Solvability of the initial value problem to the Isobe–Kakinuma model for water waves

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Abstract

We consider the initial value problem to the Isobe–Kakinuma model for water waves and the structure of the model. The Isobe–Kakinuma model is the Euler–Lagrange equations for an approximate Lagrangian which is derived from Luke's Lagrangian for water waves by approximating the velocity potential in the Lagrangian. The Isobe–Kakinuma model is a system of second order partial differential equations and is classified into a system of nonlinear dispersive equations. Since the hypersurface $t = 0$ is characteristic for the Isobe–Kakinuma model, the initial data have to be restricted in an infinite dimensional manifold for the existence of the solution. Under this necessary condition and a sign condition, which corresponds to a generalized Rayleigh–Taylor sign condition for water waves, on the initial data, we show that the initial value problem is solvable locally in time in Sobolev spaces. We also discuss the linear dispersion relation to the model.

1 Introduction

In this paper we are concerned with the solvability locally in time of the initial value problem to the Isobe–Kakinuma model for water waves. The water wave problem is mathematically formulated as a free boundary problem for an irrotational flow of an inviscid and incompressible fluid under the gravitational field. We consider the water filled in $(n+1)$-dimensional Euclidean space. Let t be the time, $x = (x_1, \ldots, x_n)$ the horizontal spatial coordinates, and z the vertical spatial coordinate. We assume that the water surface and the bottom are represented as $z = \eta(x,t)$ and $z = -h + b(x)$, respectively, where $\eta = \eta(x,t)$ is the surface elevation, h is the mean depth, and $b = b(x)$ represents the bottom topography. J. C. Luke [16] showed that the water wave problem has a variational structure, that is, he gave a Lagrangian in terms of the velocity potential $\Phi = \Phi(x,z,t)$ and the surface elevation η in the form

$$(1.1) \qquad \mathscr{L}_{\text{Luke}}(\Phi, \eta) = \int_{-h+b(x)}^{\eta(x,t)} \left(\partial_t \Phi(x,z,t) + \frac{1}{2} |\nabla_X \Phi(x,z,t)|^2 + gz \right) \mathrm{d}z$$

and the action function

$$\mathscr{J}(\Phi, \eta) = \int_{t_0}^{t_1} \int_\Omega \mathscr{L}_{\text{Luke}}(\Phi, \eta) \mathrm{d}x \mathrm{d}t,$$

where $\nabla_X = (\nabla, \partial_z) = (\partial_{x_1}, \ldots, \partial_{x_n}, \partial_z)$, g is the gravitational constant, and Ω is an appropriate region in $\mathbf{R}^n$. He showed that the corresponding Euler–Lagrange equation is exactly the basic equations for water waves. M. Isobe [8, 9] and T. Kakinuma [10, 11, 12] approximated the velocity potential Φ in Luke's Lagrangian by

$$\Phi^{\text{app}}(x,z,t) = \sum_{i=0}^N \Psi_i(z;b)\phi_i(x,t),$$

where $\{\Psi_i\}$ is an appropriate function system in the vertical coordinate z and may depend on the bottom topography b and $(\phi_0, \phi_1, \ldots, \phi_N)$ are unknown variables, and derived an approximate Lagrangian $\mathscr{L}^{\text{app}}(\phi_0, \phi_1, \ldots, \phi_N, \eta) = \mathscr{L}_{\text{Luke}}(\Phi^{\text{app}}, \eta)$. The Isobe–Kakinuma model is the

corresponding Euler–Lagrange equation for the approximated Lagrangian. Different choices of the function system $\{\Psi_i\}$ give different Isobe–Kakinuma models. In this paper we adopt the approximation

$$(1.2) \qquad \Phi^{\mathrm{app}}(x,z,t) = \sum_{i=0}^{N} (z + h - b(x))^{p_i} \phi_i(x,t),$$

where $p_0, p_1, \ldots, p_N$ are nonnegative integers satisfying $0 = p_0 < p_1 < \cdots < p_N$. As we will see later, a natural choice of these exponents is given by $p_i = 2i$ in the case of the flat bottom, that is the case $b(x) \equiv 0$, and $p_i = i$ in the case of variable bottom topographies. In order to treat the both cases at the same time, we consider such a general case. Plugging this into Luke's Lagrangian (1.1) we obtain an approximate Lagrangian

$$\begin{aligned}
&\mathscr{L}^{\mathrm{app}}(\phi_0, \phi_1, \ldots, \phi_N, \eta) \\
&= \sum_{i=0}^{N} \frac{1}{p_i + 1} H^{p_i+1} \partial_t \phi_i \\
&\quad + \frac{1}{2} \sum_{i,j=0}^{N} \left(\frac{1}{p_i + p_j + 1} H^{p_i+p_j+1} \nabla\phi_i \cdot \nabla\phi_j - \frac{2p_i}{p_i + p_j} H^{p_i+p_j} \phi_i \nabla b \cdot \nabla\phi_j \right. \\
&\qquad\qquad\qquad \left. + \frac{p_i p_j}{p_i + p_j - 1} H^{p_i+p_j-1}(1 + |\nabla b|^2) \phi_i \phi_j \right) \\
&\quad + \frac{1}{2} g(\eta^2 - (-h + b)^2),
\end{aligned}$$

where $H = H(x,t)$ is the depth of the water and is given by $H(x,t) = h + \eta(x,t) - b(x)$. Here and in what follows we use the notational convention $0/0 = 0$. Then, the corresponding Euler–Lagrange equation has the form

$$(1.3) \quad \begin{cases}
H^{p_i} \partial_t \eta + \displaystyle\sum_{j=0}^{N} \left\{ \nabla \cdot \left(\frac{1}{p_i + p_j + 1} H^{p_i+p_j+1} \nabla\phi_j - \frac{p_j}{p_i + p_j} H^{p_i+p_j} \phi_j \nabla b \right) \right. \\
\qquad\qquad \left. + \frac{p_i}{p_i + p_j} H^{p_i+p_j} \nabla b \cdot \nabla\phi_j - \frac{p_i p_j}{p_i + p_j - 1} H^{p_i+p_j-1}(1 + |\nabla b|^2) \phi_j \right\} = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 0, 1, \ldots, N, \\
\displaystyle\sum_{i=0}^{N} H^{p_i} \partial_t \phi_i + g\eta + \frac{1}{2} \sum_{i,j=0}^{N} \left(H^{p_i+p_j} \nabla\phi_i \cdot \nabla\phi_j - 2p_i H^{p_i+p_j-1} \phi_i \nabla b \cdot \nabla\phi_j \right. \\
\qquad\qquad\qquad \left. + p_i p_j H^{p_i+p_j-2}(1 + |\nabla b|^2) \phi_i \phi_j \right) = 0.
\end{cases}$$

This is the Isobe–Kakinuma model that we are going to consider in this paper. We consider the initial value problem to this Isobe–Kakinuma model (1.3) under the initial conditions

$$(1.4) \qquad (\eta, \phi_0, \ldots, \phi_N) = (\eta_{(0)}, \phi_{0(0)}, \ldots, \phi_{N(0)}) \qquad \text{at} \quad t = 0.$$

Unique solvability locally in time of the initial value problem (1.3)–(1.4) in the case $N = 1$ and $p_1 = 2$ and fundamental properties of the model are presented in Y. Murakami and T. Iguchi [17]. Therefore, this paper is a generalization of their results.

One of the interesting features of the model is its linear dispersion relation. In the following section we will consider the linearized equations of the model around the rest state in the case

of the flat bottom and calculate the linear dispersion relation together with the phase speed $c_{IK}(\xi)$ of the plane wave solution related to the wave vector $\xi \in \mathbf{R}^n$. See (2.4). It is well known that the phase speed $c_{WW}(\xi)$ of the plane wave solution to the linearized equations for water waves is given by

$$c_{WW}(\xi) = \pm\sqrt{gh\frac{\tanh(h|\xi|)}{h|\xi|}}.$$

If we choose $p_i = 2i$, then we can show that

(1.5) $\qquad\qquad (c_{IK}(\xi))^2 = [2N/2N]$ Padé approximant of $(c_{WW}(\xi))^2$,

which will be given in Theorem 2.2. Concerning Padé approximants we refer to G. A. Baker and P. Graves-Morris [2]. This relation (1.5) implies that the Isobe–Kakinuma model gives a good approximation of the basic equations for water waves in the shallow water regime $h|\xi| \ll 1$. In fact, T. Iguchi [7] gave a mathematically rigorous justification of the Isobe–Kakinuma model in the case of the flat bottom with the choice $N = 1$ and $p_1 = 2$. He showed that the Isobe–Kakinuma model gives a higher order shallow water approximation for water waves with an error of order $O(\delta^6)$, where δ is a small nondimensional parameter defined as the ratio of the mean depth h to the typical wave length. We note that the Green–Naghdi equations are known as a higher order shallow water approximation for water waves with an error of order δ^4. Therefore, the Isobe–Kakinuma model gives a better approximation than the Green–Naghdi equations in the shallow water regime. Concerning the shallow water approximations and the rigorous justifications of the Green–Naghdi equations we refer to T. Iguchi [5, 6], B. Alvarez-Samaniego and D. Lannes [1], Y. A. Li [15], H. Fujiwara and T. Iguchi [4], and D. Lannes [14]. The relation (1.5) anticipates that the Isobe–Kakinuma model (1.3) in the case of the flat bottom with the choice $p_i = 2i$ would give an approximation with an error of order $O(\delta^{4N+2})$. We postpone this subject in the future research. If we choose $p_i = i$, we do not have such a beautiful relation as (1.5) any more. However, this choice of p_i would be important in the case of the variable bottom topographies and we still have a good approximation, which will be stated in Theorem 2.3.

The Isobe–Kakinuma model (1.3) is written in the matrix form as

$$\begin{pmatrix} H^{p_0} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ H^{p_N} & 0 & \cdots & 0 \\ 0 & H^{p_0} & \cdots & H^{p_N} \end{pmatrix} \partial_t \begin{pmatrix} \eta \\ \phi_0 \\ \vdots \\ \phi_N \end{pmatrix} + \{\text{spatial derivatives}\} = \mathbf{0}.$$

Since the coefficient matrix always has the zero eigenvalue, the hypersurface $t = 0$ in the space-time $\mathbf{R}^n \times \mathbf{R}$ is characteristic for the Isobe–Kakinuma model (1.3), so that the initial value problem (1.3)–(1.4) is not solvable in general. In fact, if the problem has a solution $(\eta, \phi_0, \ldots, \phi_N)$, then by eliminating the time derivative $\partial_t \eta$ from the equations we see that the solution has to satisfy the relation

$$H^{p_i} \sum_{j=0}^{N} \nabla \cdot \left(\frac{1}{p_j + 1} H^{p_j+1} \nabla\phi_j - \frac{p_j}{p_j} H^{p_j}\phi_j\nabla b \right)$$

(1.6) $\qquad = \sum_{j=0}^{N} \left\{ \nabla \cdot \left(\frac{1}{p_i+p_j+1} H^{p_i+p_j+1}\nabla\phi_j - \frac{p_j}{p_i+p_j} H^{p_i+p_j}\phi_j\nabla b \right) \right.$

$$\left. + \frac{p_i}{p_i+p_j} H^{p_i+p_j}\nabla b \cdot \nabla\phi_j - \frac{p_ip_j}{p_i+p_j-1} H^{p_i+p_j-1}(1+|\nabla b|^2)\phi_j \right\}$$

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for $i = 1, \ldots, N$. Therefore, as a necessary condition the initial date $(\eta_{(0)}, \phi_{0(0)}, \ldots, \phi_{N(0)})$ and the bottom topography b have to satisfy the relation (1.6) for the existence of the solution. As we will see in Proposition 1.3, such initial data are constructed from the three scalar functions b, $\eta_{(0)}$, and $\phi_{(0)}$, where $\phi_{(0)}$ is the initial data for the trace of the velocity potential on the water surface.

The water wave problem has a conserved energy

$$E_{WW}(t) = \frac{\rho}{2} \int_{\mathbf{R}^n} \left\{ \int_{-h+b(x)}^{\eta(x,t)} \left| \nabla_X \Phi(x, z, t) \right|^2 \mathrm{d}z + g\big(\eta(x, t)\big)^2 \right\} \mathrm{d}x,$$

where ρ is the constant density of the water. The first term in the right-hand side is the kinetic energy and the second one is the potential energy due to the gravity. The Isobe–Kakinuma model (1.3) has also a conserved energy

$$(1.7) \qquad E(t) = \frac{\rho}{2} \int_{\mathbf{R}^n} \left\{ \int_{-h+b(x)}^{\eta(x,t)} \left| \nabla_X \Phi^{\mathrm{app}}(x, z, t) \right|^2 \mathrm{d}z + g\big(\eta(x, t)\big)^2 \right\} \mathrm{d}x,$$

where Φ^{app} is given by (1.2). See (5.6) for more explicit expression of this energy function $E(t)$. Under a physically reasonable condition on the water surface η and the bottom topography b we have an equivalence

$$E(t) \simeq \int_{\mathbf{R}^n} \left\{ |\nabla \phi_0(x, t)|^2 + \sum_{i=1}^{N} \Big(|\nabla \phi_i(x, t)|^2 + \big(\phi_i(x, t)\big)^2 \Big) + \big(\eta(x, t)\big)^2 \right\} \mathrm{d}x.$$

Therefore, it is natural to work in the function space $\eta, \nabla\phi_0 \in C([0, T]; H^m)$ and $\phi_1, \ldots, \phi_N \in C([0, T]; H^{m+1})$, where $H^m = W^{m,2}(\mathbf{R}^n)$ is the standard L^2 Sobolev space of order m on $\mathbf{R}^n$.

