^2} \tag{7.2}$$

for arbitrarily fixed ψ. Regarding ψ as a function on the water surface, we extend it into the water region by

$$\Psi(x,z) = \sum_{j=0}^{2N+2} (z + 1 - b(x))^{p_j}\psi_j^\delta(x), \tag{7.3}$$

where $\boldsymbol{\psi}^\delta = (\psi_0^\delta, \dots, \psi_{2N+2}^\delta)^{\mathrm{T}}$ is defined by

$$\mathscr{L}_0^{(2N+2)}\widetilde{\boldsymbol{\psi}}^\delta = \psi, \qquad \mathscr{L}_i^{(2N+2)}\widetilde{\boldsymbol{\psi}}^\delta = 0 \quad \text{for} \quad i = 1, \dots, 2N+2. \tag{7.4}$$

This construction of Ψ from ψ is the same as that of $\widetilde{\Phi}^{\mathrm{app}}$ from ϕ. See (6.7)–(6.8). By Green's formula, we have

$$\begin{aligned}
&((\delta^{-2}\partial_z \widetilde{\Phi}^{\mathrm{app}} - \nabla\eta\cdot\nabla\widetilde{\Phi}^{\mathrm{app}})|_{z=\eta}, \psi)_{L^2} - ((\delta^{-2}\partial_z \widetilde{\Phi}^{\mathrm{app}} - \nabla\eta\cdot\nabla\widetilde{\Phi}^{\mathrm{app}})|_{z=-1+b}, \psi_0^\delta)_{L^2} \\
&= \int_\Omega \{\nabla\cdot(\Psi\nabla\widetilde{\Phi}^{\mathrm{app}}) - \delta^{-2}\partial_z(\Psi\partial_z\widetilde{\Phi}^{\mathrm{app}})\}\,\mathrm{d}X \\
&= \int_\Omega (\Delta\widetilde{\Phi}^{\mathrm{app}} - \delta^{-2}\partial_z^2\widetilde{\Phi}^{\mathrm{app}})\Psi\,\mathrm{d}X + \int_\Omega \{\nabla\widetilde{\Phi}^{\mathrm{app}}\cdot\nabla\Psi + \delta^{-2}(\partial_z\widetilde{\Phi}^{\mathrm{app}})(\partial_z\Psi)\}\,\mathrm{d}X.
\end{aligned}$$

Since $\widetilde{\Phi}^{\mathrm{app}}$ satisfies (6.16) in the case (H1) and (6.18) in the case (H2), we obtain

$$\begin{aligned}
&((\delta^{-2}\partial_z \widetilde{\Phi}^{\mathrm{app}} - \nabla\eta\cdot\nabla\widetilde{\Phi}^{\mathrm{app}})|_{z=\eta}, \psi)_{L^2} \\
&= (r_B, \psi_0^\delta)_{L^2} + \int_\Omega R\Psi\,\mathrm{d}X + \int_\Omega \{\nabla\widetilde{\Phi}^{\mathrm{app}}\cdot\nabla\Psi + \delta^{-2}(\partial_z\widetilde{\Phi}^{\mathrm{app}})(\partial_z\Psi)\}\,\mathrm{d}X \\
&=: I_1 + I_2 + I_3,
\end{aligned} \tag{7.5}$$

where $r_B = 0$ in the case (H1). In view of (7.3) and the definition of R, that is, (6.17) in the case (H1) and (6.19) in the case (H2), we have

$$I_2 = \sum_{i,j=0}^{2N+2} \frac{1}{p_i + p_j + 1} (H^{p_i+p_j+1} r_i, \psi_j^\delta)_{L^2}. \tag{7.6}$$

In view of (6.8) and (7.3), by direct calculation we see that

$$I_3 = \sum_{i,j=0}^{2N+2} (L_{ij} \widetilde{\phi}_j^\delta, \psi_i^\delta)_{L^2}.$$

Here, we remind that $\widetilde{\boldsymbol{\phi}}^\delta$ and $\boldsymbol{\psi}^\delta$ were defined by (6.7) and (7.4), respectively, so that we have

$$\sum_{j=0}^{2N+2} H^{p_j} \widetilde{\phi}_j^\delta = \phi, \qquad \sum_{j=0}^{2N+2} L_{ij} \widetilde{\phi}_j^\delta = \sum_{j=0}^{2N+2} H^{p_i} L_{0j} \widetilde{\phi}_j^\delta \quad \text{for} \quad 1 \le i \le 2N+2,$$

and similar relations hold for $\boldsymbol{\psi}^\delta$. Moreover, $\boldsymbol{\phi}^\delta$ satisfies (6.5), so that we have also

$$\sum_{j=0}^{N} H^{p_j} \phi_j^\delta = \phi, \qquad \sum_{j=0}^{N} L_{ij} \phi_j^\delta = \sum_{j=0}^{N} H^{p_i} L_{0j} \phi_j^\delta \quad \text{for} \quad 1 \le i \le N.$$

Using these relations and $L_{ij}^* = L_{ji}$, we can rewrite I_3 as

$$\begin{aligned} I_3 &= \sum_{i,j=0}^{2N+2} (\widetilde{\phi}_j^\delta, L_{ji} \psi_i^\delta)_{L^2} = \sum_{i,j=0}^{2N+2} (\widetilde{\phi}_j^\delta, H^{p_j} L_{0i} \psi_i^\delta)_{L^2} = \sum_{i,j=0}^{2N+2} (H^{p_j} \widetilde{\phi}_j^\delta, L_{0i} \psi_i^\delta)_{L^2} \\ &= \sum_{i=0}^{2N+2} (\phi, L_{0i} \psi_i^\delta)_{L^2} = \sum_{j=0}^{N} \sum_{i=0}^{2N+2} (H^{p_j} \phi_j^\delta, L_{0i} \psi_i^\delta)_{L^2} = \sum_{j=0}^{N} \sum_{i=0}^{2N+2} (\phi_j^\delta, L_{ji} \psi_i^\delta)_{L^2} \\ &= \sum_{i,j=0}^{N} (L_{ij} \phi_j^\delta, \psi_i^\delta)_{L^2} + \sum_{j=0}^{N} \sum_{i=N+1}^{2N+2} (L_{ij} \phi_j^\delta, \psi_i^\delta)_{L^2}. \end{aligned}$$

