Isobe–Kakinuma model for water waves as a higher order shallow water approximation

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Abstract

We justify rigorously an Isobe–Kakinuma model for water waves as a higher order shallow water approximation in the case of a flat bottom. It is known that the full water wave equations are approximated by the shallow water equations with an error of order $O(\delta^2)$, where δ is a small nondimensional parameter defined as the ratio of the mean depth to the typical wavelength. The Green–Naghdi equations are known as higher order approximate equations to the water wave equations with an error of order $O(\delta^4)$. In this paper we show that the Isobe–Kakinuma model is a much higher order approximation to the water wave equations with an error of order $O(\delta^6)$.

1 Introduction

We are concerned with a mathematically rigorous justification of an Isobe–Kakinuma model for the full water wave problem as a higher order shallow water approximation in the strongly nonlinear regime in the case of a flat bottom. The water wave problem is mathematically formulated as a free boundary problem for an irrotational flow of an inviscid and incompressible fluid under the gravitational field. We consider the water filled in $(n+1)$-dimensional Euclidean space. Let t be the time, $x = (x_1, \dots, x_n)$ the horizontal spatial coordinates, and z the vertical spatial coordinate. We assume that the water surface and the bottom are represented as $z = \eta(x,t)$ and $z = -h$, respectively, where $\eta = \eta(x,t)$ is the surface elevation and h is the mean depth. J. C. Luke [14] showed that the water wave problem has a variational structure by giving a Lagrangian in terms of the velocity potential $\Phi = \Phi(x,z,t)$ and the surface elevation η. His Lagrangian has the form

$$\mathscr{L}(\Phi,\eta) = \int_{-h}^{\eta(x,t)} \left(\partial_t \Phi(x,z,t) + \frac{1}{2}|\nabla_X \Phi(x,z,t)|^2 + gz \right) \mathrm{d}z \tag{1.1}$$

and the action function is

$$\mathscr{J}(\Phi,\eta) = \int_{t_0}^{t_1} \int_\Omega \mathscr{L}(\Phi,\eta) \mathrm{d}x \mathrm{d}t,$$

where $\nabla_X = (\nabla, \partial_z) = (\partial_{x_1}, \dots, \partial_{x_n}, \partial_z)$, g is the gravitational constant, and Ω is an appropriate region in $\mathbf{R}^n$. J. C. Luke showed that the corresponding Euler–Lagrange equation is exactly the basic equations for water waves. M. Isobe [5, 6] and T. Kakinuma [7, 8, 9] approximated the velocity potential in Luke's Lagrangian as

$$\Phi(x,z,t) \simeq \sum_{k=0}^{K} \Psi_k(z) \phi_k(x,t),$$

where $\{\Psi_k\}$ is an appropriate function system in the vertical coordinate z, and derived an approximate Lagrangian for $(\eta, \phi_0, \phi_1, \dots, \phi_K)$. The Isobe–Kakinuma model is the corresponding

Euler–Lagrange equation for the approximated Lagrangian. Different choices of the function system $\{\Psi_k\}$ give different Isobe–Kakinuma models. In this paper we adopt the approximation

$$\Phi(x,z,t) \simeq \phi_0(x,t) + (h+z)^2\phi_1(x,t). \tag{1.2}$$

Plugging this into Luke's Lagrangian (1.1) we obtain an approximate Lagrangian $\mathscr{L}^{\mathrm{app}}(\phi_0,\phi_1,\eta)$. The corresponding Euler–Lagrange equation has the form

$$\begin{cases} \partial_t\eta + \nabla\cdot\left(H\nabla\phi_0 + \frac{1}{3}H^3\nabla\phi_1\right) = 0, \\ H^2\partial_t\eta + \nabla\cdot\left(\frac{1}{3}H^3\nabla\phi_0 + \frac{1}{5}H^5\nabla\phi_1\right) - \frac{4}{3}H^3\phi_1 = 0, \\ \partial_t\phi_0 + H^2\partial_t\phi_1 + g\eta + \frac{1}{2}|\nabla\phi_0|^2 + H^2\nabla\phi_0\cdot\nabla\phi_1 + \frac{1}{2}H^4|\nabla\phi_1|^2 + 2H^2(\phi_1)^2 = 0, \end{cases} \tag{1.3}$$

where $H = H(x,t)$ is the depth of the water and is given by $H(x,t) = h + \eta(x,t)$. This is the Isobe–Kakinuma model that we are going to consider in this paper. For the detailed derivation of this model we refer to Y. Murakami and T. Iguchi [17].