It is well known that the well-posedness of the initial value problem to the water wave problem may be broken unless a generalized Rayleigh–Taylor sign condition $-\frac{\partial P}{\partial N} \geq c_0 > 0$ on the water surface is satisfied, where P is the pressure and N is the unit outward normal on the water surface. For example, we refer to S. Wu [18, 19] and D. Lannes [13]. This sign condition is equivalent to $-\partial_z P \geq c_0 > 0$ because the pressure P is equal to the constant atmospheric pressure P_0 on the water surface. By using Bernoulli's law

$$\partial_t \Phi + \frac{1}{2} |\nabla_X \Phi|^2 + \frac{1}{\rho}(P - P_0) + gz \equiv 0,$$

the sign condition can be written in term of our unknowns $(\eta, \phi_0, \ldots, \phi_N)$ and b as $a(x, t) \geq c_0 > 0$, where

$$(1.8) \qquad a = g + \sum_{i=0}^{N} p_i H^{p_i - 1} \partial_t \phi_i$$

$$+ \frac{1}{2} \sum_{i,j=0}^{N} \Big\{ (p_i + p_j) H^{p_i + p_j - 1} \nabla \phi_i \cdot \nabla \phi_j - 2p_i(p_i + p_j - 1) H^{p_i + p_j - 2} \phi_i \nabla b \cdot \nabla \phi_j$$

$$+ p_i p_j (p_i + p_j - 2) H^{p_i + p_j - 3} (1 + |\nabla b|^2) \phi_i \phi_j \Big\}.$$

In fact, we have $-\frac{1}{\rho} \partial_z P^{\mathrm{app}} = g + \partial_z \partial_t \Phi^{\mathrm{app}} + \nabla_X \partial_z \Phi^{\mathrm{app}} \cdot \nabla_X \Phi^{\mathrm{app}} = a$ on $z = \eta(x, t)$. In this paper, we assume that this sign condition is satisfied at the initial time $t = 0$.

Now, we are ready to give our main result in this paper.

Theorem 1.1 *Let g, h, c_0, M_0 be positive constants and m an integer such that $m > n/2 + 1$. There exists a time $T > 0$ such that if the initial data $(\eta_{(0)}, \phi_{0(0)}, \ldots, \phi_{N(0)})$ and b satisfy the relation* (1.6) *and*

$$(1.9) \qquad \begin{cases} \|\eta_{(0)}\|_m + \|\nabla\phi_{0(0)}\|_m + \|(\phi_{1(0)}, \ldots, \phi_{N(0)})\|_{m+1} + \|b\|_{W^{m+2,\infty}} \leq M_0, \\ h + \eta_{(0)}(x) - b(x) \geq c_0, \quad a(x, 0) \geq c_0 \qquad for \quad x \in \mathbf{R}^n, \end{cases}$$

then the initial value problem (1.3)–(1.4) *has a unique solution $(\eta, \phi_0, \ldots, \phi_N)$ satisfying*

$$\eta, \nabla\phi_0 \in C([0, T]; H^m), \quad \phi_1, \ldots, \phi_N \in C([0, T]; H^{m+1}).$$

Moreover, the energy function $E(t)$ defined by (1.7) *is a conserved quantity.*

Remark 1.2 (1) If we impose an additional condition $\phi_{(0)} \in L^2(\mathbf{R}^n)$, then the solution satisfies an additional integrability $\phi_0 \in C([0, T]; H^{m+1})$.

(2) In the sign condition $a(x, 0) \geq c_0 > 0$ we use the quantities $\partial_t\phi_1(x, 0), \ldots, \partial_t\phi_N(x, 0)$ which should be written in terms of the initial data. Although the hypersurface $t = 0$ is characteristic for the Isobe–Kakinuma model (1.3), we can express $\partial_t\phi_1(x, 0), \ldots, \partial_t\phi_N(x, 0)$ in terms of the initial data and b. For details, we refer to Remark 4.1.

(3) Let ϕ be the trace of the velocity potential Φ on the water surface. In view of our approximation (1.2), ϕ should be related to our variables approximately by the formula

$$(1.10) \qquad \phi = \sum_{i=0}^{N} H^{p_i}\phi_i.$$

Given the initial data $\eta_{(0)}$ and $\phi_{(0)}(= \phi|_{t=0})$ and the bottom topography b, the necessary condition (1.6) and the relation (1.10) determine uniquely the initial data $\phi_{0(0)}, \ldots, \phi_{N(0)}$. In fact, we have the following proposition.

Proposition 1.3 *Let h, c_0, M_0 be positive constants and m an integer such that $m > n/2 + 1$. There exists a positive $C > 0$ such that if the initial data $(\eta_{(0)}, \phi_{(0)})$ and b satisfy*

$$\begin{cases} \|\eta_{(0)}\|_m + \|b\|_{W^{m,\infty}} \leq M_0, \quad \|\nabla\phi_{(0)}\|_{m-1} < \infty, \\ h + \eta_{(0)}(x) - b(x) \geq c_0 \qquad for \quad x \in \mathbf{R}^n, \end{cases}$$

then the necessary condition (1.6) *and the relation* (1.10) *determine uniquely the initial data $\phi_{0(0)}, \ldots, \phi_{N(0)}$, which satisfy*

$$\|\nabla\phi_{0(0)}\|_{m-1} + \|(\phi_{1(0)}, \ldots, \phi_{N(0)})\|_m \leq C\|\nabla\phi_{(0)}\|_{m-1}.$$

The contents of this paper are as follows. In Section 2 we consider the linearized equations of the Isobe–Kakinuma model around the rest state in the case of the flat bottom and analyze the linear dispersion relation. Especially, we show the beautiful relation (1.5). In Section 3 we consider the linearized equations of the Isobe–Kakinuma model around an arbitrary flow, reveal a hidden symmetric structure of the model, and give an idea to obtain an energy estimate for the solution of the nonlinear equations. In Section 4 we transform the Isobe–Kakinuma model to a system of equations for which the hypersurface $t = 0$ is noncharacteristic by using the necessary condition (1.6) and construct the solution of the initial value problem to the transformed system by using a standard parabolic regularization. In Section 5 we show that the

solution constructed in Section 4 is the solution of the Isobe–Kakinuma model (1.3) if the initial data satisfy the necessary condition (1.6).

Notation. We denote by $W^{m,p}(\mathbf{R}^n)$ the L^p Sobolev space of order m on $\mathbf{R}^n$. The norms of the Lebesgue space $L^p(\mathbf{R}^n)$ and the Sobolev space $H^m = W^{m,2}(\mathbf{R}^n)$ are denoted by $|\cdot|_p$ and $\|\cdot\|_m$, respectively. The L^2-norm and the L^2-inner product are simply denoted by $\|\cdot\|$ and $(\cdot,\cdot)_{L^2}$, respectively. We put $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, and $\partial_z = \partial/\partial z$. For a multi-index $\alpha = (\alpha_1,\ldots,\alpha_n)$ we put $\partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$. $[P,Q] = PQ - QP$ denotes the commutator. For a matrix A we denote by A^{T} the transpose of A. For a vector $\boldsymbol{\phi} = (\phi_0,\phi_1,\ldots,\phi_N)^{\mathrm{T}}$ we denote the last N components by $\boldsymbol{\phi}' = (\phi_1,\ldots,\phi_N)^{\mathrm{T}}$. We use the notational convention $0/0 = 0$.

2 Linear dispersion relation

In this section we consider the linearized equations of the Isobe–Kakinuma model (1.3) in the case of the flat bottom. The linearized equations have the form

(2.1)
$$
\begin{cases}
\partial_t \eta + \sum_{j=0}^N \left(\frac{1}{p_i + p_j + 1} h^{p_j+1}\Delta\phi_j - \frac{p_i p_j}{p_i + p_j - 1} h^{p_j-1}\phi_j \right) = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 0,1,\ldots,N, \\
\sum_{i=0}^N h^{p_i}\partial_t\phi_i + g\eta = 0.
\end{cases}
$$

Putting $\boldsymbol{\psi} = (h^{p_0}\phi_0,\ldots,h^{p_N}\phi_N)^{\mathrm{T}}$ we can rewrite the above equations as the following simple matrix form
$$
\begin{pmatrix} 0 & h\mathbf{1}^{\mathrm{T}} \\ -h\mathbf{1} & O \end{pmatrix} \partial_t \begin{pmatrix} \eta \\ \boldsymbol{\psi} \end{pmatrix} + \begin{pmatrix} gh & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & A(hD) \end{pmatrix} \begin{pmatrix} \eta \\ \boldsymbol{\psi} \end{pmatrix} = \mathbf{0},
$$
where $\mathbf{1} = (1,\ldots,1)^{\mathrm{T}}$ and $A(hD) = -A_0 h^2\Delta + A_1$. The $(N+1)\times(N+1)$ matrices A_0 and A_1 are given by
$$
A_0 = \left(\frac{1}{p_i + p_j + 1} \right)_{0\le i,j\le N}, \qquad A_1 = \left(\frac{p_i p_j}{p_i + p_j - 1} \right)_{0\le i,j\le N},
$$
where we used a rather nonstandard notation for matrices. Since it might not cause any confusion, we will continue to use this notation in the following. Therefore, the linear dispersion relation is given by
$$
\det \begin{pmatrix} gh & \sqrt{-1}h\omega\mathbf{1}^{\mathrm{T}} \\ -\sqrt{-1}h\omega\mathbf{1} & A(h\xi) \end{pmatrix} = 0,
$$
where $\xi \in \mathbf{R}^n$ is the wave vector, $\omega \in \mathbf{C}$ is the angular frequency, and $A(h\xi) = (h|\xi|)^2 A_0 + A_1$. We can rewrite the above dispersion relation as

(2.2)
$$
h^2\omega^2 \det \tilde{A}(h\xi) - gh \det A(h\xi) = 0.
$$

Throughout this section we use the notation
$$
\tilde{A} = \begin{pmatrix} 0 & \mathbf{1}^{\mathrm{T}} \\ -\mathbf{1} & A \end{pmatrix}
$$

for a matrix A. Concerning the determinants in the dispersion relation (2.2) we have the following proposition.

Proposition 2.1   *1. For any $\xi \in \mathbf{R}^n \setminus \{\mathbf{0}\}$, the symmetric matrix $A(h\xi)$ is positive.*

2. There exists $c_0 > 0$ such that for any $\xi \in \mathbf{R}^n$ we have $\det \tilde{A}(h\xi) \geq c_0$.

3. $(h|\xi|)^{-2} \det A(h\xi)$ is a polynomial in $(h|\xi|)^2$ of degree N and the coefficient of $(h|\xi|)^{2N}$ is $\det A_0$.

4. $\det \tilde{A}(h\xi)$ is a polynomial in $(h|\xi|)^2$ of degree N and the coefficient of $(h|\xi|)^{2N}$ is $\det \tilde{A}_0$.

Proof. For any $\boldsymbol{\psi} = (\psi_0, \ldots, \psi_N)^{\mathrm{T}} \in \mathbf{R}^{N+1}$ we see that

$$(2.3) \qquad (A(h\xi)\boldsymbol{\psi}) \cdot \boldsymbol{\psi} = \int_0^1 \left\{ (h|\xi|)^2 \left(\sum_{i=0}^N \psi_i z^{p_i} \right)^2 + \left(\sum_{i=1}^N p_i \psi_i z^{p_i - 1} \right)^2 \right\} \mathrm{d}z$$

$$\simeq (h|\xi|)^2 |\boldsymbol{\psi}|^2 + |\boldsymbol{\psi}'|^2,$$

where $\boldsymbol{\psi}' = (\psi_1, \ldots, \psi_N)^{\mathrm{T}}$. This shows a positivity of $A(h\xi)$ for $\xi \neq \mathbf{0}$.