Here, for the first term of the right-hand side we see that

$$\begin{aligned} \sum_{i,j=0}^{N} (L_{ij} \phi_j^\delta, \psi_i^\delta)_{L^2} &= \sum_{i,j=0}^{N} (H^{p_i} L_{0j} \phi_j^\delta, \psi_i^\delta)_{L^2} = \sum_{i,j=0}^{N} (L_{0j} \phi_j^\delta, H^{p_i} \psi_i^\delta)_{L^2} \\ &= \sum_{j=0}^{N} (L_{0j} \phi_j^\delta, \psi)_{L^2} - \sum_{j=0}^{N} \sum_{i=N+1}^{2N+2} (L_{0j} \phi_j^\delta, H^{p_i} \psi_i^\delta)_{L^2}, \end{aligned}$$

so that

$$\begin{aligned} I_3 - \sum_{j=0}^{N} (L_{0j} \phi_j^\delta, \psi)_{L^2} &= \sum_{j=0}^{N} \sum_{i=N+1}^{2N+2} ((L_{ij} - H^{p_i} L_{0j}) \phi_j^\delta, \psi_i^\delta)_{L^2} \\ &= \sum_{j=0}^{N} \sum_{i=N+1}^{2N+2} ((L_{ij} - H^{p_i} L_{0j})(\phi_j^\delta - \widetilde{\phi}_j^\delta), \psi_i^\delta)_{L^2} + \sum_{j=0}^{N} \sum_{i=N+1}^{2N+2} ((L_{ij} - H^{p_i} L_{0j}) \widetilde{\phi}_j^\delta, \psi_i^\delta)_{L^2} \\ &= \sum_{j=0}^{N} \sum_{i=N+1}^{2N+2} ((L_{ij} - H^{p_i} L_{0j}) \varphi_j^\delta, \psi_i^\delta)_{L^2} - \sum_{j=N+1}^{2N+2} \sum_{i=N+1}^{2N+2} ((L_{ij} - H^{p_i} L_{0j}) \widetilde{\phi}_j^\delta, \psi_i^\delta)_{L^2} \\ &=: I_{3,1} + I_{3,2}, \end{aligned} \tag{7.7}$$

where $\boldsymbol{\varphi}^\delta = (\varphi_0^\delta, \dots, \varphi_N^\delta)^{\rm T}$ was defined by (6.9). Summarizing the above calculations, the quantity I defined by (7.2) is decomposed as

$$I = I_1 + I_2 + I_{3,1} + I_{3,2}.$$

By using this expression, we will evaluate the quantity I in the following.

7.1 The case $p_i = 2i$ with the flat bottom

Lemma 7.1 *Choose $p_i = 2i$ $(i = 0, 1, \dots, N)$ and suppose that $b = 0$ and that (η, ϕ) satisfy* (6.6). *For any $j = 0, 1, \dots, 2N+2$, if an integer k satisfies $|k+2j| \leq m$ and $|k+1| \leq m-1$, then we have*

$$\|(\psi_j^\delta, \psi_{j+1}^\delta, \dots, \psi_{2N+2}^\delta)\|_{-(k+2j)} \leq C\delta^{2j}\|\psi\|_{-k},$$

where $C = C(M, c_0, m, j, k, N)$ is a positive constant independent of $\delta \in (0, 1]$.

Proof. By Lemma 3.4, particularly, the second estimate in (3.20) with k replaced by $k-1$, we have $\|\boldsymbol{\psi}^\delta\|_k \lesssim \|\psi\|_k$ if $|k-1| \leq m-1$. On the other hand, it follows from Lemma 5.2 that $\|(\psi_j^\delta, \psi_{j+1}^\delta, \dots, \psi_N^\delta)\|_k \lesssim \delta^{2j}\|\boldsymbol{\psi}^\delta\|_{k+2j}$ if $|k| \vee |k+2(j-1)| \leq m$. These two estimates give $\|(\psi_j^\delta, \psi_{j+1}^\delta, \dots, \psi_N^\delta)\|_k \lesssim \delta^{2j}\|\psi\|_{k+2j}$ if $|k| \vee |k+2j-1| + 1 \leq m$. Replacing k with $-(k+2j)$, we obtain the desired result. $\square$

Lemma 7.2 *Choose $p_i = 2i$ $(i = 0, 1, \dots, N)$ and suppose that $b = 0$ and that (η, ϕ) and $\boldsymbol{\phi}^\delta$ satisfy* (6.5)–(6.6). *For any $l = 0, 1, \dots, 2N+1$, if $m \geq l + 1 + \delta_{l1}$, then we have*

$$\|(\delta^{-2}\partial_z\widetilde{\Phi}^{\rm app} - \nabla\eta\cdot\nabla\widetilde{\Phi}^{\rm app})|_{z=\eta} - \sum_{j=0}^N L_{0j}\phi_j^\delta\|_{m-2(l+1)} \leq C\delta^{2l},$$

where $C = C(M, c_0, m, l, N)$ is a positive constant independent of $\delta \in (0, 1]$ and δ_{l1} is the Kronecker delta.

Proof. In the case $l = 0$, we do not need to use the duality argument, and by direct evaluation and Lemma 6.1 we obtain the estimate of the lemma. Therefore, we will consider the case $1 \leq l \leq 2N+1$. By assumption we have $I_1 = 0$, so that it is sufficient to evaluate I_2, $I_{3,1}$, and $I_{3,2}$. It follows from (7.6) that

$$|I_2| \lesssim \|(r_0, r_1, \dots, r_{2N+2})\|_k\|\boldsymbol{\psi}^\delta\|_{-k} \quad \text{if} \quad |k| \leq m.$$

Here, by Lemmas 6.4 and 7.1 we have

$$\begin{cases} \|(r_0, r_1, \dots, r_{2N+2})\|_k \lesssim \delta^{2l} & \text{if} \quad |k| \vee |k+2| \leq m, \ |k+2l+1| \leq m-1, \\ \|\boldsymbol{\psi}^\delta\|_{-k} \lesssim \|\psi\|_{-k} & \text{if} \quad |k| \leq m, \ |k+1| \leq m-1, \end{cases}$$

so that $|I_2| \lesssim \delta^{2l}\|\psi\|_{-k}$ if $|k| \vee |k+2| \leq m$ and $|k+1| \vee |k+2l+1| \leq m-1$. Since these conditions on k are equivalent to $-m \leq k \leq m - 2(l+1)$, if $m \geq l+1$, then we can take $k = m - 2(l+1)$ and obtain

$$|I_2| \lesssim \delta^{2l}\|\psi\|_{-(m-2(l+1))}. \tag{7.8}$$

We proceed to evaluate $I_{3,1}$ and $I_{3,2}$. To this end, we note that in the case of the flat bottom we have

$$\|(L_{ij} - H^{p_i}L_{0j})\varphi\|_k \leq C(\|\eta\|_{|k| \vee t_0})(\|\varphi\|_{k+2} + \delta^{-2}\|\varphi\|_k).$$

We decompose l into a sum of two integers l_1 and l_2 satisfying $0 \leq l_1 \leq N+1$ and $1 \leq l_2 \leq N$. It follows from (7.7) that

$$|I_{3,1}| \lesssim (\|\boldsymbol{\varphi}^\delta\|_{k+2l_1+2} + \delta^{-2}\|\boldsymbol{\varphi}^\delta\|_{k+2l_1})\|(\psi_{N+1}^\delta, \dots, \psi_{2N+2}^\delta)\|_{-(k+2l_1)} \quad \text{if} \quad |k+2l_1| \leq m.$$