In order to compare this Isobe–Kakinuma model with the full water wave problem in the shallow water regime, we need to rewrite (1.3) in an appropriate nondimensional form. Let λ be the typical wave length and introduce a nondimensional parameter δ by the aspect ratio $\delta = h/\lambda$, which measures the shallowness of the water. We rescale the independent and the dependent variables by

$$x = \lambda\tilde{x}, \quad z = h\tilde{z}, \quad t = \frac{\lambda}{\sqrt{gh}}\tilde{t}, \quad \phi_0 = \lambda\sqrt{gh}\tilde{\phi}_0, \quad \phi_1 = \frac{\sqrt{gh}}{\lambda}\tilde{\phi}_1, \quad \eta = h\tilde{\eta}.$$

Here we note that these rescaling of dependent variables are related to the strongly nonlinear regime of the wave. Plugging these into (1.3) and dropping the tilde sign in the notation we obtain

$$\begin{cases} \partial_t\eta + \nabla\cdot\left(H\nabla\phi_0 + \frac{1}{3}\delta^2H^3\nabla\phi_1\right) = 0, \\ H^2\partial_t\eta + \nabla\cdot\left(\frac{1}{3}H^3\nabla\phi_0 + \frac{1}{5}\delta^2H^5\nabla\phi_1\right) - \frac{4}{3}H^3\phi_1 = 0, \\ \partial_t\phi_0 + \delta^2H^2\partial_t\phi_1 + \eta \\ \quad + \frac{1}{2}|\nabla\phi_0|^2 + \delta^2H^2\nabla\phi_0\cdot\nabla\phi_1 + \frac{1}{2}\delta^4H^4|\nabla\phi_1|^2 + 2\delta^2H^2(\phi_1)^2 = 0, \end{cases} \tag{1.4}$$

where $H(x,t) = 1+\eta(x,t)$. We consider the initial value problem to this Isobe–Kakinuma model (1.4) under the initial conditions

$$(\eta,\phi_0,\phi_1) = (\eta_{(0)},\phi_{0(0)},\phi_{1(0)}) \qquad \text{at} \quad t = 0. \tag{1.5}$$

Unique solvability locally in time of the initial value problem (1.4)–(1.5) and fundamental properties of the model, especially, the linear dispersion relation are presented in Y. Murakami and T. Iguchi [17].

On the other hand, the initial value problem to the full water wave problem in Zakharov–Craig–Sulem formulation in the nondimensional form is written as

$$\begin{cases} \partial_t\eta - \Lambda(\eta,\delta)\phi = 0, \\ \partial_t\phi + \eta + \frac{1}{2}|\nabla\phi|^2 - \delta^2\frac{(\Lambda(\eta,\delta)\phi + \nabla\eta\cdot\nabla\phi)^2}{2(1+\delta^2|\nabla\eta|^2)} = 0, \end{cases} \tag{1.6}$$

$$(\eta, \phi) = (\eta_{(0)}, \phi_{(0)}), \tag{1.7}$$

where $\phi = \phi(x,t)$ is the trace of the velocity potential Φ on the water surface and $\Lambda(\eta, \delta)$ is the Dirichlet-to-Neumann map for Laplace's equation. More precisely, the linear operator $\Lambda(\eta, \delta)$ depending nonlinearly on the surface elevation η and the parameter δ is defined by

$$\Lambda(\eta, \delta)\phi = (\delta^{-2}\partial_z \Phi - \nabla\eta \cdot \nabla\Phi)|_{z=\eta(x,t)}, \tag{1.8}$$

where Φ is a unique solution to the boundary value problem for Laplace's equation

$$\begin{cases} \delta^2 \Delta\Phi + \partial_z^2 \Phi = 0 & \text{in} \quad -1 < z < \eta(x,t), \\ \Phi = \phi & \text{on} \quad z = \eta(x,t), \\ \partial_z \Phi = 0 & \text{on} \quad z = -1. \end{cases} \tag{1.9}$$