Let $\xi \neq \mathbf{0}$ and $C(h\xi) = (C_{ij}(h\xi))_{0 \leq i,j \leq N}$ be the cofactor matrix of $A(h\xi)$. Since $A(h\xi)$ is symmetric, $C(h\xi)$ is also symmetric and we have $C(h\xi) = (\det A(h\xi))A(h\xi)^{-1}$. Then, expanding the first row of the matrix $\tilde{A}(h\xi)$ we see that

$$\det \tilde{A}(h\xi) = \sum_{i,j=0}^N C_{ij}(h\xi) = (C(h\xi)\mathbf{1}) \cdot \mathbf{1} = (\det A(h\xi))(A(h\xi)^{-1}\mathbf{1}) \cdot \mathbf{1},$$

which implies the positivity of $\det \tilde{A}(h\xi)$ due to the positivity of $A(h\xi)$. We proceed to consider the case $\xi = \mathbf{0}$. It holds that $\tilde{A}(\mathbf{0}) = \tilde{A}_1$ and

$$A_1 = \begin{pmatrix} 0 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & A_1' \end{pmatrix}, \quad A_1' = \left(\frac{p_i p_j}{p_i + p_j - 1} \right)_{1 \leq i,j \leq N}.$$

By the similar calculation as (2.3) we see the positivity of the symmetric matrix A_1'. Moreover, we have $\det \tilde{A}(\mathbf{0}) = \det A_1' > 0$. Therefore, we obtain the strict positivity of $\det \tilde{A}(h\xi)$.

It is easy to see that $\det A(h\xi)$ is a polynomial in $(h|\xi|)^2$ of degree less than or equal to $N+1$ and that the coefficient of $(h|\xi|)^{2(N+1)}$ is $\det A_0$. By the similar calculation as (2.3) we see the positivity of the symmetric matrix A_0 so that the degree is in fact $N+1$. Moreover, $A(\mathbf{0})$ has the zero eigenvalue with an eigenvector $(1, 0, \ldots, 0)^{\mathrm{T}}$, so that $\det A(\mathbf{0}) = 0$. Therefore, the term of degree 0 in $\det A(h\xi)$ vanishes so that $(h|\xi|)^{-2} \det A(h\xi)$ is a polynomial in $(h|\xi|)^2$ of degree N. Similarly we see that $\det \tilde{A}(h\xi)$ is a polynomial in $(h|\xi|)^2$ of degree less than or equal to N and that the coefficient of $(h|\xi|)^{2N}$ is $\det \tilde{A}_0$, which is also positive, so that the degree is in fact N. $\square$

Thanks of this proposition and the dispersion relation (2.2), the linearized system (2.1) is classified into the dispersive system, so that the Isobe–Kakinuma model is a nonlinear dispersive system of equations. Therefore, we can define the phase speed $c_{IK}(\xi)$ of the plane wave solution to (2.1) related to the wave vector $\xi \in \mathbf{R}^n$ by

$$(2.4) \qquad c_{IK}(\xi) = \pm \sqrt{gh \frac{(h|\xi|)^{-2} \det A(h\xi)}{\det \tilde{A}(h\xi)}}.$$

It follows from Proposition 2.1 that

$$\lim_{|\xi| \to +\infty} c_{IK}(\xi) = \pm \sqrt{gh \frac{\det A_0}{\det \tilde{A}_0}},$$

which is not zero. It is not consistent with the linear water waves: $\lim_{|\xi|\to+\infty} c_{WW}(\xi) = 0$. This implies that the Isobe–Kakinuma model (1.3) cannot give a good approximation to the water waves in deep water. However, as is shown by the following theorems the Isobe–Kakinuma model gives a very precise approximation in the shallow water regime $h|\xi| \ll 1$.

Theorem 2.2 *If we choose $p_i = 2i$ ($i = 0, 1, \ldots, N$), then $(c_{IK}(\xi))^2$ becomes the $[2N/2N]$ Padé approximant of $(c_{WW}(\xi))^2$. More precisely, there exists a positive constant C depending only on N such that for any $\xi \in \mathbf{R}^n$ and any $h, g > 0$ we have*

$$\left| \left(\frac{c_{WW}(\xi)}{\sqrt{gh}} \right)^2 - \left(\frac{c_{IK}(\xi)}{\sqrt{gh}} \right)^2 \right| \le C(h|\xi|)^{4N+2}.$$

Proof. Without loss of generality it is sufficient to consider the case $h = 1$ and to show that

$$(2.5) \qquad\qquad |\xi| \tanh |\xi| = \frac{\det A(\xi)}{\det \tilde{A}(\xi)} + O(|\xi|^{4N+4}) \quad \text{as} \quad |\xi| \to 0.$$

Let $\hat{\Phi}(z) = \frac{1}{\cosh |\xi|} \cosh(|\xi|z)$. In view of $\hat{\Phi}''(z) = |\xi|^2 \hat{\Phi}(z)$, $\hat{\Phi}(1) = 1$, $\hat{\Phi}'(1) = |\xi| \tanh |\xi|$, and $\hat{\Phi}'(0) = 0$, we have

$$(2.6) \qquad\qquad |\xi| \tanh |\xi| = \int_0^1 \left\{ |\xi|^2 (\hat{\Phi}(z))^2 + (\hat{\Phi}'(z))^2 \right\} \mathrm{d}z.$$

By the Taylor series expansion of $\cosh(|\xi|z)$ we have

$$\hat{\Phi}(z) = \sum_{i=0}^{2N} \hat{\psi}_i z^{2i} + O(|\xi|^{4N+2}), \quad \hat{\Phi}'(z) = \sum_{i=0}^{2N} 2i\hat{\psi}_i z^{2i-1} + O(|\xi|^{4N+2})$$

uniformly with respect to $z \in [0, 1]$, where

$$(2.7) \qquad\qquad \hat{\psi}_i = \frac{|\xi|^{2i}}{(2i)! \cosh |\xi|}, \quad i = 0, 1, \ldots.$$

Plugging these expansions into (2.6) we see that

$$|\xi| \tanh |\xi| = \sum_{i,j=0}^{N} \left(\frac{1}{2(i+j)+1} |\xi|^2 + \frac{4ij}{2(i+j)-1} \right) \hat{\psi}_i \hat{\psi}_j + 2R + O(|\xi|^{4N+4}),$$

where

$$\begin{aligned}
R &= \sum_{i=0}^{N-1} \sum_{j=N+1}^{2N} \frac{|\xi|^2}{2(i+j)+1} \hat{\psi}_i \hat{\psi}_j + \sum_{i=0}^{N} \sum_{j=N+1}^{2N} \frac{4ij}{2(i+j)-1} \hat{\psi}_i \hat{\psi}_j \\
&= \sum_{i=0}^{N-1} \sum_{j=N+1}^{2N} \left(\frac{|\xi|^2}{2(i+j)+1} \hat{\psi}_i \hat{\psi}_j + \frac{4(i+1)j}{2(i+j)+1} \hat{\psi}_{i+1} \hat{\psi}_j \right) \\
&= \frac{|\xi|}{\cosh^2 |\xi|} \sum_{i=0}^{N-1} \sum_{j=N+1}^{2N} \frac{|\xi|^{2i+1}}{(2i+1)!} \frac{|\xi|^{2j}}{(2j)!} \\
&= \frac{|\xi| \tanh |\xi|}{\cosh |\xi|} \sum_{j=N+1}^{\infty} \frac{|\xi|^{2j}}{(2j)!} + O(|\xi|^{4N+4}).
\end{aligned}$$

Here we used the explicit form (2.7) of $\hat{\psi}_i$. Therefore, we obtain

$$(2.8) \qquad (|\xi|\tanh|\xi|)\left(1 - \frac{2}{\cosh|\xi|}\sum_{j=N+1}^{\infty}\frac{|\xi|^{2j}}{(2j)!}\right)$$

$$= \sum_{i,j=0}^{N}\left(\frac{1}{2(i+j)+1}|\xi|^2 + \frac{4ij}{2(i+j)-1}\right)\hat{\psi}_i\hat{\psi}_j + O(|\xi|^{4N+4}).$$

Up to now we did not use the specific choice of p_i. From now on, we use the advantage of the choice $p_i = 2i$. Then, we can rewrite the above identity (2.8) as

$$(2.9) \qquad (|\xi|\tanh|\xi|)\left(1 - \frac{2}{\cosh|\xi|}\sum_{j=N+1}^{\infty}\frac{|\xi|^{2j}}{(2j)!}\right) = (A(\xi)\hat{\psi})\cdot\hat{\psi} + O(|\xi|^{4N+4}),$$

where $\hat{\psi} = (\hat{\psi}_0,\ldots,\hat{\psi}_N)^{\mathrm{T}}$. On the other hand, we consider the following equations for $\hat{\eta}$ and $\hat{\phi} = (\hat{\phi}_0,\ldots,\hat{\phi}_N)^{\mathrm{T}}$.

$$(2.10) \qquad \begin{pmatrix} 0 & \mathbf{1}^{\mathrm{T}} \\ -\mathbf{1} & A(\xi) \end{pmatrix}\begin{pmatrix} \hat{\eta} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}.$$

Since the coefficient matrix $\tilde{A}(\xi)$ is nonsingular, we have a unique solution $(\hat{\eta},\hat{\phi})$ of these equations. Moreover, we see that

$$(2.11) \qquad (A(\xi)\hat{\phi})\cdot\hat{\phi} = \hat{\eta} = \frac{\det A(\xi)}{\det \tilde{A}(\xi)}.$$

In view of this and (2.9), it is sufficient to compare $\hat{\psi}$ with $\hat{\phi}$. Let $A(\xi) = (\boldsymbol{a}_0,\ldots,\boldsymbol{a}_N)$. Then, by using the explicit form (2.7) of $\hat{\psi}_i$ we see that

$$\boldsymbol{a}_i\cdot\hat{\psi} = \frac{1}{\cosh|\xi|}\left(\sum_{j=0}^{N-1}\frac{|\xi|^{2j+2}}{(2j+1)!} + \frac{1}{2(i+N)+1}\frac{|\xi|^{2N+2}}{(2N)!}\right)$$

for $i = 0,1,\ldots,N$, and that

$$\mathbf{1}\cdot\hat{\psi} = \frac{1}{\cosh|\xi|}\sum_{i=0}^{N}\frac{|\xi|^{2i}}{(2i)!} = 1 + O(|\xi|^{2N+2}).$$

Therefore, if we put $\hat{\zeta} = \frac{1}{\cosh|\xi|}\sum_{j=0}^{N-1}\frac{|\xi|^{2j+2}}{(2j+1)!}$, then we have

$$(2.12) \qquad \begin{pmatrix} 0 & \mathbf{1}^{\mathrm{T}} \\ -\mathbf{1} & A(\xi) \end{pmatrix}\begin{pmatrix} \hat{\zeta} \\ \hat{\psi} \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} + O(|\xi|^{2N+2}).$$

By Proposition 2.1 the determinant of the coefficient matrix is strictly positive, so that this together with (2.10) implies $\hat{\psi} = \hat{\phi} + O(|\xi|^{2N+2})$. Moreover, it holds that

$$(A(\xi)\hat{\phi})\cdot\hat{\psi} = \frac{\det A(\xi)}{\det \tilde{A}(\xi)}\frac{1}{\cosh|\xi|}\sum_{i=0}^{N}\frac{|\xi|^{2i}}{(2i)!},$$

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which together with (2.11) implies

$$(A(\xi)\hat{\phi}) \cdot (\hat{\phi} - \hat{\psi}) = \frac{\det A(\xi)}{\det \tilde{A}(\xi)} \frac{1}{\cosh |\xi|} \sum_{i=N+1}^{\infty} \frac{|\xi|^{2i}}{(2i)!},$$

so that

$$(2.13) \qquad (A(\xi)\hat{\psi}) \cdot \hat{\psi} = (A(\xi)\hat{\phi}) \cdot \hat{\phi} - 2(A(\xi)\hat{\phi}) \cdot (\hat{\phi} - \hat{\psi}) + (A(\xi)(\hat{\phi} - \hat{\psi})) \cdot (\hat{\phi} - \hat{\psi})$$

$$= \frac{\det A(\xi)}{\det \tilde{A}(\xi)} \left(1 - \frac{2}{\cosh |\xi|} \sum_{i=N+1}^{\infty} \frac{|\xi|^{2i}}{(2i)!} \right) + O(|\xi|^{4N+4}).$$

Plugging this into (2.9) we obtain the desired relation (2.5). □

As was shown by J. Boussinesq [3], in the case of the flat bottom the velocity potential Φ for the water wave problem can be expanded in a Taylor series with respect to the vertical spatial variable z around the bottom $z = -h$ as

$$\Phi(x, z, t) = \sum_{i=0}^{\infty} \frac{(z + h)^{2i}}{(2i)!} (-\Delta)^i \phi_0(x, t),$$

where ϕ_0 is the trace of the velocity potential Φ on the bottom. Therefore, it is natural to choose $p_i = 2i$ in such a case. However, in the case of variable bottom topographies the Taylor series with respect to z around the bottom $z = -h + b(x)$ contains terms of odd degree too, so that the choice $p_i = i$ would be important to such cases. If we choose $p_i = i$, we do not have such a beautiful result as Theorem 2.2 but we still have the following theorem, which asserts that the terms of odd degree in the approximation of the velocity potential Φ do not affect the precision of the linear dispersion relation in the shallow water regime $h|\xi| \ll 1$.