Here, by Lemma 6.2 with (k,j) replaced by $(k+2l_1+2, l_2)$ and $(k+2l_2, l_2+1)$

$$\begin{cases} \|\boldsymbol{\varphi}^\delta\|_{k+2l_1+2} \lesssim \delta^{2l_2} & \text{if} \quad |k+2l_1+1| \vee |k+2l_1+2| \vee |k+2(l_1+l_2)+1| \leq m-1, \\ \|\boldsymbol{\varphi}^\delta\|_{k+2l_1} \lesssim \delta^{2l_2+2} & \text{if} \quad |k+2l_1-1| \vee |k+2l_1| \vee |k+2(l_1+l_2)+1| \leq m-1, \end{cases}$$

and by Lemma 7.1 with l replaced by l_1

$$\|(\psi_{N+1}^\delta, \dots, \psi_{2N+2}^\delta)\|_{-(k+2l_1)} \lesssim \delta^{2l_1}\|\psi\|_{-k} \quad \text{if} \quad |k+2l_1| \leq m, \ |k+1| \leq m-1,$$

so that $|I_{3,1}| \lesssim \delta^{2l}\|\psi\|_{-k}$ if $|k+1| \vee |k+2l_1-1| \vee |k+2(l_1+l_2)+1| \leq m-1$. In the case $l \geq 2$, we can take $l_1 \geq 1$ so that these conditions on k is equivalent to $-m \leq k \leq m-2l-2$. Therefore, if $m \geq l+1$, then we can take $k = m-2(l+1)$. In the case $l=1$, we have $l_1 = 0$ and $l_2 = 1$ so that these conditions on k is equivalent to $-m+2 \leq k \leq m-4$. Therefore, if $m \geq 3$, then we can take $k = m-4$. In any case, if $m \geq l+1+\delta_{l1}$, then we obtain

$$|I_{3,1}| \lesssim \delta^{2l}\|\psi\|_{-(m-2(l+1))}. \tag{7.9}$$

Similarly, it follows from (7.7) that

$$\begin{aligned} |I_{3,2}| \lesssim (&\|(\widetilde{\phi}_{N+1}^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_{k+2l_1+2} + \delta^{-2}\|(\widetilde{\phi}_{N+1}^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_{k+2l_1}) \\ &\times \|(\psi_{N+1}^\delta, \dots, \psi_{2N+2}^\delta)\|_{-(k+2l_1)} \qquad \text{if} \quad |k+2l_1| \leq m. \end{aligned}$$

Here, by Lemma 6.1 with (k,j) replaced by $(k+2l_1+2, l_2)$ and $(k+2l_2, l_2+1)$

$$\begin{cases} \|(\widetilde{\phi}_{N+1}^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_{k+2l_1+2} \lesssim \delta^{2l_2} & \text{if} \quad |k+2l_1+2| \leq m, \ |k+2(l_1+l_2)+1| \leq m-1, \\ \|(\widetilde{\phi}_{N+1}^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_{k+2l_1} \lesssim \delta^{2l_2+2} & \text{if} \quad |k+2l_1| \leq m, \ |k+2(l_1+l_2)+1| \leq m-1, \end{cases}$$

so that $|I_{3,2}| \lesssim \delta^{2l}\|\psi\|_{-k}$ if $|k+2l_1| \vee |k+2l_1+2| \leq m$ and $|k+1| \vee |k+2(l_1+l_2)+1| \leq m-1$. Since these conditions on k are equivalent to $-m \leq k \leq m-2(l+1)$, if $m \geq l+1$, then we can take $k = m-2(l+1)$ and obtain $|I_{3,2}| \lesssim \delta^{2l}\|\psi\|_{-(m-2(l+1))}$. This together with (7.8) and (7.9) yields $|I| \lesssim \delta^{2l}\|\psi\|_{-(m-2(l+1))}$, that is,

$$|((\delta^{-2}\partial_z\widetilde{\Phi}^{\mathrm{app}} - \nabla\eta \cdot \nabla\widetilde{\Phi}^{\mathrm{app}})|_{z=\eta} - \sum_{j=0}^{N} L_{0j}\phi_j^\delta, \psi)_{L^2}| \lesssim \delta^{2l}\|\psi\|_{-(m-2(l+1))}$$

for any ψ. Therefore, by the duality $(H^{m-2(l+1)})^* = H^{-(m-2(l+1))}$ we obtain the desired estimate. □

Remark 7.3 Lemma 7.2 implies that for the solution $(\eta, \boldsymbol{\phi}^\delta)$ of the Isobe–Kakinuma model, $(\eta, \widetilde{\Phi}^{\mathrm{app}})$ satisfies the second condition in (6.2) with an error of order $O(\delta^{4N+2})$ if m is sufficiently large.

7.2 The case $p_i = i$ with general bottom topographies

Lemma 7.4 *Choose $p_i = i$ $(i = 0, 1, \dots, N)$ and suppose that (η, ϕ) and b satisfy (6.6). For any $j = 0, 1, \dots, N+1$, if an integer k satisfies $|k+2j| \le m$ and $|k+1| \le m-1$, then we have*

$$\|(\psi^\delta_{2j-1}, \psi^\delta_{2j}, \dots, \psi^\delta_{2N+1}, \psi^\delta_{2N+2})\|_{-(k+2j)} \le C\delta^{2j}\|\psi\|_{-k},$$

where $C = C(M, c_0, m, j, k, N)$ is a positive constant independent of $\delta \in (0, 1]$ and we used a notational convention $\psi^\delta_{-1} = 0$. Particularly, for any $j = 0, 1, \dots, [N/2]+1$, if an integer k satisfies $|k+2j| \le m$ and $|k+1| \le m-1$, then we have

$$\|(\psi^\delta_{N+1}, \psi^\delta_{N+2}, \dots, \psi^\delta_{2N+2})\|_{-(k+2j)} \le C\delta^{2j}\|\psi\|_{-k}.$$

Proof. As in the proof of Lemma 7.1, we have $\|\boldsymbol{\psi}^\delta\|_k \lesssim \|\psi\|_k$ if $|k-1| \le m-1$. It follows from Lemma 5.4 that $\|(\psi^\delta_{2j-1}, \psi^\delta_{2j}, \dots, \psi^\delta_{2N+2})\|_k \lesssim \delta^{2j}\|\boldsymbol{\psi}^\delta\|_{k+2j}$ if $|k| \vee |k+2(j-1)| \vee |k+2j-1| \le m$. These two estimates give $\|(\psi^\delta_{2j-1}, \psi^\delta_{2j}, \dots, \psi^\delta_{2N+2})\|_k \lesssim \delta^{2j}\|\psi\|_{k+2j}$ if $|k| \le m$ and $|k+2j-1| \le m-1$. Replacing k with $-(k+2j)$ we obtain the desired result. The later part of the lemma comes from the former one as in the proof of Lemma 6.7. □

Lemma 7.5 *Choose $p_i = i$ $(i = 0, 1, \dots, N)$ and suppose that (η, ϕ) and b satisfy (6.5)–(6.6). For any $l = 0, 1, \dots, 2[N/2]+1$, if $m \ge l+1+\delta_{l1}$, then we have*

$$\|(\delta^{-2}\partial_z\widetilde{\Phi}^{\rm app} - \nabla\eta\cdot\nabla\widetilde{\Phi}^{\rm app})|_{z=\eta} - \sum_{j=0}^{N} L_{0j}\phi^\delta_j\|_{m-2(l+1)-\delta_{l1}} \le C\delta^{2l},$$

where $C = C(M, c_0, m, l, N)$ is a positive constant independent of $\delta \in (0, 1]$ and δ_{l1} is the Kronecker delta.