It has already been established that the solution to the full water wave problem is approximated by the solution to the shallow water equations up to order $O(\delta^2)$, that is, we have

$$|\eta^{\mathrm{WW}}(x,t) - \eta^{\mathrm{SW}}(x,t)| \lesssim \delta^2$$

on some time interval independent of δ, where η^{WW} and η^{SW} are the solutions to the full water wave and to the shallow water equations, respectively. For this rigorous justification of the shallow water equations, we refer to L. V. Ovsjannikov [18, 19] and T. Kano and T. Nishida [10] in the case of analytic initial data and Y. A. Li [13], T. Iguchi [3, 4], and B. Alvarez-Samaniego and D. Lannes [1] in the case of initial data in Sobolev spaces. Green–Naghdi equations are known as higher order approximate equations to the full water wave equations in the shallow water regime, that is, we have

$$|\eta^{\mathrm{WW}}(x,t) - \eta^{\mathrm{GN}}(x,t)| \lesssim \delta^4$$

on some time interval independent of δ, where η^{GN} is a solution to the Green–Naghdi equations. For this approximation we refer to Y. A. Li [13], B. Alvarez-Samaniego and D. Lannes [1], and H. Fujiwara and T. Iguchi [2]. In this paper we will show that the Isobe–Kakinuma model (1.4) is a much higher order approximation to the full water wave equations in the shallow water regime, that is, we have

$$|\eta^{\mathrm{WW}}(x,t) - \eta^{\mathrm{IK}}(x,t)| \lesssim \delta^6 \tag{1.10}$$

on some time interval independent of δ, where η^{IK} is a solution to the Isobe–Kakinuma model (1.4). Here we remark that Y. Matsuno [15, 16] derived extended Green–Naghdi equations as higher order shallow water approximations in the strongly nonlinear regime. His δ^{2N} model is an approximation of the full water wave equations with an error of order δ^{2N+2}. Since the linear dispersion relation of his δ^4 model does not have good structures, we cannot expect the well-posedness of the initial value problem so that it is hopeless to obtain an error estimate of the solutions such as (1.10). The linear part of his δ^6 model has a good structure and the solution might approximate the solution to the full water wave equations up to order $O(\delta^8)$. However, it contains 7th order derivative terms, which are troublesome in a numerical computation. Although the Isobe–Kakinuma model (1.4) is a higher order shallow water approximation, it is a system of second order partial differential equations and does not contain such higher order derivative terms. This is a strong advantage of the Isobe–Kakinuma model.

The contents of this paper are as follows. In Section 2 we present our main results in this paper, that is, uniform estimates of the solution of the initial value problem to the Isobe–Kakinuma model (1.4)–(1.5) on some time interval independent of the parameter δ, the consistency of the Isobe–Kakinuma model at order $O(\delta^6)$, and the rigorous justification of the Isobe–Kakinuma model by establishing an error estimate of the solutions such as (1.10). In Section 3 we consider linearized equations of the Isobe–Kakinuma model around the rest state in order to explaine a hidden symmetric structure of the model and to give an idea to obtain uniform estimates of the solution. In Section 4 we derive a symmetric system of quasilinear equations for derivatives of the solution. In Section 5 we give uniform estimates of the solution by using the symmetric structure of the model. In Section 6 we show that the Isobe–Kakinuma model is consistent at order (δ^6), that is, the solution to the Isobe–Kakinuma model satisfies the full water wave equations with an error of order $O(\delta^6)$. In Section 7 we derive an error estimate of the solutions by using the stability of the full water wave problem.