Theorem 2.3 *If we choose $p_i = i$ $(i = 0, 1, \ldots, N)$, then for any $\xi \in \mathbf{R}^n$ and any $h, g > 0$ we have*

$$\left| \left(\frac{c_{WW}(\xi)}{\sqrt{gh}} \right)^2 - \left(\frac{c_{IK}(\xi)}{\sqrt{gh}} \right)^2 \right| \leq C(h|\xi|)^{4[N/2]+2},$$

where C is a positive constant depending only on N and $[N/2]$ is the integer part of $N/2$.

Proof. We will give the proof in the case of even integer N. As in the proof of the previous theorem, it is sufficient to show that

$$(2.14) \qquad |\xi| \tanh |\xi| = \frac{\det A(\xi)}{\det \tilde{A}(\xi)} + O(|\xi|^{4[N/2]+4}) \quad \text{as} \quad |\xi| \to 0.$$

In place of (2.7), we define $\hat{\psi}_i$ by

$$\hat{\psi}_i = \begin{cases} \dfrac{|\xi|^i}{i! \cosh |\xi|} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd} \end{cases}$$

and put $\hat{\psi} = (\hat{\psi}_0, \ldots, \hat{\psi}_N)^{\mathrm{T}}$. Then, we have

$$(A(\xi)\hat{\psi}) \cdot \hat{\psi} = \sum_{i,j=0}^{[N/2]} \left(\frac{1}{2(i + j) + 1} |\xi|^2 + \frac{4ij}{2(i + j) - 1} \right) \hat{\psi}_{2i} \hat{\psi}_{2j},$$

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so that we can rewrite the identity (2.8) as

$$(2.15) \qquad (|\xi| \tanh |\xi|) \left(1 - \frac{2}{\cosh |\xi|} \sum_{j=[N/2]+1}^{\infty} \frac{|\xi|^{2j}}{(2j)!} \right) = (A(\xi)\hat{\psi}) \cdot \hat{\psi} + O(|\xi|^{4[N/2]+4}).$$

Let $(\hat{\eta}, \hat{\phi})$ be the solution of (2.10) as before. Then, we have the identity (2.11). If we put $\hat{\zeta} = \frac{1}{\cosh |\xi|} \sum_{j=0}^{[N/2]-1} \frac{|\xi|^{2j+2}}{(2j+1)!}$, then in place of (2.12) we have

$$\begin{pmatrix} 0 & \mathbf{1}^{\mathrm{T}} \\ -\mathbf{1} & A(\xi) \end{pmatrix} \begin{pmatrix} \hat{\zeta} \\ \hat{\psi} \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} + O(|\xi|^{2[N/2]+2}).$$

This together with (2.10) implies $\hat{\psi} = \hat{\phi} + O(|\xi|^{2[N/2]+2})$. Therefore, in place of (2.13) we have

$$(A(\xi)\hat{\psi}) \cdot \hat{\psi} = \frac{\det A(\xi)}{\det \tilde{A}(\xi)} \left(1 - \frac{2}{\cosh |\xi|} \sum_{i=[N/2]+1}^{\infty} \frac{|\xi|^{2i}}{(2i)!} \right) + O(|\xi|^{4[N/2]+4}).$$

Plugging this into (2.15) we obtain the desired relation (2.14). The case of odd integer N can be proved in the same way, so we omit it. $\square$

3 Analysis of a linearized system

In this section we consider the linearized equations of the Isobe–Kakinuma model (1.3) around an arbitrary flow $(\eta, \phi_0, \ldots, \phi_N)$, which is assumed to be sufficiently smooth. The hypersurface $t = 0$ is still characteristic for the linearized equations. We will transform the equations to a symmetric positive system of partial differential equations for which the hypersurface $t = 0$ is noncharacteristic and give an idea to derive a priori estimates for the solution to the nonlinear equations.

We introduce second order differential operators $L_{ij} = L_{ij}(H, b)$ $(i, j = 0, 1, \ldots, N)$ depending on the water depth H and the bottom topography b by

$$(3.1) \qquad L_{ij}\psi_j = -\nabla \cdot \left(\frac{1}{p_i + p_j + 1} H^{p_i+p_j+1} \nabla \psi_j - \frac{p_j}{p_i + p_j} H^{p_i+p_j} \psi_j \nabla b \right)$$
$$- \frac{p_i}{p_i + p_j} H^{p_i+p_j} \nabla b \cdot \nabla \psi_j + \frac{p_i p_j}{p_i + p_j - 1} H^{p_i+p_j-1} (1 + |\nabla b|^2) \psi_j.$$

Then, we have $L_{ij}^* = L_{ji}$, where L_{ij}^* is the adjoint operator of L_{ij} in $L^2(\mathbf{R}^n)$. In addition to the function a defined by (1.8) we introduce the functions $\boldsymbol{u}$ and w by

$$(3.2) \qquad \boldsymbol{u} = \sum_{i=0}^{N} (H^{p_i} \nabla \phi_i - p_i H^{p_i-1} \phi_i \nabla b), \quad w = \sum_{i=0}^{N} p_i H^{p_i-1} \phi_i.$$

Since $\boldsymbol{u} = \nabla \Phi^{\mathrm{app}}|_{z=\eta}$ and $w = \partial_z \Phi^{\mathrm{app}}|_{z=\eta}$, where Φ^{app} is the approximate velocity potential defined by (1.2), $\boldsymbol{u}$ and w represent approximately the horizontal and the vertical components of the velocity field on the water surface, respectively. Then, the Isobe–Kakinuma model (1.3) can be written simply as

$$\begin{cases} H^{p_i} \partial_t \eta - \displaystyle\sum_{j=0}^{N} L_{ij}\phi_j = 0 \qquad \text{for} \quad i = 0, 1, \ldots, N, \\ \displaystyle\sum_{j=0}^{N} H^{p_j} \partial_t \phi_j + g\eta + \frac{1}{2}(|\boldsymbol{u}|^2 + w^2) = 0. \end{cases}$$

Now, let us linearize the above equations around $(\eta, \phi_0, \ldots, \phi_N)$. We denote by $(\zeta, \psi_0, \ldots, \psi_N)$ the variation from $(\eta, \phi_0, \ldots, \phi_N)$. After a tedious but straightforward calculation we obtain the linearized equations

(3.3)
$$
\begin{cases}
H^{p_i}(\partial_t \zeta + \nabla \cdot (\boldsymbol{u}\zeta)) + p_i H^{p_i-1}(\partial_t \eta + \boldsymbol{u} \cdot \nabla \eta - w)\zeta - \sum_{j=0}^{N} L_{ij}\psi_j = f_i \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 0, 1, \ldots, N, \\
\sum_{j=0}^{N} \Big\{ H^{p_j}(\partial_t \psi_j + \boldsymbol{u} \cdot \nabla \psi_j) - p_j H^{p_j-1}(\boldsymbol{u} \cdot \nabla b - w)\psi_j \Big\} + a\zeta = f_{N+1},
\end{cases}
$$

where $f_0, \ldots, f_{N+1}$ are given functions.

The hypersurface $t = 0$ is still characteristic for these linearized equations. In fact, by eliminating the time derivative $\partial_t \zeta$ from these equations we have

$$
\sum_{j=0}^{N}(L_{ij} - H^{p_i}L_{0j})\psi_j = p_i H^{p_i-1}(\partial_t \eta + \boldsymbol{u} \cdot \nabla \eta - w)\zeta + H^{p_i}f_0 - f_i
$$

for $i = 1, \ldots, N$. Now, we differentiate this with respect to the time t and use the first equation in (3.3) to eliminate the time derivative $\partial_t \zeta$. Then, we obtain

(3.4)
$$
\sum_{j=0}^{N}(L_{ij} - H^{p_i}L_{0j})\partial_t \psi_j = p_i H^{p_i-1}(\partial_t \eta + \boldsymbol{u} \cdot \nabla \eta - w)\Big(\sum_{j=0}^{N} L_{0j}\psi_j - \nabla \cdot (\boldsymbol{u}\zeta) \Big)
$$
$$
- \sum_{j=0}^{N}[\partial_t, L_{ij} - H^{p_i}L_{0j}]\psi_j + [\partial_t, p_i H^{p_i-1}(\partial_t \eta + \boldsymbol{u} \cdot \nabla \eta - w)]\zeta
$$
$$
+ \partial_t(H^{p_i}f_0 - f_i) + p_i H^{p_i-1}(\partial_t \eta + \boldsymbol{u} \cdot \nabla \eta - w)f_0.
$$

On the other hand, it follows from the second equation in (3.3) that

(3.5)
$$
\sum_{j=0}^{N} H^{p_j}\partial_t \psi_j = -\sum_{j=0}^{N}\Big\{ H^{p_j}\boldsymbol{u} \cdot \nabla \psi_j - p_j H^{p_j-1}(\boldsymbol{u} \cdot \nabla b - w)\psi_j \Big\} - a\zeta + f_{N+1}.
$$

Here we note that the right hand sides of (3.4) and (3.5) do not include any time derivatives of $(\zeta, \psi_0, \ldots, \psi_N)$. In view of these equations we introduce linear operators $\mathscr{L}_i = \mathscr{L}_i(H, b)$ $(i = 0, 1, \ldots, N)$ depending on the water depth H and the bottom topography b and acting on $\boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_N)^{\mathrm{T}}$ by

(3.6)
$$
\mathscr{L}_0\boldsymbol{\varphi} = \sum_{j=0}^{N} H^{p_j}\varphi_j, \qquad \mathscr{L}_i\boldsymbol{\varphi} = \sum_{j=0}^{N}(L_{ij} - H^{p_i}L_{0j})\varphi_j \quad \text{for} \quad i = 1, \ldots, N,
$$

and put $\mathscr{L}\boldsymbol{\varphi} = (\mathscr{L}_0\boldsymbol{\varphi}, \ldots, \mathscr{L}_N\boldsymbol{\varphi})^{\mathrm{T}}$. For a given $\boldsymbol{F} = (F_0, \ldots, F_N)^{\mathrm{T}}$ we consider the equation

(3.7)
$$
\mathscr{L}\boldsymbol{\varphi} = \boldsymbol{F}.
$$

Once we show the solvability of $\boldsymbol{\varphi}$ of this equation, we could express the time derivatives $\partial_t \psi_i$ $(i = 0, 1, \ldots, N)$ in terms of the spatial derivatives so that we could avoid the difficulty arising from the fact that the hypersurface $t = 0$ is characteristic for the linearized equations (3.3).

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Let $\boldsymbol{\varphi}$ be a solution of (3.7). It follows from the first component of (3.7) that

$$(3.8) \qquad \varphi_0 = F_0 - \sum_{j=1}^{N} H^{p_j} \varphi_j.$$

Plugging this into the other components of (3.7) we obtain

$$(3.9) \qquad P_i \boldsymbol{\varphi}' = F_i - (L_{i0} - H^{p_i} L_{00}) F_0 \quad \text{for} \quad i = 1, \ldots, N,$$

where $\boldsymbol{\varphi}' = (\varphi_1, \ldots, \varphi_N)^{\mathrm{T}}$ and $P_j = P_j(H, b)$ $(j = 1, \ldots, N)$ are second order differential operators defined by

$$(3.10) \qquad P_i \boldsymbol{\varphi}' = \sum_{j=1}^{N} \left\{ (L_{ij} - H^{p_i} L_{0j}) \varphi_j - (L_{i0} - H^{p_i} L_{00})(H^{p_j} \varphi_j) \right\}.$$

We further introduce the operator $P\boldsymbol{\varphi}' = (P_1 \boldsymbol{\varphi}', \ldots, P_N \boldsymbol{\varphi}')^{\mathrm{T}}$. Since $L_{ij}^* = L_{ji}$, we see easily that P is symmetric in $L^2(\mathbf{R}^n)$. Moreover, P is positive in $L^2(\mathbf{R}^n)$ as shown in the following lemma.

Lemma 3.1 Let c_0, c_1 be positive constants. There exists a positive constant $C = C(c_0, c_1) > 0$ depending only on c_0 and c_1 such that if $H, \nabla b \in L^\infty(\mathbf{R}^n)$ satisfy $H(x) \geq c_0$ and $|\nabla b(x)| \leq c_1$, then we have

$$(P\boldsymbol{\varphi}', \boldsymbol{\varphi}')_{L^2} \geq C^{-1} \|\boldsymbol{\varphi}'\|_1^2.$$

Proof. Introducing $\varphi_0 = -\sum_{j=1}^{N} H^{p_j} \varphi_j$, we have

$$(3.11) \qquad (P\boldsymbol{\varphi}', \boldsymbol{\varphi}')_{L^2} = \sum_{i,j=0}^{N} (L_{ij} \varphi_j, \varphi_i)_{L^2}.$$

Although this equality can be derived by direct calculation, it is also derived by the following argument which help us to understand the structure of the equations. Put $P\boldsymbol{\varphi}' = (G_1, \ldots, G_N)^{\mathrm{T}}$. Then, we have

$$\sum_{j=0}^{N} H^{p_j} \varphi_j = 0, \qquad \sum_{j=0}^{N} (L_{ij} - H^{p_i} L_{0j}) \varphi_j = G_i \quad \text{for} \quad i = 1, \ldots, N.$$

If we further introduce $\zeta = \sum_{j=0}^{N} L_{0j} \varphi_j$, then the above equations can be written as

$$\begin{pmatrix} 0 & \boldsymbol{l}^{\mathrm{T}} \\ -\boldsymbol{l} & L \end{pmatrix} \begin{pmatrix} \zeta \\ \boldsymbol{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{G} \end{pmatrix},$$

where $\boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_N)^{\mathrm{T}}$, $\boldsymbol{G} = (0, G_1, \ldots, G_N)^{\mathrm{T}}$, $L = (L_{ij})_{0 \leq i,j \leq N}$, and

$$(3.12) \qquad \boldsymbol{l} = \boldsymbol{l}(H) = (H^{p_0}, \ldots, H^{p_N})^{\mathrm{T}}.$$

By taking the L^2-inner product of the above equation with $(\zeta, \boldsymbol{\varphi}^{\mathrm{T}})^{\mathrm{T}}$, we obtain $(L\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L^2} = (\boldsymbol{G}, \boldsymbol{\varphi})_{L^2} = (P\boldsymbol{\varphi}', \boldsymbol{\varphi}')_{L^2}$, which implies (3.11).