Proof. In the case $l = 0$, we do not need to use the duality argument, and by direct evaluation and Lemma 6.6 we obtain the estimate of the lemma. Therefore, we will consider the case $1 \le l \le 2[N/2]+1$. It follows from (7.5)–(7.6) that

$$|I_1| + |I_2| \lesssim (\|r_B\|_k + \|(r_0, r_1, \dots, r_{2N+2})\|_k)\|\boldsymbol{\psi}^\delta\|_{-k} \quad \text{if} \quad |k| \le m.$$

Here, in view of $2[N/2]+1 \le N+1$ by Lemmas 6.9 and 7.4 we have

$$\begin{cases} \|r_B\|_k + \|(r_0, r_1, \dots, r_{2N+2})\|_k \lesssim \delta^{2l} & \text{if} \quad |k| \vee |k+2| \le m, \ |k+2l+1| \le m-1, \\ \|\boldsymbol{\psi}^\delta\|_{-k} \lesssim \|\psi\|_{-k} & \text{if} \quad |k| \le m, \ |k+1| \le m-1. \end{cases}$$

Therefore, as in the proof of Lemma 7.2, we have $|I_1| + |I_2| \lesssim \delta^{2l}\|\psi\|_{-(m-2(l+1))}$.

We proceed to evaluate $I_{3,1}$ and $I_{3,2}$. To this end, we note that

$$\|(L_{ij} - H^{p_i}L_{0j})\varphi\|_k \le C(\|\eta\|_{|k|\vee t_0}, \|b\|_{W^{|k|+1\vee|k+1|+1,\infty}})(\|\varphi\|_{k+2} + \delta^{-2}\|\varphi\|_k).$$

As before, we decompose l into a sum of two integers l_1 and l_2 satisfying $1 \le l_1 \le [N/2]+1$ and $0 \le l_2 \le [N/2]$. It follows from (7.7) that

$$\begin{aligned} |I_3| \lesssim \{ &\|(\widetilde{\phi}^\delta_{N+1}, \dots, \widetilde{\phi}^\delta_{2N+2})\|_{k+2l_1+2} + \delta^{-2}\|(\widetilde{\phi}^\delta_{N+1}, \dots, \widetilde{\phi}^\delta_{2N+2})\|_{k+2l_1} \\ &+ \|\boldsymbol{\varphi}^\delta\|_{k+2l_1+2} + \delta^{-2}\|\boldsymbol{\varphi}^\delta\|_{k+2l_1}\}\|(\psi^\delta_{N+1}, \dots, \psi^\delta_{2N+2})\|_{-(k+2l_1)} \end{aligned}$$

if $|k+2l_1| \vee |k+2l_1+1| \le m$. Here, by Lemma 6.7 with (k,j) replaced by $(k+2l_1+2, l_2)$ and $(k+2l_2, l_2+1)$ we have

$$\begin{cases} \|(\widetilde{\phi}_{N+1}^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_{k+2l_1+2} + \|\boldsymbol{\varphi}^\delta\|_{k+2l_1+2} \lesssim \delta^{2l_2}, \\ \|(\widetilde{\phi}_{N+1}^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_{k+2l_1} + \|\boldsymbol{\varphi}^\delta\|_{k+2l_1} \lesssim \delta^{2l_2+2} \end{cases}$$

if $|k+2l_1-1| \vee |k+2l_1+2| \vee |k+2(l_1+l_2)+1| \le m-1$. By Lemma 7.4 we have also

$$\|(\psi_{N+1}^\delta, \dots, \psi_{2N+2}^\delta)\|_{-(k+2l_1)} \lesssim \delta^{2l_1}\|\psi\|_{-k} \quad \text{if} \quad |k+2l_1| \le m, \ |k+1| \le m-1.$$

Therefore, we obtain $|I_3| \lesssim \delta^{2l}\|\psi\|_{-k}$ if $|k+1| \vee |k+2l_1-1| \vee |k+2l_1+2| \vee |k+2(l_1+l_2)+1| \le m-1$. In the case $l \ge 2$, we can take $l_2 \ge 1$ so that these conditions on k is equivalent to $-m \le k \le m-2l-2$. Therefore, if $m \ge l+1$, then we can take $k = m-2(l+1)$. In the case $l=1$, we have $l_1 = 1$ and $l_2 = 0$ so that these conditions on k is equivalent to $-m \le k \le m-5$. Therefore, if $m \ge 3$, then we can take $k = m-5$. In any case, we obtain $|I_3| \lesssim \delta^{2l}\|\psi\|_{-(m-2(l+1)-\delta_{l1})}$ if $m \ge l+1+\delta_{l1}$.

Summarizing the above estimate, we obtain $|I| \le \delta^{2l}\|\psi\|_{-(m-2(l+1)-\delta_{l1})}$ if $m \ge l+1+\delta_{l1}$. This implies the desired estimate. □

Remark 7.6 Lemma 7.5 implies that for the solution $(\eta, \boldsymbol{\phi}^\delta)$ of the Isobe–Kakinuma model, $(\eta, \widetilde{\Phi}^{\rm app})$ satisfies the second condition in (6.2) with an error of order $O(\delta^{4[N/2]+2})$ if m is sufficiently large.

8 Consistency of the Isobe–Kakinuma model III

In this section we will finish to prove Theorem 2.2, that is, a consistency of the Isobe–Kakinuma model (1.4) with the water wave equations (1.6) in Zakharov–Craig–Sulem formulation. To this end, in view of (7.1) we need to correlate $\sum_{j=0}^N L_{0j}\phi_j^\delta$ with $\Lambda(\eta,b,\delta)\phi$, where $\Lambda(\eta,b,\delta)$ is the Dirichlet-to-Neumann map for Laplace's equation defined by (1.8)–(1.9). We remind that the modified approximate velocity potential $\widetilde{\Phi}^{\rm app}$ satisfies the boundary value problem (6.16) or (6.18) and that ϕ was defined by (2.8) from the solution $(\eta, \boldsymbol{\phi}^\delta)$ to the Isobe–Kakinuma model.

Let Φ be the unique solution to the boundary value problem (1.9) and put

$$\Phi^{\rm res} = \Phi - \widetilde{\Phi}^{\rm app}. \tag{8.1}$$

Then, $\Phi^{\rm res}$ satisfies the boundary value problem

$$\begin{cases} \Delta\Phi^{\rm res} + \delta^{-2}\partial_z^2\Phi^{\rm res} = -R & \text{in} \quad \Omega, \\ \Phi^{\rm res} = 0 & \text{on} \quad \Gamma, \\ \delta^{-2}\partial_z\Phi^{\rm res} - \nabla b\cdot\nabla\Phi^{\rm res} = -r_B & \text{on} \quad \Sigma, \end{cases} \tag{8.2}$$

where R and r_B were defined by (6.17) and $r_B = 0$ in the case (H1) and by (6.19)–(6.20) in the case (H2). Applying the identity

$$\begin{aligned} \nabla\cdot\int_{-1+b}^{\eta}\nabla\Psi\,{\rm d}z &= \int_{-1+b}^{\eta}(\Delta\Psi + \delta^{-2}\partial_z^2\Psi)\,{\rm d}z \\ &\quad - (\delta^{-2}\partial_z\Psi - \nabla\eta\cdot\nabla\Psi)|_{z=\eta} + (\delta^{-2}\partial_z\Psi - \nabla\eta\cdot\nabla\Psi)|_{z=-1+b} \end{aligned}$$

to $\Psi = \Phi^{\rm res}$ and noting (1.8), we obtain

$$(8.3)\qquad (\delta^{-2}\partial_z\widetilde{\Phi}^{\rm app} - \nabla\eta\cdot\nabla\widetilde{\Phi}^{\rm app})|_{z=\eta} - \Lambda(\eta,b,\delta)\phi = \nabla\cdot\int_{-1+b}^{\eta}\nabla\Phi^{\rm res}\,{\rm d}z + \int_{-1+b}^{\eta}R\,{\rm d}z + r_B =: I_1 + I_2 + I_3.$$