By direct calculation we have

(3.13) $$I := \int_{\mathbf{R}^n} \mathrm{d}x \int_0^H \left\{ \left| \sum_{i=0}^N (z^{p_i} \nabla \varphi_i - p_i z^{p_i-1} \varphi_i \nabla b) \right|^2 + \left(\sum_{i=0}^N p_i z^{p_i-1} \varphi_i \right)^2 \right\} \mathrm{d}z$$

$$= \sum_{i,j=0}^N (L_{ij} \varphi_i, \varphi_j)_{L^2}.$$

In the case $p_i = 2i$ $(i = 0, 1, \dots, N)$, $\{z^{p_i}, z^{p_i-1}\}_{0 \le i \le N}$ are linearly independent so that

$$I \simeq \int_{\mathbf{R}^n} \mathrm{d}x \int_0^H \sum_{i=0}^N \left\{ \left(z^{2p_i} |\nabla \varphi_i|^2 + p_i^2 z^{2p_i-2} |\nabla b|^2 \varphi_i^2 \right) + p_i^2 z^{2p_i-2} \varphi_i^2 \right\} \mathrm{d}z$$

$$\simeq \int_{\mathbf{R}^n} \sum_{i=0}^N \left\{ H^{2p_i+1} |\nabla \varphi_i|^2 + p_i^2 H^{2p_i-1} (1 + |\nabla b|^2) \varphi_i^2 \right\} \mathrm{d}x,$$

which together with (3.11) and (3.13) implies the desired estimate in that case. We remark that the constant C can be taken independent of c_1. In the case $p_i = i$ $(i = 0, 1, \dots, N)$, we see that

$$I = \int_{\mathbf{R}^n} \mathrm{d}x \int_0^H \left\{ \left| \sum_{i=0}^{N-1} z^i \left(\nabla \varphi_i - (i+1) \varphi_{i+1} \nabla b \right) + z^N \nabla \varphi_N \right|^2 + \left(\sum_{i=1}^N i z^{i-1} \varphi_i \right)^2 \right\} \mathrm{d}z$$

$$\simeq \int_{\mathbf{R}^n} \left\{ \sum_{i=0}^{N-1} H^{2i+1} |\nabla \varphi_i - (i+1) \varphi_{i+1} \nabla b|^2 + H^{N+1} |\nabla \varphi_N|^2 + \sum_{i=1}^N H^{2i-1} \varphi_i^2 \right\} \mathrm{d}z$$

$$\gtrsim \sum_{i=0}^{N-1} \|\nabla \varphi_i - (i+1) \varphi_{i+1} \nabla b\|^2 + \|\nabla \varphi_N\|^2 + \sum_{i=1}^N \|\varphi_i\|^2$$

which together with (3.11) and (3.13) implies the desired estimate in that case. The other cases can be treated in the same way, so we omit it. $\square$

By this lemma, the explicit expression (3.10) of the operator P (see also (3.1)), and the standard theory of elliptic partial differential equations, we can obtain the following lemma.

Lemma 3.2 *Let h, c_0, M be positive constants and m an integer such that $m > n/2 + 1$. There exists a positive constant $C = C(h, c_0, M)$ such that if η and b satisfy*

$$\begin{cases} \|\eta\|_m + \|b\|_{W^{m,\infty}} \le M, \\ c_0 \le H(x) = h + \eta(x) - b(x) \quad for \quad x \in \mathbf{R}^n, \end{cases}$$

then for $1 \le k \le m$ we have

$$\|P^{-1} \mathbf{G}'\|_k \le C \|\mathbf{G}'\|_{k-2}.$$

We proceed to consider equation (3.7). Thanks of this lemma, we see that for a given $\mathbf{F}$ there exists a unique solution $\boldsymbol{\varphi}' = (\varphi_1, \dots, \varphi_N)^{\mathrm{T}}$ of (3.9). If we define φ_0 by (3.8), then $\boldsymbol{\varphi} = (\varphi_0, \dots, \varphi_N)^{\mathrm{T}}$ is a solution of (3.7). More precisely, we have the following lemma.

Lemma 3.3 *Under the hypothesis of Lemma 3.2, for any $\mathbf{F} = (F_0, \dots, F_N)^{\mathrm{T}}$ satisfying $\nabla F_0 \in H^{k-1}$ and $(F_1, \dots, F_N) \in H^{k-2}$ with $1 \le k \le m$ there exists a unique solution $\boldsymbol{\varphi} = (\varphi_0, \dots, \varphi_N)^{\mathrm{T}}$ of (3.7) satisfying*

$$\|\nabla \varphi_0\|_{k-1} + \|(\varphi_1, \dots, \varphi_N)\|_k \le C (\|\nabla F_0\|_{k-1} + \|(F_1, \dots, F_N)\|_{k-2}),$$

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where $C = C(h, c_0, M) > 0$. If, in addition, $F_0 \in L^2(\mathbf{R}^n)$, then we have

$$\|\varphi\|_k \le C(\|F_0\|_k + \|(F_1, \ldots, F_N)\|_{k-2}).$$

Now, we have established the solvability of φ to equation (3.7). We go back to consider the linearized equations in (3.3). In the following of this section we denote lower order terms and terms related with the given functions $f_0, \ldots, f_{N+1}$ by the same symbol LOT, which may change from line to line. We introduce a symmetric matrix $A(H)$ depending on the water depth H by

$$(3.14) \qquad A(H) = (a_{ij}(H))_{0 \le i,j \le N}, \quad a_{ij}(H) = \frac{1}{p_i + p_j + 1} H^{p_i + p_j + 1}.$$

We also use a matrix $\tilde{A}(H)$ defined by

$$(3.15) \qquad \tilde{A}(H) = \begin{pmatrix} 0 & \boldsymbol{l}(H)^{\mathrm{T}} \\ -\boldsymbol{l}(H) & A(H) \end{pmatrix},$$

where $\boldsymbol{l}(H)$ is defined by (3.12). Since $L_{ij}(H, b) = -a_{ij}(H)\Delta + LOT$, it follows from (3.4) and (3.5) that

$$\begin{cases} \displaystyle\sum_{j=0}^{N} (a_{ij}(H) - H^{p_i} a_{0j}(H))\Delta\partial_t\psi_j = \nabla \cdot (LOT) + LOT \quad \text{for} \quad i = 1, \ldots, N, \\ \displaystyle\sum_{j=0}^{N} H^{p_j}\Delta\partial_t\psi_j + \sum_{j=0}^{N} H^{p_j}(\boldsymbol{u} \cdot \nabla)\Delta\psi_j + \Delta(a\zeta) = \nabla \cdot (LOT) + LOT. \end{cases}$$

We can rewrite these equations in a matrix form as

$$(3.16) \qquad \tilde{A}(H) \begin{pmatrix} \sum_{j=0}^{N} a_{0j}(H)\Delta\partial_t\psi_j \\ \Delta\partial_t\boldsymbol{\psi} \end{pmatrix} + \begin{pmatrix} \sum_{j=0}^{N} H^{p_j}(\boldsymbol{u} \cdot \nabla)\Delta\psi_j + \Delta(a\zeta) \\ \mathbf{0} \end{pmatrix}$$
$$= \nabla \cdot (LOT) + LOT,$$

where $\boldsymbol{\psi} = (\psi_0, \ldots, \psi_N)^{\mathrm{T}}$. Since $\det \tilde{A}(H) = H^{2\sum_{i=0}^{N} p_i + N} \det \tilde{A}_0$, where $\tilde{A}_0$ is the matrix defined in Section 2, the matrix $\tilde{A}(H)$ is nonsingular and its inverse matrix can be written as

$$(3.17) \qquad \tilde{A}(H)^{-1} = \begin{pmatrix} q(H) & \boldsymbol{q}(H)^{\mathrm{T}} \\ -\boldsymbol{q}(H) & Q(H) \end{pmatrix}$$

with a symmetric matrix $Q(H)$. If it causes no confusion, we omit the dependence of H in the notation. Then, it holds that

$$(3.18) \qquad \boldsymbol{l} \cdot \boldsymbol{q} = -1, \quad A\boldsymbol{q} = -q\boldsymbol{l}, \quad q = \frac{\det A}{\det \tilde{A}} = H\frac{\det A_0}{\det \tilde{A}_0}.$$

Therefore, it follows from (3.16) that

$$(3.19) \qquad -A\Delta\partial_t\boldsymbol{\psi} = q\boldsymbol{l}\{\boldsymbol{l}^{\mathrm{T}}(\boldsymbol{u} \cdot \nabla)\Delta\boldsymbol{\psi} + \Delta(a\zeta)\} + \nabla \cdot (LOT) + LOT.$$

On the other hand, it follows from the first equation in (3.3) that

$$\boldsymbol{l}(\partial_t\zeta + \boldsymbol{u} \cdot \nabla\zeta) + A\Delta\boldsymbol{\psi} = LOT.$$

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Taking the Euclidean inner product of this equation with $-a\boldsymbol{q}$ and using the relations in (3.18) we obtain

$$a(\partial_t \zeta + \boldsymbol{u} \cdot \nabla \zeta) + a\Delta(q\boldsymbol{l}^{\mathrm{T}}\boldsymbol{\psi}) = LOT.$$

We can write this equation and (3.19) in a matrix form as

$$(3.20) \qquad \begin{pmatrix} a & \boldsymbol{0}^{\mathrm{T}} \\ \boldsymbol{0} & -A\Delta \end{pmatrix} \partial_t \begin{pmatrix} \zeta \\ \boldsymbol{\psi} \end{pmatrix} + \begin{pmatrix} a\boldsymbol{u} \cdot \nabla & a\Delta(q\boldsymbol{l}^{\mathrm{T}} \cdot) \\ -q\boldsymbol{l}\Delta(a \cdot) & -q\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}(\boldsymbol{u} \cdot \nabla)\Delta \end{pmatrix} \begin{pmatrix} \zeta \\ \boldsymbol{\psi} \end{pmatrix}$$

$$= \begin{pmatrix} LOT \\ \nabla \cdot (LOT) + LOT \end{pmatrix}.$$

Here the operator in the second term in the left-hand side is skew-symmetric in $L^2(\mathbf{R}^n)$ modulo lower order terms. Since the matrix $A = A(H)$ can be written as

$$A(H) = H\mathrm{diag}(H^{p_0}, \ldots, H^{p_N})A_0\mathrm{diag}(H^{p_0}, \ldots, H^{p_N})$$

and we have shown the positivity of the matrix A_0 in Proposition 2.1, $A(H)$ is strictly positive under the strict positivity of the water depth H. Therefore, under the sign condition $a > 0$ (3.20) forms a symmetric positive system, so that the corresponding energy function is defined by

$$E(\zeta, \boldsymbol{\psi}) = (a\zeta, \zeta)_{L^2} + \sum_{k=1}^{n}(A\partial_k\boldsymbol{\psi}, \partial_k\boldsymbol{\psi})_{L^2} + \|\boldsymbol{\psi}'\|^2.$$

In the following sections we use this symmetric structure of the Isobe–Kakinuma model to construct the solution and derive an energy estimate.

4   Construction of the solution to a reduced system

In this section we transform the Isobe–Kakinuma model (1.3) into a system of equations for which the hypersurface $t = 0$ is noncharacteristic by using the necessary condition (1.6) for the existence of the solution and construct the solution of the initial value problem to the transformed system by using a standard parabolic regularization.