In view of (6.17) and (6.19), we have

$$(8.4)\qquad I_2 = \sum_{j=0}^{2N+2}\frac{1}{p_j+1}H^{p_j+1}r_j.$$

Therefore, we can evaluate I_2 and I_3 directly by using the estimates obtained in Section 6. To evaluate I_1 we will use an estimate for the boundary value problem (8.2) of elliptic type. To this end, it is convenient to transform the problem (8.2) in the water region Ω into a problem in a simple domain $\Omega_0 = \mathbf{R}^n\times(0,1)$ by using a diffeomorphism $\Theta(x,z) = (x,\theta(x,z)) : \Omega_0 \to \Omega$, which simply stretches the vertical direction, where $\theta(x,z) = \eta(x)(z+1) + (1-b(x))z$. We put $\widetilde{\Phi}^{\rm res} = \Phi^{\rm res}\circ\Theta$. Then, we have

$$(8.5)\qquad I_1 = \nabla\cdot\int_{-1}^{0}(H\nabla\widetilde{\Phi}^{\rm res} - (\partial_z\widetilde{\Phi}^{\rm res})\nabla\theta)\,{\rm d}z,$$

and the boundary value problem (8.2) is transformed into

$$(8.6)\qquad \begin{cases} \nabla_X\cdot\mathcal{P}\nabla_X\widetilde{\Phi}^{\rm res} = -\widetilde{R} & \text{in}\quad -1<z<0, \\ \widetilde{\Phi}^{\rm res} = 0 & \text{on}\quad z=0, \\ \boldsymbol{e}_z\cdot\mathcal{P}\nabla_X\widetilde{\Phi}^{\rm res} = -r_B & \text{on}\quad z=-1, \end{cases}$$

where the coefficient matrix $\mathcal{P}$ is defined by

$$\mathcal{P} = \det\Bigl(\frac{\partial\Theta}{\partial X}\Bigr)\Bigl(\frac{\partial\Theta}{\partial X}\Bigr)^{-1}\begin{pmatrix} E_n & \mathbf{0} \\ \mathbf{0}^{\rm T} & \delta^{-2}\end{pmatrix}\Bigl(\Bigl(\frac{\partial\Theta}{\partial X}\Bigr)^{-1}\Bigr)^{\rm T},$$

$\boldsymbol{e}_z = (0,\ldots,0,1)^{\rm T}$, and

$$(8.7)\qquad \widetilde{R} = R\circ\Theta = \sum_{j=0}^{2N+2}(z+1)^{p_j}H^{p_j}r_j.$$

By applying the standard theory of elliptic partial differential equations to (8.6), we obtain the following lemma. For details, we refer to T. Iguchi [5, 6] and D. Lannes [14].

Lemma 8.1 *Let c_0, M be positive constant and m an integer such that $m > n/2$. There exists a positive constant $C = C(c_0, M, m)$ such that if η and b satisfy*

$$\begin{cases} \|\eta\|_m + \|b\|_{W^{m,\infty}} \le M, \\ c_0 < H(x) = 1 + \eta(x) - b(x) \quad \text{for} \quad x\in\mathbf{R}^n, \end{cases}$$

and $\widetilde{\Phi}^{\rm res}$ is a solution to (8.6), then for $k = 0,1,\ldots,m-1$ and $\delta\in(0,1]$ we have

$$\|J^k\nabla\widetilde{\Phi}^{\rm res}\|_{L^2(\Omega_0)} + \delta^{-1}\|J^k\partial_z\widetilde{\Phi}^{\rm res}\|_{L^2(\Omega_0)} \le C\delta(\|J^k\widetilde{R}\|_{L^2(\Omega_0)} + \|r_B\|_k).$$

We remind that we have assumed (6.6). Thanks of this lemma and (8.7), we see that

$$\|I_1\|_k \lesssim \|J^{k+2}\partial_z\widetilde{\Phi}^{\rm res}\|_{L^2(\Omega_0)} \lesssim \delta^2(\|(r_0,r_1,\dots,r_{2N+2})\|_{k+2} + \|r_B\|_{k+2}) \tag{8.8}$$

if $0 \le k+2 \le m-1$. In the above calculation, we used the Poincaré inequality.

Now, we suppose that $(\eta, \boldsymbol{\phi}^\delta)$ is a solution to the Isobe–Kakinuma model (2.2) obtained in Theorem 2.1 and define ϕ by (2.8). Then, we put

$$\begin{cases} \mathfrak{r}_1 = \partial_t\eta - \Lambda(\eta,b,\delta)\phi, \\ \mathfrak{r}_2 = \partial_t\phi + \eta + \dfrac{1}{2}|\nabla\phi|^2 - \delta^2\dfrac{(\Lambda(\eta,b,\delta)\phi + \nabla\eta\cdot\nabla\phi)^2}{2(1+\delta^2|\nabla\eta|^2)}. \end{cases} \tag{8.9}$$

We will evaluate these remainder terms $\mathfrak{r}_1$ and $\mathfrak{r}_2$ in the following. To this end, we put also

$$\begin{cases} \mathfrak{r}_3 = (\delta^{-2}\partial_z\widetilde{\Phi}^{\rm app} - \nabla\eta\cdot\nabla\widetilde{\Phi}^{\rm app})|_{z=\eta} - \Lambda(\eta,b,\delta)\phi, \\ \mathfrak{r}_4 = (\delta^{-2}\partial_z\widetilde{\Phi}^{\rm app} - \nabla\eta\cdot\nabla\widetilde{\Phi}^{\rm app})|_{z=\eta} - \displaystyle\sum_{j=0}^N L_{0j}\phi_j^\delta, \\ \mathfrak{r}_5 = ((\delta^{-2}\partial_z - \nabla\eta\cdot\nabla)(\Phi^{\rm app} - \widetilde{\Phi}^{\rm app}))|_{z=\eta}. \end{cases} \tag{8.10}$$

It follows from (8.3), (8.4), and (8.8) that

$$\|\mathfrak{r}_3\|_k \lesssim \delta^2(\|(r_0,r_1,\dots,r_{2N+2})\|_{k+2} + \|r_B\|_{k+2}) \quad \text{if} \quad 0\le k+2 \le m-1. \tag{8.11}$$

We have evaluated $\mathfrak{r}_4$ in Lemmas 7.2 and 7.5. By direct calculation, we see that

$$\begin{aligned} \mathfrak{r}_5 &= \sum_{j=0}^N \{p_jH^{p_j-1}(\delta^{-2} + \nabla\eta\cdot\nabla b)\varphi_j^\delta - H^{p_j}\nabla\eta\cdot\nabla\varphi_j^\delta\} \\ &\quad - \sum_{j=N+1}^{2N+2}\{p_jH^{p_j-1}(\delta^{-2} + \nabla\eta\cdot\nabla b)\widetilde{\phi}_j^\delta - H^{p_j}\nabla\eta\cdot\nabla\widetilde{\phi}_j^\delta\}, \end{aligned}$$

so that

$$\begin{aligned} \|\mathfrak{r}_5\|_k &\lesssim \delta^{-2}(\|\boldsymbol{\varphi}^\delta\|_k + \|(\widetilde{\phi}_{N+1}^\delta,\dots,\widetilde{\phi}_{2N+2}^\delta)\|_k) \\ &\quad + \|\boldsymbol{\varphi}^\delta\|_{k+1} + \|(\widetilde{\phi}_{N+1}^\delta,\dots,\widetilde{\phi}_{2N+2}^\delta)\|_{k+1} \quad \text{if} \quad |k|+1\le m. \end{aligned} \tag{8.12}$$