By using the notation introduced in the previous section, the Isobe–Kakinuma model (1.3) can be written simply as

$$(4.1) \qquad \begin{cases} H^{p_i}\partial_t\eta - \displaystyle\sum_{j=0}^{N}L_{ij}\phi_j = 0 \qquad \text{for} \quad i = 0, 1, \ldots, N, \\ \displaystyle\sum_{j=0}^{N}H^{p_j}\partial_t\phi_j = F_0, \end{cases}$$

where

$$(4.2) \qquad F_0 = -g\eta - \frac{1}{2}(|\boldsymbol{u}|^2 + w^2).$$

The necessary condition (1.6) can also be written simply as

$$\sum_{j=0}^{N}(L_{ij} - H^{p_i}L_{0j})\phi_j = 0 \qquad \text{for} \quad i = 1, \ldots, N.$$

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We first derive an evolution equation for $\boldsymbol{\phi} = (\phi_0, \ldots, \phi_N)^{\mathrm{T}}$. Since the left-hand side of the above equation does not contain the term ∇H, differentiating this with respect to the time t we obtain

$$\sum_{j=0}^{N}(L_{ij} - H^{p_i}L_{0j})\partial_t\phi_j = f_i\partial_t\eta \qquad \text{for} \quad i = 1, \ldots, N,$$

where

$$(4.3) \qquad f_i = -\frac{\partial}{\partial H}\sum_{j=0}^{N}(L_{ij} - H^{p_i}L_{0j})\phi_j$$

$$= p_iH^{p_i-1}\left\{\nabla b\cdot\boldsymbol{u} - w - \sum_{j=0}^{N}\left(\frac{1}{p_j+1}H^{p_j+1}\Delta\phi_j - \frac{p_j}{p_j}H^{p_j}\nabla\cdot(\phi_j\nabla b)\right)\right\}$$

for $i = 1, \ldots, N$. We use the first equation in (4.1) with $i = 0$ to remove the time derivative $\partial_t\eta$ from the above equation and obtain

$$(4.4) \qquad \sum_{j=0}^{N}(L_{ij} - H^{p_i}L_{0j})\partial_t\phi_j = F_i \qquad \text{for} \quad i = 1, \ldots, N,$$

where

$$(4.5) \qquad F_i = f_i\sum_{j=0}^{N}L_{0j}\phi_j \qquad \text{for} \quad i = 1, \ldots, N.$$

By using the operator $\mathscr{L}$ introduced by (3.6), the second equation in (4.1) and (4.4) constitute the equation

$$\mathscr{L}\partial_t\boldsymbol{\phi} = \boldsymbol{F},$$

where $\boldsymbol{F} = (F_0, \ldots, F_N)^{\mathrm{T}}$. This is the evolution equation for $\boldsymbol{\phi}$. We proceed to derive an appropriate evolution equation for η. Let $\boldsymbol{q} = (q_0, \ldots, q_N)^{\mathrm{T}} = (q_0(H), \ldots, q_N(H))^{\mathrm{T}}$ be the rational functions of H defined by (3.17). In view of the first relation in (3.18) and the arguments in the previous section, we multiply the first equation in (4.1) by q_i and adding the resulting equations over $i = 0, 1, \ldots, N$ to obtain

$$\partial_t\eta = F_{N+1},$$

where

$$(4.6) \qquad F_{N+1} = -\sum_{i,j=0}^{N}q_iL_{ij}\phi_j.$$

To summarize we have reduced the Isobe–Kakinuma model (4.1) to

$$(4.7) \qquad \begin{cases} \mathscr{L}\partial_t\boldsymbol{\phi} = \boldsymbol{F}, \\ \partial_t\eta = F_{N+1}. \end{cases}$$

We note that $\boldsymbol{F}$ and F_{N+1} do not contain any time derivatives and that $\mathscr{L}$ is invertible thanks to Lemma 3.3. Therefore, the hypersurface $t = 0$ is not characteristic any more for the above reduced equations. In the rest of this section we consider the initial value problem to (4.7) under the initial conditions

$$(4.8) \qquad (\eta, \phi_0, \ldots, \phi_N) = (\eta_{(0)}, \phi_{0(0)}, \ldots, \phi_{N(0)}) \qquad \text{at} \quad t = 0.$$

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Remark 4.1 *As mentioned in Remark 1.2 (2), we have to express $\partial_t\phi_1(x,0),\ldots,\partial_t\phi_N(x,0)$ in terms of the initial data and b. Such an expression is given by $\partial_t\boldsymbol{\phi}(\cdot,0) = \mathscr{L}^{-1}\boldsymbol{F}|_{t=0}$.*

We will construct the solution to the initial value problem for the reduced system (4.7)–(4.8) by a standard parabolic regularization of the equations, that is,

$$(4.9) \qquad \begin{cases} \mathscr{L}(\partial_t\boldsymbol{\phi} - \varepsilon\Delta\boldsymbol{\phi}) = \boldsymbol{F}, \\ \partial_t\eta - \varepsilon\Delta\eta = F_{N+1}, \end{cases}$$

where $\varepsilon > 0$ is a small regularized parameter. By the definitions (4.2), (4.5), and (4.6) of $\boldsymbol{F} = (F_0,\ldots,F_N)^{\mathrm{T}}$ and F_{N+1} (see also (4.3), (3.1), and (3.2)) we see that

$$(4.10) \qquad \begin{cases} F_0 = F_0(\eta, H, \boldsymbol{\phi}', \nabla\boldsymbol{\phi}, \nabla b), \\ F_i = F_i(H, \nabla H, \boldsymbol{\phi}', \nabla\boldsymbol{\phi}, \Delta\boldsymbol{\phi}, \nabla b, \Delta b) \quad \text{for} \quad i = 1,\ldots,N+1, \end{cases}$$

where $\boldsymbol{\phi}' = (\phi_1,\ldots,\phi_N)^{\mathrm{T}}$. Therefore, thanks to Lemma 3.3, $\mathscr{L}^{-1}\boldsymbol{F}$ behaves as if it is a function of $(\eta, H, \boldsymbol{\phi}', \nabla\boldsymbol{\phi}, \nabla b)$, so that we can show the following lemma.

Lemma 4.2 *Let g, h, c_0, M_0 be positive constants and m an integer such that $m > n/2 + 1$. Suppose that the initial data $(\eta_{(0)}, \phi_{0(0)}, \ldots, \phi_{N(0)})$ and b satisfy the conditions in (1.9), then for any $\varepsilon > 0$ there exists a maximal existence time $T_\varepsilon > 0$ such that the initial value problem (4.9) and (4.8) has a unique solution $(\eta^\varepsilon, \boldsymbol{\phi}^\varepsilon)$ satisfying*

$$\eta^\varepsilon, \nabla\phi_0^\varepsilon \in C([0,T_\varepsilon); H^m), \quad \phi_1^\varepsilon, \ldots, \phi_N^\varepsilon \in C([0,T_\varepsilon); H^{m+1}).$$

We proceed to derive uniform estimates of the solution $(\eta^\varepsilon, \boldsymbol{\phi}^\varepsilon)$ with respect to the regularized parameter $\varepsilon \in (0,1]$ for a time interval $[0,T]$ independent of ε. To this end, we make use of a good symmetric structure of the Isobe–Kakinuma model discussed in the previous section. In order to simplify the notation we write $(\eta, \boldsymbol{\phi})$ in place of $(\eta^\varepsilon, \boldsymbol{\phi}^\varepsilon)$ in the following. Concerning the generalized Rayleigh–Taylor sign condition, a regularized version of the function a defined by (1.8) is given by

$$(4.11) \qquad a^\varepsilon = g + \sum_{i=0}^{N} p_i H^{p_i-1}(\partial_t\phi_i - \varepsilon\Delta\phi_i)$$

$$+ \frac{1}{2}\sum_{i,j=0}^{N}\Big\{(p_i + p_j)H^{p_i+p_j-1}\nabla\phi_i \cdot \nabla\phi_j - 2p_i(p_i + p_j - 1)H^{p_i+p_j-2}\phi_i\nabla b \cdot \nabla\phi_j$$

$$+ p_i p_j(p_i + p_j - 2)H^{p_i+p_j-3}(1 + |\nabla b|^2)\phi_i\phi_j\Big\}.$$

Here, we have $\partial_t\boldsymbol{\phi} - \varepsilon\Delta\boldsymbol{\phi} = \mathscr{L}^{-1}\boldsymbol{F}$ so that $a^\varepsilon(x,0) = a(x,0)$. See also Remark 4.1. Therefore, by the sign condition in (1.9) we have

$$(4.12) \qquad\qquad a^\varepsilon(x,0) \geq c_0 \quad \text{for} \quad x \in \mathbf{R}^n.$$

We will derive a regularized version of the symmetric positive system (3.20) for $(\eta, \boldsymbol{\phi})$. First, we derive an evolution equation for $\boldsymbol{\phi}$. The first equation in (4.9) is written as

$$(4.13) \qquad \begin{cases} \boldsymbol{l} \cdot (\partial_t\boldsymbol{\phi} - \varepsilon\Delta\boldsymbol{\phi}) = F_0, \\ \mathscr{L}_i(\partial_t\boldsymbol{\phi} - \varepsilon\Delta\boldsymbol{\phi}) = F_i \quad \text{for} \quad i = 1,\ldots,N, \end{cases}$$

where $\boldsymbol{l} = \boldsymbol{l}(H) = (H^{p_0}, \ldots, H^{p_N})^{\mathrm{T}}$. Here we remark that F_i $(i = 1, \ldots, N)$ is a correction of lower order terms and its explicit form has no importance whereas F_0 contains principal terms and we have to treat it carefully. In other words, the second equation in (4.13) is quasilinear whereas the first one is fully nonlinear. Applying Δ to the first equation in (4.13) we have

$$\boldsymbol{l}(H) \cdot \Delta(\partial_t \boldsymbol{\phi} - \varepsilon \Delta \boldsymbol{\phi}) + (\boldsymbol{l}^{(1)}(H) \cdot (\partial_t \boldsymbol{\phi} - \varepsilon \Delta \boldsymbol{\phi})) \Delta H - \Delta F_0$$
$$= -([\Delta, \boldsymbol{l}(H)] - \boldsymbol{l}^{(1)}(H)(\Delta H)) \cdot \mathscr{L}^{-1} \boldsymbol{F},$$

where $\boldsymbol{l}^{(1)}(H) = (0, p_1 H^{p_1 - 1}, \ldots, p_N H^{p_N - 1})^{\mathrm{T}}$ is the derivative of $\boldsymbol{l}(H)$ with respect to H. Here we see that

$$-\Delta F_0 = g \Delta \eta + \boldsymbol{u} \cdot \Delta \boldsymbol{u} + w \Delta w + \frac{1}{2} \{ (\Delta(\boldsymbol{u} \cdot \boldsymbol{u}) - 2\boldsymbol{u} \cdot \Delta \boldsymbol{u}) + (\Delta(w^2) - 2w \Delta w) \},$$

where

$$\Delta \boldsymbol{u} = \sum_{i=0}^{N} \{ H^{p_i} \nabla \Delta \phi_i + p_i (H^{p_i - 1} \nabla \phi_i - (p_i - 1) H^{p_i - 2} \phi_i \nabla b)(\Delta \eta)$$
$$+ ([\Delta, H^{p_i}] - p_i H^{p_i - 1}(\Delta H) - p_i H^{p_i - 1}(\Delta b)) \nabla \phi_i - p_i H^{p_i - 1} \Delta(\phi_i \nabla b)$$
$$- p_i ([\Delta, H^{p_i - 1}] - (p_i - 1) H^{p_i - 2}(\Delta H) - (p_i - 1) H^{p_i - 2}(\Delta b))(\phi_i \nabla b) \},$$

$$\Delta w = \sum_{i=0}^{N} \{ p_i (p_i - 1) H^{p_i - 2} \phi_i \Delta \eta + p_i H^{p_i - 1} \Delta \phi_i$$
$$+ p_i ([\Delta, H^{p_i - 1}] - (p_i - 1) H^{p_i - 2}(\Delta H) - (p_i - 1) H^{p_i - 2}(\Delta b)) \phi_i \}.$$

Therefore, we obtain

(4.14) $$\qquad \boldsymbol{l}(H) \cdot \Delta(\partial_t \boldsymbol{\phi} - \varepsilon \Delta \boldsymbol{\phi}) + \boldsymbol{l}(H) \cdot (\boldsymbol{u} \cdot \nabla) \Delta \boldsymbol{\phi} + a^\varepsilon \Delta \eta = G_{00},$$

where

(4.15) $$\quad G_{00} = -([\Delta, \boldsymbol{l}(H)] - \boldsymbol{l}^{(1)}(H)(\Delta H)) \cdot \mathscr{L}^{-1} \boldsymbol{F} + (\boldsymbol{l}^{(1)}(H) \cdot (\mathscr{L}^{-1} \boldsymbol{F})) \Delta b$$
$$- \frac{1}{2} \{ (\Delta(\boldsymbol{u} \cdot \boldsymbol{u}) - 2\boldsymbol{u} \cdot \Delta \boldsymbol{u}) + (\Delta(w^2) - 2w \Delta w) \}$$
$$- \boldsymbol{u} \cdot \sum_{i=0}^{N} \{ ([\Delta, H^{p_i}] - p_i H^{p_i - 1}(\Delta H) - p_i H^{p_i - 1}(\Delta b)) \nabla \phi_i - p_i H^{p_i - 1} \Delta(\phi_i \nabla b)$$
$$- p_i ([\Delta, H^{p_i - 1}] - (p_i - 1) H^{p_i - 2}(\Delta H) - (p_i - 1) H^{p_i - 2}(\Delta b))(\phi_i \nabla b) \}$$
$$- w \sum_{i=0}^{N} \{ p_i H^{p_i - 1} \Delta \phi_i$$
$$+ p_i ([\Delta, H^{p_i - 1}] - (p_i - 1) H^{p_i - 2}(\Delta H) - (p_i - 1) H^{p_i - 2}(\Delta b)) \phi_i \}.$$