Therefore, we can evaluate $\mathfrak{r}_3$, $\mathfrak{r}_4$, and $\mathfrak{r}_5$ by the results obtained in Sections 6–7, so that it is sufficient to express $\mathfrak{r}_1$ and $\mathfrak{r}_2$ in terms of these quantities. It is clear that $\mathfrak{r}_1 = \mathfrak{r}_3 - \mathfrak{r}_4$. Differentiating the identity $\phi = \Phi^{\rm app}|_{z=\eta}$ with respect to t and x, we have

$$\begin{cases} \partial_t\phi = (\partial_t\Phi^{\rm app} + (\partial_z\Phi^{\rm app})\partial_t\eta)|_{z=\eta}, \\ \nabla\phi = (\nabla\Phi^{\rm app} + (\partial_z\Phi^{\rm app})\nabla\eta)|_{z=\eta}. \end{cases} \tag{8.13}$$

Plugging these into (6.4) to eliminate $(\partial_t\Phi^{\rm app})|_{z=\eta}$ and $(\nabla\Phi^{\rm app})|_{z=\eta}$ and using the first equation in (8.9) to eliminate $\partial_t\eta$, we obtain

$$\begin{aligned} &\partial_t\phi + \eta + \frac{1}{2}|\nabla\phi|^2 \\ &= (\partial_z\Phi^{\rm app})|_{z=\eta}(\Lambda\phi + \nabla\eta\cdot\nabla\phi + \mathfrak{r}_1) - \frac{1}{2}\delta^{-2}(1+\delta^2|\nabla\eta|^2)(\partial_z\Phi^{\rm app})^2|_{z=\eta} \\ &= (\partial_z\Phi^{\rm app})|_{z=\eta}\mathfrak{r}_1 + \frac{1}{2}(\partial_z\Phi^{\rm app})|_{z=\eta}\{2(\Lambda\phi + \nabla\eta\cdot\nabla\phi) - \delta^{-2}(1+\delta^2|\nabla\eta|^2)(\partial_z\Phi^{\rm app})|_{z=\eta}\}, \end{aligned}$$

where $\Lambda = \Lambda(\eta, b, \delta)$. On the other hand, it follows from the definition of $\mathfrak{r}_3$ and $\mathfrak{r}_5$ that

$$(\delta^{-2}\partial_z \Phi^{\mathrm{app}} - \nabla\eta \cdot \nabla\Phi^{\mathrm{app}})|_{z=\eta} = \Lambda\phi + \mathfrak{r}_3 + \mathfrak{r}_5,$$

which together with the second equation in (8.13) yields

$$(1+\delta^2|\nabla\eta|^2)(\partial_z \Phi^{\mathrm{app}})|_{z=\eta} = \delta^2(\Lambda\phi + \nabla\eta\cdot\nabla\phi + \mathfrak{r}_3 + \mathfrak{r}_5).$$

Therefore,

$$\begin{aligned}
&\partial_t\phi + \eta + \frac{1}{2}|\nabla\phi|^2 \\
&= (\partial_z \Phi^{\mathrm{app}})|_{z=\eta}\mathfrak{r}_1 + \delta^2 \frac{(\Lambda\phi + \nabla\eta\cdot\nabla\phi + (\mathfrak{r}_3+\mathfrak{r}_5))(\Lambda\phi + \nabla\eta\cdot\nabla\phi - (\mathfrak{r}_3+\mathfrak{r}_5))}{2(1+\delta^2|\nabla\eta|^2)} \\
&= (\partial_z \Phi^{\mathrm{app}})|_{z=\eta}\mathfrak{r}_1 + \delta^2 \frac{(\Lambda\phi + \nabla\eta\cdot\nabla\phi)^2 - (\mathfrak{r}_3+\mathfrak{r}_5)^2}{2(1+\delta^2|\nabla\eta|^2)},
\end{aligned}$$

so that

$$\begin{cases}
\mathfrak{r}_1 = \mathfrak{r}_3 - \mathfrak{r}_4, \\
\mathfrak{r}_2 = (\partial_z \Phi^{\mathrm{app}})|_{z=\eta}\mathfrak{r}_1 - \delta^2 \dfrac{(\mathfrak{r}_3+\mathfrak{r}_5)^2}{2(1+\delta^2|\nabla\eta|^2)}.
\end{cases} \tag{8.14}$$

Here, in view of $(\partial_z \Phi^{\mathrm{app}})|_{z=\eta} = \sum_{i=1}^N p_i H^{p_i-1}\phi_i^\delta = \delta^2 w$, we have $\|(\partial_z \Phi^{\mathrm{app}})|_{z=\eta}\|_m \lesssim \delta$.

8.1 The case $p_i = 2i$ with the flat bottom

It follows directly from Lemma 7.2 that $\|\mathfrak{r}_4\|_{m-4(N+1)} \lesssim \delta^{4N+2}$ if $m \geq 2(N+1)$. By (8.11) and Lemma 6.4 with (k,j) replaced by $(k+2, l-1)$, for $l = 1, 2, \ldots, 2N+3$ we have $\|\mathfrak{r}_3\|_k \lesssim \delta^{2l}$ if $0 \leq k+2 \leq m-1$, $|k+2| \vee |k+4| \leq m$, and $|k+2l+1| \leq m-1$. These conditions on k are equivalent to $-2 \leq k \leq m-2(l+1)$. Particularly, we have

$$\begin{cases}
\|\mathfrak{r}_3\|_{m-4(N+1)} \lesssim \delta^{4N+2} & \text{if} \quad m \geq 4N+2, \\
\|\mathfrak{r}_3\|_{m-2(N+1)} \lesssim \delta^{2N} & \text{if} \quad m \geq 2N,
\end{cases}$$

so that

$$\|\mathfrak{r}_1\|_{m-4(N+1)} \lesssim \delta^{4N+2} \quad \text{if} \quad m \geq 4N+2.$$

By (8.12) and Lemma 6.2 with (k,j) replaced by $(k, N+1)$ and $(k+1, N)$ we have $\|\mathfrak{r}_5\|_k \lesssim \delta^{2N}$ if $|k| \leq m-1$, $|k-1| \vee |k| \vee |k+2N+1| \leq m-1$, and $|k| \vee |k+1| \vee |k+2N| \leq m-1$. These conditions on k are equivalent to $-m+2 \leq k \leq m-2(N+1)$. Particularly, we have

$$\|\mathfrak{r}_5\|_{m-2(N+1)} \lesssim \delta^{2N} \quad \text{if} \quad m \geq N+2.$$

Therefore, if $m \geq 4N+2$ and $m - 2(N+1) > n/2$, then

$$\|\mathfrak{r}_2\|_{m-4(N+1)} \lesssim \|\mathfrak{r}_1\|_{m-4(N+1)} + \delta^2(\|\mathfrak{r}_3\|_{m-2(N+1)} + \|\mathfrak{r}_5\|_{m-2(N+1)})^2 \lesssim \delta^{4N+2},$$

so that we obtain the desired estimate in Theorem 2.2 in the case (H1).

8.2 The case $p_i = i$ with general bottom topographies

It follows directly from Lemma 7.5 that $\|\mathfrak{r}_4\|_{m-4([N/2]+1)} \lesssim \delta^{4[N/2]+2}$ if $m \geq 2([N/2]+1)+\delta_{N1}$. By (8.11) and Lemma 6.9 with (k,j) replaced by $(k+2, l-1)$, for $l = 1, 2, \dots, N+2$ we have $\|\mathfrak{r}_3\|_k \lesssim \delta^{2l}$ if $0 \leq k+2 \leq m-1$, $|k+2| \vee |k+4| \leq m$, and $|k+2l+1| \leq m-1$. These conditions on k are equivalent to $-2 \leq k \leq m-2(l+1)$. Particularly, we have

$$\begin{cases} \|\mathfrak{r}_3\|_{m-4([N/2]+1)} \lesssim \delta^{4[N/2]+2} & \text{if} \quad m \geq 4[N/2]+2, \\ \|\mathfrak{r}_3\|_{m-2([N/2]+1)} \lesssim \delta^{2[N/2]} & \text{if} \quad m \geq 2[N/2]. \end{cases}$$

Here, we note that the later estimate in the case $N = 1$ comes from a direct evaluation. Therefore,

$$\|\mathfrak{r}_1\|_{m-4([N/2]+1)} \lesssim \delta^{4[N/2]+2} \quad \text{if} \quad m \geq 4[N/2]+2+\delta_{N1}.$$

By (8.12) and Lemma 6.7 with (k,j) replaced by $(k, [N/2]+1)$ and $(k+1, [N/2])$ we have $\|\mathfrak{r}_5\|_k \lesssim \delta^{2[N/2]}$ if $|k| \leq m-1$, $|k-1| \vee |k| \vee |k+2[N/2]+1| \leq m-1$, and $|k| \vee |k+1| \vee |k+2[N/2]| \leq m-1$. These conditions on k are equivalent to $-m+2 \leq k \leq m-2([N/2]+1)$. Particularly, we have

$$\|\mathfrak{r}_5\|_{m-2([N/2]+1)} \lesssim \delta^{2[N/2]} \quad \text{if} \quad m \geq [N/2]+2.$$

Therefore, if $m \geq 4[N/2]+2+\delta_{N1}$ and $m-2([N/2]+1) > n/2$, then

$$\|\mathfrak{r}_2\|_{m-4([N/2]+1)} \lesssim \|\mathfrak{r}_1\|_{m-4([N/2]+1)} + \delta^2 (\|\mathfrak{r}_3\|_{m-2([N/2]+1)} + \|\mathfrak{r}_5\|_{m-2([N/2]+1)})^2 \lesssim \delta^{4[N/2]+2},$$

so that we obtain the desired estimate in Theorem 2.2 in the case (H2).

The proof of Theorem 2.2 is complete.

9 Rigorous justification of the Isobe–Kakinuma model

In this section we will prove Theorem 2.5. To this end we take advantage of the stability of the water wave equations (1.6), which is given by the following theorem. Although the statement is not explicitly given in T. Iguchi [5], we can prove it in exactly the same way as the proof of Theorem 2.3, so that we omit the proof. See also D. Lannes [15].

Theorem 9.1 *In addition to hypothesis of Theorem 2.3 we assume that $(\eta^{\mathrm{app}}, \phi^{\mathrm{app}})$ satisfy the equations*

$$\begin{cases} \partial_t \eta^{\mathrm{app}} - \Lambda(\eta^{\mathrm{app}}, b, \delta)\phi^{\mathrm{app}} = f_1^{\mathrm{err}}, \\ \partial_t \phi^{\mathrm{app}} + \eta^{\mathrm{app}} + \dfrac{1}{2}|\nabla \phi^{\mathrm{app}}|^2 - \delta^2 \dfrac{(\Lambda(\eta^{\mathrm{app}}, b, \delta)\phi^{\mathrm{app}} + \nabla \eta^{\mathrm{app}} \cdot \nabla \phi^{\mathrm{app}})^2}{2(1+\delta^2 |\nabla \eta^{\mathrm{app}}|^2)} = f_2^{\mathrm{err}}, \end{cases}$$

on a time interval $[0, T_1]$, the initial condition (1.7), and the uniform bound:

$$\begin{cases} \|\eta^{\mathrm{app}}(t)\|_{m+3+1/2} + \|\nabla \phi^{\mathrm{app}}(t)\|_{m+3} \leq M_1, \\ 1 + \eta^{\mathrm{app}}(x,t) - b(x) \geq c_0/2 \qquad \textit{for} \quad x \in \mathbf{R}^n, \ 0 \leq t \leq T_1. \end{cases}$$

Let $(\eta^{\mathrm{WW}}, \phi^{\mathrm{WW}})$ be the solution obtained in Theorem 2.3 and put $T_* = \min\{T_1, T_2\}$, where T_2 is that in Theorem 2.3. Then, we have

$$\sup_{0 \leq t \leq T_*} \big(\|\eta^{\mathrm{WW}}(t) - \eta^{\mathrm{app}}(t)\|_{m+2} + \|\phi^{\mathrm{WW}}(t) - \phi^{\mathrm{app}}(t)\|_{m+2} \big) \\ \leq C_2 \sup_{0 \leq t \leq T_*} (\|f_1^{\mathrm{err}}(t)\|_{m+2} + \|\Lambda_0(\delta)^{1/2} f_2^{\mathrm{err}}(t)\|_{m+2}),$$

where $\Lambda_0(\delta) = \Lambda(0,0,\delta)$ and C_2 is a positive constant independent of $\delta \in (0, \delta_*]$.

Proof of Theorem 2.5. Suppose that the hypotheses in Theorem 2.5 are satisfied for the initial data $(\eta_{(0)}, \phi_{(0)})$ and the bottom topography b. We will construct the initial data $\boldsymbol{\phi}^{\delta}_{(0)} = (\phi^{\delta}_{0(0)}, \ldots, \phi^{\delta}_{N(0)})^{\mathrm{T}}$ as a unique solution to

$$\mathscr{L}^{(N)}_0 \boldsymbol{\phi}^{\delta}_{(0)} = \phi_{(0)}, \qquad \mathscr{L}^{(N)}_i \boldsymbol{\phi}^{\delta}_{(0)} = 0 \quad \text{for} \quad i = 1, \ldots, N.$$