In view of (3.6) and (3.1) we divide the operator $\mathscr{L}_i$ into its principal part and the remainder part $\mathscr{L}_i^{\mathrm{low}}$ as

$$\mathscr{L}_i \boldsymbol{\psi} = (-\boldsymbol{a}_i(H) + H^{p_i} \boldsymbol{a}_0(H)) \cdot \Delta \boldsymbol{\psi} + \mathscr{L}_i^{\mathrm{low}} \boldsymbol{\psi} \quad \text{for} \quad i = 1, \ldots, N,$$

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where $A(H) = (\boldsymbol{a}_0(H), \ldots, \boldsymbol{a}_N(H))$ is the matrix defined by (3.14) and

$$(4.16) \qquad \mathscr{L}_i^{\text{low}} \boldsymbol{\psi} = \sum_{j=0}^N \left(-\frac{p_i p_j}{p_j(p_i + p_j)} H^{p_i + p_j} \nabla \cdot (\psi_j \nabla b) - \frac{p_i}{p_i + p_j} H^{p_i + p_j} \nabla b \cdot \nabla \psi_j \right.$$
$$\left. + \frac{p_i p_j}{p_i + p_j - 1} H^{p_i + p_j - 1}(1 + |\nabla b|^2)\psi_j \right).$$

Therefore, we obtain

$$(4.17) \qquad (\boldsymbol{a}_i(H) - H^{p_i}\boldsymbol{a}_0(H)) \cdot \Delta(\partial_t \boldsymbol{\phi} - \varepsilon \Delta \boldsymbol{\phi}) = G_{0i} \quad \text{for} \quad i = 1, \ldots, N,$$

where

$$(4.18) \qquad G_{0i} = -F_i + \mathscr{L}_i^{\text{low}} \mathscr{L}^{-1} \boldsymbol{F}.$$

We can rewrite (4.14) and (4.17) in a matrix form as

$$\tilde{A}(H) \begin{pmatrix} \boldsymbol{a}_0 \cdot \Delta(\partial_t \boldsymbol{\phi} - \varepsilon \Delta \boldsymbol{\phi}) \\ \Delta(\partial_t \boldsymbol{\phi} - \varepsilon \Delta \boldsymbol{\phi}) \end{pmatrix} + \begin{pmatrix} \boldsymbol{l}(H)^{\mathrm{T}}(\boldsymbol{u} \cdot \nabla)\Delta \boldsymbol{\phi} + a^\varepsilon \Delta \eta \\ \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} G_{00} \\ \boldsymbol{G}_0 \end{pmatrix},$$

where $\tilde{A}(H)$ is the matrix in (3.15) and $\boldsymbol{G}_0 = (0, G_{01}, \ldots, G_{0N})^{\mathrm{T}}$. Therefore, using the notation in (3.17) and the relations in (3.18) we have

$$A(H)\Delta(\partial_t \boldsymbol{\phi} - \varepsilon \Delta \boldsymbol{\phi}) + q(H)\boldsymbol{l}(H)\{\boldsymbol{l}(H)^{\mathrm{T}}(\boldsymbol{u} \cdot \nabla)\Delta \boldsymbol{\phi} + a^\varepsilon \Delta \eta\} = \boldsymbol{G},$$

where

$$(4.19) \qquad \boldsymbol{G} = q(H)\boldsymbol{l}(H)G_{00} + A(H)Q(H)\boldsymbol{G}_0.$$

This is the desired equation for $\boldsymbol{\phi}$.

Secondly, we derive an evolution equation for η. In view of (3.1) we divide the operator L_{ij} into its principal part and the remainder part L_{ij}^{low} as

$$L_{ij}\phi_j = -a_{ij}(H)\Delta \phi_j - H^{p_i}(H^{p_j}\nabla \phi_j - p_j H^{p_j - 1}\phi_j \nabla b) \cdot \nabla \eta + L_{ij}^{\text{low}}\phi_j,$$

where

$$(4.20) \qquad L_{ij}^{\text{low}}\phi_j = H^{p_i}(H^{p_j}\nabla \phi_j - p_j H^{p_j - 1}\phi_j \nabla b) \cdot \nabla b + \frac{p_j}{p_i + p_j} H^{p_i + p_j} \nabla \cdot (\phi_j \nabla b)$$
$$- \frac{p_i}{p_i + p_j} H^{p_i + p_j} \nabla b \cdot \nabla \phi_j + \frac{p_i p_j}{p_i + p_j - 1} H^{p_i + p_j - 1}(1 + |\nabla b|^2)\phi_j.$$

Thanks to the relations in (3.18), the second equation in (4.9) can be written as

$$\partial_t \eta - \varepsilon \Delta \eta + \boldsymbol{u} \cdot \nabla \eta + q(H)\boldsymbol{l}(H)^{\mathrm{T}}\Delta \boldsymbol{\phi} = G_0,$$

where

$$(4.21) \qquad G_0 = -\sum_{i,j=0}^N q_i(H)L_{ij}^{\text{low}}\phi_j.$$

This is the desired equation for η.

To summarize, we have derived the equations

$$(4.22) \qquad \begin{cases} \partial_t \eta - \varepsilon \Delta \eta + \boldsymbol{u} \cdot \nabla \eta + q(H)\boldsymbol{l}(H)^{\mathrm{T}}\Delta \boldsymbol{\phi} = G_0, \\ A(H)\Delta(\partial_t \boldsymbol{\phi} - \varepsilon \Delta \boldsymbol{\phi}) + q(H)\boldsymbol{l}(H)\{\boldsymbol{l}(H)^{\mathrm{T}}(\boldsymbol{u} \cdot \nabla)\Delta \boldsymbol{\phi} + a^\varepsilon \Delta \eta\} = \boldsymbol{G}. \end{cases}$$

Using this we can derive uniform estimate of the solution $(\eta^\varepsilon, \boldsymbol{\phi}^\varepsilon)$ of the initial value problem to the regularized equation (4.9).

Lemma 4.3 *Under the hypothesis of Lemma 4.2, there exist a time $T > 0$ and a constant $C > 0$ independent of ε such that the solution $(\eta^\varepsilon, \phi^\varepsilon)$ obtained in Lemma 4.2 satisfies the uniform estimate*

$$(4.23) \qquad \sup_{0 \le t \le T} \left(\|\eta^\varepsilon(t)\|_m^2 + \|\nabla\phi_0^\varepsilon(t)\|_m^2 + \|(\phi_1^\varepsilon(t), \ldots, \phi_N^\varepsilon(t))\|_{m+1}^2 \right)$$

$$+ \varepsilon \int_0^T \left(\|\eta^\varepsilon(t)\|_{m+1}^2 + \|\nabla\phi_0^\varepsilon(t)\|_{m+1}^2 + \|(\phi_1^\varepsilon(t), \ldots, \phi_N^\varepsilon(t))\|_{m+2}^2 \right) \mathrm{d}t \le C$$

for $0 < \varepsilon \le 1$.

Proof. Once again we simply write (η, ϕ) in place of $(\eta^\varepsilon, \phi^\varepsilon)$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index satisfying $1 \le |\alpha| \le m$. Without loss of generality, we can assume that $\alpha_1 \ge 1$. Applying the differential operator $a^\varepsilon \partial^\alpha$ to the first equation in (4.22) we have

$$(4.24) \qquad a^\varepsilon \partial_t \partial^\alpha \eta - \varepsilon \nabla \cdot (a^\varepsilon \nabla \partial^\alpha \eta) + a^\varepsilon \boldsymbol{u} \cdot \nabla \partial^\alpha \eta + \sum_{k=1}^n \partial_k (a^\varepsilon q \boldsymbol{l}^\mathrm{T} \partial_k \partial^\alpha \phi) = F_{0,\alpha},$$

where

$$(4.25) \quad F_{0,\alpha} = a^\varepsilon (\partial^\alpha G_0 - [\partial^\alpha, \boldsymbol{u}] \cdot \nabla\eta - [\partial^\alpha, q\boldsymbol{l}^\mathrm{T}]\Delta\phi) - \varepsilon\nabla a^\varepsilon \cdot \nabla\partial^\alpha\eta + \sum_{k=1}^n (\partial_k(a^\varepsilon q\boldsymbol{l}^\mathrm{T}))\partial_k\partial^\alpha\phi.$$

Here and in what follows, for simplicity, we omit the dependence of H in the notation. Applying the differential operator ∂^α to the second equation in (4.22) we have

$$(4.26) \qquad -\sum_{k=1}^n \partial_k(A\partial_k\partial_t\partial^\alpha\phi) + \varepsilon\Delta(A\Delta\partial^\alpha\phi) - \sum_{k=1}^n \partial_k\left\{ q\boldsymbol{l}\boldsymbol{l}^\mathrm{T}(\boldsymbol{u} \cdot \nabla)\partial_k\partial^\alpha\phi + a^\varepsilon q\boldsymbol{l}\partial_k\partial^\alpha\eta \right\}$$

$$= \sum_{k=1}^n \partial_k \boldsymbol{F}_{k,\alpha},$$

where

$$(4.27) \qquad \boldsymbol{F}_{k,\alpha} = -\delta_{1k}\partial^{\alpha'}\left\{ \boldsymbol{G} + \sum_{l=1}^n \left((\partial_l A)\partial_l \mathscr{L}^{-1}\boldsymbol{F} + [\partial_l, q\boldsymbol{l}\boldsymbol{l}^\mathrm{T}(\boldsymbol{u} \cdot \nabla)]\partial_l\phi + (\partial_l(a^\varepsilon q\boldsymbol{l}))\partial_l\eta \right) \right\}$$

$$+ \varepsilon(\partial_k A)\Delta\partial^\alpha\phi + [\partial^\alpha, A]\partial_k\mathscr{L}^{-1}\boldsymbol{F} + [\partial^\alpha, q\boldsymbol{l}\boldsymbol{l}^\mathrm{T}(\boldsymbol{u} \cdot \nabla)]\partial_k\phi + [\partial^\alpha, a^\varepsilon q\boldsymbol{l}]\partial_k\eta.$$

Here, $\alpha' = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_n)$ and δ_{1k} is the Kronecker delta. (4.24) and (4.26) can be written in the matrix form as

$$(4.28) \quad \begin{pmatrix} a^\varepsilon & \boldsymbol{0}^\mathrm{T} \\ \boldsymbol{0} & -\sum_{k=1}^n \partial_k(A\partial_k \,\cdot\,) \end{pmatrix} \partial_t \begin{pmatrix} \partial^\alpha\eta \\ \partial^\alpha\phi \end{pmatrix} + \varepsilon \begin{pmatrix} -\nabla\cdot(a^\varepsilon\nabla\,\cdot\,) & \boldsymbol{0}^\mathrm{T} \\ \boldsymbol{0} & \Delta(A\Delta\,\cdot\,) \end{pmatrix} \begin{pmatrix} \partial^\alpha\eta \\ \partial^\alpha\phi \end{pmatrix}$$

$$+ \begin{pmatrix} a^\varepsilon\boldsymbol{u}\cdot\nabla & \sum_{k=1}^n \partial_k(a^\varepsilon q\boldsymbol{l}^\mathrm{T}\partial_k\,\cdot\,) \\ -\sum_{k=1}^n \partial_k(a^\varepsilon q\boldsymbol{l}\partial_k\,\cdot\,) & -\sum_{k=1}^n \partial_k(q\boldsymbol{l}\boldsymbol{l}^\mathrm{T}(\boldsymbol{u}\cdot\nabla)\partial_k\,\cdot\,) \end{pmatrix} \begin{pmatrix} \partial^\alpha\eta \\ \partial^\alpha\phi \end{pmatrix}$$

$$= \begin{pmatrix} F_{0,\alpha} \\ \sum_{k=1}^n \partial_k \boldsymbol{F}_{k,\alpha} \end{pmatrix},$$

which forms a symmetric positive system.