We note that the second relation is nothing but the necessary condition (2.4) whereas the first one corresponding the approximate relation (2.8). By Lemma 3.4 with m replaced by $m+4N+8$ in the case (H1) and by $m + 4[N/2] + 8$ in the case (H2), we see that

$$\begin{cases} \|\nabla \boldsymbol{\phi}^{\delta}_{(0)}\|_{m+4N+7} + \delta^{-1}\|\boldsymbol{\phi}^{\delta\prime}_{(0)}\|_{m+4N+7} \le C & \text{in the case (H1)}, \\ \|\nabla \boldsymbol{\phi}^{\delta}_{(0)}\|_{m+4[N/2]+7} + \delta^{-1}\|\boldsymbol{\phi}^{\delta\prime}_{(0)}\|_{m+4[N/2]+7} \le C & \text{in the case (H2)}, \end{cases}$$

where $C = C(c_0, M_0, m, N)$ does not depend on $\delta \in (0, 1]$. Then, by Lemma 4.4 we have $\|a(\cdot, 0) - 1\|_{m+6} \le C\delta$. Therefore, by Sobolev imbedding theorem $a(x, 0) \ge 1/2$ for $x \in \mathbf{R}$ if we take δ_* sufficiently small, so that the initial data $(\eta_{(0)}, \boldsymbol{\phi}^{\delta}_{(0)})$ satisfy the condition (2.6) in Theorem 2.1 is satisfied and the solution $(\eta^{\mathrm{IK}}, \boldsymbol{\phi}^{\delta})$ to the Isobe–Kakinuma model exists on the time interval $[0, T_1]$ satisfying

$$\begin{cases} \|\eta^{\mathrm{IK}}(t)\|_{m+4N+7} + \|\nabla \boldsymbol{\phi}^{\delta}(t)\|_{m+4N+7} + \delta^{-1}\|\boldsymbol{\phi}^{\delta\prime}(t)\|_{m+4N+7} \le M & \text{in the case (H1)}, \\ \|\eta^{\mathrm{IK}}(t)\|_{m+4[N/2]+7} + \|\nabla \boldsymbol{\phi}^{\delta}(t)\|_{m+4[N/2]+7} + \delta^{-1}\|\boldsymbol{\phi}^{\delta\prime}(t)\|_{m+4[N/2]+7} \le M & \text{in the case (H2)}, \\ 1 + \eta^{\mathrm{IK}}(x, t) - b(x) \ge c_0/2 \quad \text{for} \quad x \in \mathbf{R}^n,\ 0 \le t \le T_1. \end{cases}$$

Now, we define ϕ^{IK} by (2.8), that is, $\phi^{\mathrm{IK}} = \sum_{i=0}^{N} H^{p_j} \phi^{\delta}_j$. Then, we also have

$$\begin{cases} \|\nabla \phi^{\mathrm{IK}}(t)\|_{m+4N+6} & \text{in the case (H1)}, \\ \|\nabla \phi^{\mathrm{IK}}(t)\|_{m+4[N/2]+6} & \text{in the case (H2)}. \end{cases}$$

Moreover, by Theorem 2.2 with m replaced by $m+4N+7$ in the case (H1) and by $m+4[N/2]+7$ in the case (H2), we see that $(\eta^{\mathrm{IK}}, \phi^{\mathrm{IK}})$ satisfy

$$\begin{cases} \partial_t \eta^{\mathrm{IK}} - \Lambda(\eta^{\mathrm{IK}}, \delta)\phi^{\mathrm{IK}} = \mathfrak{r}_1, \\ \partial_t \phi^{\mathrm{IK}} + \eta^{\mathrm{IK}} + \dfrac{1}{2}|\nabla \phi^{\mathrm{IK}}|^2 - \delta^2 \dfrac{(\Lambda(\eta^{\mathrm{IK}}, \delta)\phi^{\mathrm{IK}} + \nabla \eta^{\mathrm{IK}} \cdot \nabla \phi^{\mathrm{IK}})^2}{2(1 + \delta^2 |\nabla \eta^{\mathrm{IK}}|^2)} = \mathfrak{r}_2, \end{cases}$$

where $(\mathfrak{r}_1, \mathfrak{r}_2)$ satisfy

$$\begin{cases} \|(\mathfrak{r}_1(t), \mathfrak{r}_2(t))\|_{m+3} \le C\delta^{4N+2} & \text{in the case (H1)}, \\ \|(\mathfrak{r}_1(t), \mathfrak{r}_2(t))\|_{m+3} \le C\delta^{4[N/2]+2} & \text{in the case (H2)}, \end{cases}$$

Therefore, applying Theorem 9.1 we obtain the desired estimate (2.12). $\square$

References

[1] B. Alvarez-Samaniego and D. Lannes, Large time existence for 3D water-waves and asymptotics, Invent. Math., **171** (2008), 485–541.

[2] J. Boussinesq, Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond, J. Math. Pure. Appl., **17** (1872), 55–108.

[3] W. Craig and C. Sulem, Numerical simulation of gravity waves, J. Comput. Phys., **108** (1993), 73–83.

[4] H. Fujiwara and T. Iguchi, A shallow water approximation for water waves over a moving bottom, Adv. Stud. Pure Math., **64** (2015), 77–88.

[5] T. Iguchi, A shallow water approximation for water waves, J. Math. Kyoto Univ., **49** (2009), 13–55.

[6] T. Iguchi, A mathematical analysis of tsunami generation in shallow water due to seabed deformation, Proc. Roy. Soc. Edinburgh Sect. A., **141** (2011), 551–608.

[7] T. Iguchi, Isobe–Kakinuma model for water waves as a higher order shallow water approximation, to appear in J. Differential Equations.

[8] M. Isobe, A proposal on a nonlinear gentle slope wave equation, Proceedings of Coastal Engineering, Japan Society of Civil Engineers, **41** (1994), 1–5 [Japanese].

[9] M. Isobe, Time-dependent mild-slope equations for random waves, Proceedings of 24th International Conference on Coastal Engineering, ASCE, 285–299, 1994.

[10] T. Kakinuma, [title in Japanese], Proceedings of Coastal Engineering, Japan Society of Civil Engineers, **47** (2000), 1–5 [Japanese].

[11] T. Kakinuma, A set of fully nonlinear equations for surface and internal gravity waves, Coastal Engineering V: Computer Modelling of Seas and Coastal Regions, 225–234, WIT Press, 2001.

[12] T. Kakinuma, A nonlinear numerical model for surface and internal waves shoaling on a permeable beach, Coastal engineering VI: Computer Modelling and Experimental Measurements of Seas and Coastal Regions, 227–236, WIT Press, 2003.

[13] H. Lamb, Hydrodynamics, 6th edition, Cambridge University Press, Cambridge, 1993.

[14] D. Lannes, Well-posedness of the water-waves equations, J. Amer. Math. Soc., **18** (2005), 605–654.

[15] D. Lannes, The water waves problem: mathematical analysis and asymptotics, Math. Surveys Monogr., **188**, American Mathematical Society, Providence, RI, 2013.

[16] Y. A. Li, A shallow-water approximation to the full water wave problem, Comm. Pure Appl. Math., **59** (2006), 1225–1285.

[17] J. C. Luke, A variational principle for a fluid with a free surface, J. Fluid Mech., **27** (1967), 395–397.

[18] Y. Matsuno, Hamiltonian formulation of the extended Green–Naghdi equations, Phys. D, **301/302** (2015), 1–7.

[19] Y. Matsuno, Hamiltonian structure for two-dimensional extended Green–Naghdi equations, Proc. A., **472** (2016), no. 2190, 20160127.

[20] Y. Murakami and T. Iguchi, Solvability of the initial value problem to a model system for water waves, Kodai Math. J., **38** (2015), 470–491.

[21] R. Nemoto and T. Iguchi, Solvability of the initial value problem to the Isobe-Kakinuma model for water waves, to appear in J. Math. Fluid Mech.

[22] J. J. Stoker, Water waves: the mathematical theory with application, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1992.

[23] V. E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, J. Appl. Mech. Tech. Phys., **9** (1968), 190–194.

Tatsuo Iguchi
Department of Mathematics
Faculty of Science and Technology, Keio University
3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522, Japan
E-mail: iguchi@math.keio.ac.jp