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In view of (4.28) we define an energy function $\mathscr{E}_m(t)$ and a dissipation function $\mathscr{D}_m(t)$ by

$$\mathscr{E}_m(t) = \sum_{|\alpha|\leq m}\left\{(a^\varepsilon\partial^\alpha\eta(t),\partial^\alpha\eta(t))_{L^2} + \sum_{k=1}^n(A\partial_k\partial^\alpha\phi(t),\partial_k\partial^\alpha\phi(t))_{L^2}\right\} + \|\phi'(t)\|^2,$$

$$\mathscr{D}_m(t) = \sum_{|\alpha|\leq m}\left\{(a^\varepsilon\nabla\partial^\alpha\eta(t),\nabla\partial^\alpha\eta(t))_{L^2} + (A\Delta\partial^\alpha\phi(t),\Delta\partial^\alpha\phi(t))_{L^2}\right\},$$

which will be equivalent to

$$E_m(t) = \|\eta(t)\|_m^2 + \|\nabla\phi_0(t)\|_m^2 + \|\phi'(t)\|_{m+1}^2, \quad D_m(t) = \|\nabla\eta(t)\|_m^2 + \|\Delta\phi(t)\|_m^2,$$

respectively, where $\phi' = (\phi_1,\ldots,\phi_N)^{\mathrm{T}}$. In view of (1.9), (4.10), (4.11), (4.12), and Lemma 3.3, we see that there exists a constant $C_0 = C_0(g,h,c_0,M_0) > 0$ such that

$$c_0 \leq H(x,0) \leq C_0, \quad c_0 \leq a^\varepsilon(x,0) \leq C_0 \qquad \text{for} \quad x \in \mathbf{R}^n.$$

Now, we assume that

$$(4.29) \qquad E_m(t) + \varepsilon\int_0^t E_{m+1}(\tau)\mathrm{d}\tau \leq M_1, \quad \frac{c_0}{2} \leq H(x,t) \leq 2C_0, \quad \frac{c_0}{2} \leq a^\varepsilon(x,t) \leq 2C_0$$

for $0 \leq t \leq T$ and $x \in \mathbf{R}^n$, where the constant M_1 and the time T will be determined later. In the following we simply write the constants depending only on (g,h,c_0,C_0,M_0) by C_1 and the constants depending also on M_1 by C_2, which may change from line to line. Then, it holds that

$$C_1^{-1}E_m(t) \leq \mathscr{E}_m(t) \leq C_1 E_m(t), \quad C_1^{-1}D_m(t) \leq \mathscr{D}_m(t) \leq C_1 D_m(t)$$

for $0 \leq t \leq T$. We are going to evaluate the evolution of the energy function $\mathscr{E}_m(t)$. To this end, we make use of the symmetric form (4.28) for the case $1 \leq |\alpha| \leq m$ and of (4.9) directly for the case $|\alpha| = 0$. Then, by integration by parts we see that

$$(4.30) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_m(t) + 2\varepsilon(\mathscr{D}_m(t) + \|\nabla\phi'(t)\|^2)$$

$$= \sum_{|\alpha|\leq m}\left\{((\partial_t a^\varepsilon)\partial^\alpha\eta,\partial^\alpha\eta)_{L^2} + \sum_{k=1}^n((\partial_t A)\partial_k\partial^\alpha\phi,\partial_k\partial^\alpha\phi)_{L^2}\right\}$$

$$+ \sum_{1\leq|\alpha|\leq m}\left\{((\nabla\cdot(a^\varepsilon\boldsymbol{u}))\partial^\alpha\eta,\partial^\alpha\eta)_{L^2} + \sum_{k=1}^n(\left(\sum_{l=1}^n\partial_l(u_l q\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}})\right)\partial_k\partial^\alpha\phi,\partial_k\partial^\alpha\phi)_{L^2}\right.$$

$$\left. + 2(F_{0,\alpha},\partial^\alpha\eta)_{L^2} - 2\sum_{k=1}^n(\boldsymbol{F}_{k,\alpha},\partial_k\partial^\alpha\phi)_{L^2}\right\}$$

$$+ 2(a^\varepsilon F_{N+1},\eta)_{L^2} + 2\sum_{k=1}^n(\partial_k\mathscr{L}^{-1}\boldsymbol{F},A\partial_k\phi)_{L^2} + 2((\mathscr{L}^{-1}\boldsymbol{F})',\phi')_{L^2}$$

$$- 2\varepsilon(\nabla\eta,\eta\nabla a^\varepsilon)_{L^2} - 2\varepsilon\sum_{k=1}^n(\Delta\phi,(\partial_k A)\partial_k\phi)_{L^2},$$

where $(\mathscr{L}^{-1}\boldsymbol{F})'$ is the last N components of $\mathscr{L}^{-1}\boldsymbol{F}$. Here, we see easily that $\|(\boldsymbol{u},w)\|_m \leq C_2$, $\|\mathscr{L}^{-1}\boldsymbol{F}\|_m \leq C_2$, and $\|\nabla a^\varepsilon\|_{m-1} \leq C_2$. It follows from (4.25) and (4.27) (see also (4.15), (4.16), and (4.18)–(4.21)) that $\|F_{0,\alpha}\| \leq C_2(1+\varepsilon\|\nabla\eta\|_m)$ and $\|\boldsymbol{F}_{k,\alpha}\| \leq C_2(1+\varepsilon\|\Delta\phi\|_m)$ for $k = 1,\ldots,n$.

We need to evaluate $\partial_t a^\varepsilon$. Note that we can write formally as $[\partial_t, \mathscr{L}] = (\partial_t \eta)(\frac{\partial}{\partial H}\mathscr{L})$ because the coefficients of the differential operator $\mathscr{L}$ do not depend on ∇H but on H. By the relation

$$\partial_t(\partial_t\phi - \varepsilon\Delta\phi) = -\mathscr{L}^{-1}[\partial_t, \mathscr{L}]\mathscr{L}^{-1}\boldsymbol{F} + \mathscr{L}^{-1}\partial_t\boldsymbol{F}$$

and Lemma 3.3 we see that

$$\|\partial_t(\partial_t\phi - \varepsilon\Delta\phi)\|_{m-1} \le C_1\bigg(\|[\partial_t, \mathscr{L}_0]\mathscr{L}^{-1}\boldsymbol{F}\|_{m-1} + \sum_{i=1}^{N}\|[\partial_t, \mathscr{L}_i]\mathscr{L}^{-1}\boldsymbol{F}\|_{m-3}$$

$$+ \|\partial_t F_0\|_{m-1} + \|(\partial_t F_1, \ldots, \partial_t F_N)\|_{m-3}\bigg)$$

$$\le C_1(\|\partial_t\phi\|_m + \|\partial_t\eta\|_{m-1}).$$

Therefore, we have $\|\partial_t a^\varepsilon\|_{m-1} \le C_2(\|\partial_t\phi\|_m + \|\partial_t\eta\|_{m-1})$. In view of $\|\partial_t\phi - \varepsilon\Delta\phi\|_m \le C_2$ and $\|\partial_t\eta - \varepsilon\Delta\eta\|_{m-1} \le C_2$, we also have $\|\partial_t\phi\|_m \le C_2(1+\varepsilon\|\Delta\phi\|_m)$ and $\|\partial_t\eta\|_{m-1} \le C_2(1+\varepsilon\|\nabla\eta\|_m)$. Thus, by (4.30) we have $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_m(t) + 2\varepsilon\mathscr{D}_m(t) \le C_2(1 + \varepsilon(\|\Delta\phi\|_m + \varepsilon\|\nabla\eta\|_m))$, which implies that $\mathscr{E}_m(t) + 2\varepsilon\int_0^t \mathscr{D}_m(\tau)\mathrm{d}\tau \le C_1 + C_2(t + \sqrt{t})$. To summarize, we have obtained the estimate

$$\begin{cases} E_m(t) + \varepsilon\displaystyle\int_0^t E_{m+1}(\tau)\mathrm{d}\tau \le C_1 + C_2(t + \sqrt{t}), \\ |H(x,t) - H(x,0)| + |a^\varepsilon(x,t) - a^\varepsilon(x,0)| \le C_2(t + \sqrt{t}) \quad \text{for} \quad x \in \mathbf{R}^n. \end{cases}$$

Now, we define the constant M_1 by $M_1 = 2C_1$, and then the time T sufficiently small so that $C_2(T + \sqrt{T}) \ll 1$. Then, we see that (4.29) holds. The proof is complete. $\square$

Once we obtain this kind of uniform estimate (4.23), we can pass to the limit $\varepsilon \to +0$ in the regularized problem (4.9) and (4.8) and obtain the following lemma.

Lemma 4.4 *Under the hypothesis of Theorem 1.1, there exists a time $T > 0$ such that the initial value problem (4.7)–(4.8) has a unique solution (η, ϕ) satisfying*

$$\eta, \nabla\phi_0 \in C([0,T]; H^m), \quad \phi_1, \ldots, \phi_N \in C([0,T]; H^{m+1}).$$

5 Proof of the main theorem

We will show that the solution to the transformed problem (4.7)–(4.8) is the solution of the Isobe–Kakinuma model (1.3)–(1.4) if the initial data $(\eta_{(0)}, \phi_{0(0)}, \ldots, \phi_{N(0)})$ and the bottom topography b satisfy the necessary condition (1.6). Let (η, ϕ) be the solution of (4.7)–(4.8) obtained in Lemma 4.4. Then, we have $\mathscr{L}_0\partial_t\phi = F_0$, which is exactly the second equation in (4.1). Therefore, it is sufficient to show that the first equation in (4.1) holds for $i = 0, 1, \ldots, N$. To this end, putting

$$(5.1) \qquad R_i = H^{p_i}\partial_t\eta - \sum_{j=0}^{N} L_{ij}\phi_j \quad \text{for} \quad i = 0, 1, \ldots, N,$$

we are going to show $R_i(x,t) \equiv 0$ for $i = 0, 1, \ldots, N$. We also introduce auxiliary functions

$$(5.2) \qquad \tilde{R}_i = \mathscr{L}_i\phi \quad \text{for} \quad i = 1, \ldots, N.$$

Then, we see that the necessary condition (1.6) is equivalent to $\tilde{R}_i = 0$ for $i = 1, \ldots, N$. We have also the relations

(5.3) $$\tilde{R}_i = H^{p_i} R_0 - R_i \quad \text{for} \quad i = 1, \ldots, N.$$

Differentiate (5.2) with respect to the time t and using (5.1) with $i = 0$ to eliminate the time derivative $\partial_t \eta$ and the equations $\mathscr{L}_i \partial_t \phi = F_i$, we obtain

(5.4) $$\partial_t \tilde{R}_i = -f_i R_0 \quad \text{for} \quad i = 1, \ldots, N,$$

where f_i is the function defined by (4.3). Let $\boldsymbol{q} = \boldsymbol{q}(H) = (q_0(H), \ldots, q_N(H))$ be rational functions of H defined by (3.17). Multiplying (5.1) by q_i, adding the resulting equations over $i = 0, 1, \ldots, N$, and using the relation (3.18) we obtain

$$\sum_{i=0}^{N} q_i R_i = -\partial_t \eta - \sum_{i,j=0}^{N} q_i L_{ij} \phi_j = 0,$$

where we used the second equation in (4.7). This together with (5.3) implies

(5.5) $$R_0 = -\sum_{j=1}^{N} q_j \tilde{R}_j.$$

Here, we used the relation $\boldsymbol{l} \cdot \boldsymbol{q} = -1$ once again. Plugging this into (5.4) we obtain a system of linear homogeneous ordinary differential equations for $(\tilde{R}_1, \ldots, \tilde{R}_N)$:

$$\partial_t \tilde{R}_i = f_i \sum_{j=1}^{N} q_j \tilde{R}_j \quad \text{for} \quad i = 1, \ldots, N.$$

The necessary condition (1.6) for the initial data $(\eta_{(0)}, \phi_{0(0)}, \ldots, \phi_{N(0)})$ and the bottom topography b is equivalent to $\tilde{R}_i(x, 0) \equiv 0$ for $i = 1, \ldots, N$, so that the uniqueness of the solution to the initial value problem implies $\tilde{R}_i = 0$ for $i = 1, \ldots, N$. Then, by (5.5) and (5.3) we see in turn that $R_0 = 0$ and $R_i = 0$ for $i = 1, \ldots, N$. Therefore, we have shown that $(\eta, \boldsymbol{\phi})$ is the solution to the Isobe–Kakinuma model (1.3)–(1.4).

It remains to show that the energy function $E(t)$ defined by (1.7) is conserved in time. The energy function can be written explicitly as

(5.6) $$E(t) = \frac{\rho}{2} \int_{\mathbf{R}^n} \left\{ \sum_{i,j=0}^{N} \left(\frac{1}{p_i + p_j + 1} H^{p_i + p_j + 1} \nabla \phi_i \cdot \nabla \phi_j - \frac{2 p_i}{p_i + p_j} H^{p_i + p_j} \phi_i \nabla b \cdot \nabla \phi_j \right.\right.$$
$$\left.\left. + \frac{p_i p_j}{p_i + p_j - 1} H^{p_i + p_j - 1} (1 + |\nabla b|^2) \phi_i \phi_j \right) + g \eta^2 \right\} \mathrm{d}x.$$

Therefore, by the direct calculation we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = \rho \left\{ \sum_{i,j=0}^{N} (L_{ij} \phi_j, \partial_t \phi_i)_{L^2} + (g\eta + \frac{1}{2}(|\boldsymbol{u}|^2 + w^2), \partial_t \eta)_{L^2} \right\}$$
$$= \rho \left\{ \sum_{i=0}^{N} (H^{p_i} \partial_t \eta, \partial_t \phi_i)_{L^2} - (F_0, \partial_t \eta)_{L^2} \right\} = 0.$$

The proof of Theorem 1.1 is complete.

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