Notation. We denote by $W^{m,p}(\mathbf{R}^n)$ the L^p Sobolev space of order m on $\mathbf{R}^n$. The norms of the Lebesgue space $L^p(\mathbf{R}^n)$ and the Sobolev space $H^m = W^{m,2}(\mathbf{R}^n)$ are denoted by $|\cdot|_p$ and $\|\cdot\|_m$, respectively. The L^2-norm and the L^2-inner product are simply denoted by $\|\cdot\|$ and $(\cdot,\cdot)_{L^2}$, respectively. We put $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, and $\partial_z = \frac{\partial}{\partial z}$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ we put $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. $[P, Q] = PQ - QP$ denotes the commutator.

2 Main results

The Isobe–Kakinuma model (1.4) is written in the matrix form as

$$\begin{pmatrix} 1 & 0 & 0 \\ H^2 & 0 & 0 \\ 0 & 1 & \delta^2 H^2 \end{pmatrix} \partial_t \begin{pmatrix} \eta \\ \phi_0 \\ \phi_1 \end{pmatrix} + \{\text{spatial derivatives}\} = \mathbf{0}.$$

Since the coefficient matrix has always the zero eigenvalue, the hypersurface $t = 0$ in the space-time $\mathbf{R}^n \times \mathbf{R}$ is characteristic for the Isobe–Kakinuma model (1.4), so that the initial value problem (1.4)–(1.5) is not solvable in general. In fact, if the problem has a solution (η, ϕ_0, ϕ_1), then by eliminating the time derivative $\partial_t \eta$ from the first two equations in (1.4) we see that the solution has to satisfy the relation

$$H^2 \nabla \cdot \left(H \nabla \phi_0 + \frac{1}{3} \delta^2 H^3 \nabla \phi_1 \right) = \nabla \cdot \left(\frac{1}{3} H^3 \nabla \phi_0 + \frac{1}{5} \delta^2 H^5 \nabla \phi_1 \right) - \frac{4}{3} H^3 \phi_1, \tag{2.1}$$

which is equivalent to

$$\frac{2}{3} \Delta \phi_0 + \frac{2}{15} \delta^2 H^2 \Delta \phi_1 + \frac{4}{3} \phi_1 = 0. \tag{2.2}$$

Therefore, as a necessary condition the initial data $(\eta_{(0)}, \phi_{0(0)}, \phi_{1(0)})$ have to satisfy this relation for the existence of the solution.

We also need to mention that the initial value problem for the full water wave problem (1.6) may be broken unless a so-called generalized Rayleigh–Taylor sign condition $-\frac{\partial p}{\partial N} \geq c_0 > 0$ on the water surface is satisfied, where p is the pressure and N is the unit outward normal on the water surface. For the Isobe–Kakinuma model (1.4) the corresponding sign condition is written as $a(x,t) \geq c_0 > 0$, where

$$a = 1 + 2\delta^2 H \partial_t \phi_1 + 2\delta^2 H \nabla \phi_0 \cdot \nabla \phi_1 + 2\delta^4 H^3 |\nabla \phi_1|^2 + 4\delta^2 H (\phi_1)^2. \tag{2.3}$$

The following theorem is one of the main results in this paper and asserts the existence of the solution to the initial value problem (1.4)–(1.5) with uniform bounds of the solution on a time interval independent of the small parameter δ.

Theorem 2.1 *Let $M_0, c_0 > 0$ and m be an integer such that $m > n/2+1$. There exist a time $T_1 > 0$ and constants $C, \delta_1 > 0$ such that for any $\delta \in (0, \delta_1]$ if the initial data $(\eta_{(0)}, \phi_{0(0)}, \phi_{1(0)})$ satisfy the relation* (2.1) *and*

$$\left\{\begin{array}{l} \|\eta_{(0)}\|_m + \delta\|\eta_{(0)}\|_{m+1} + \|\nabla\phi_{0(0)}\|_m + \delta\|\nabla\phi_{0(0)}\|_{m+1} \\ \qquad\qquad + \delta\|\phi_{1(0)}\|_m + \delta^2\|\phi_{1(0)}\|_{m+1} + \delta^3\|\phi_{1(0)}\|_{m+2} \le M_0, \\ 1+\eta_{(0)}(x) \ge c_0 \qquad \textit{for} \quad x \in \mathbf{R}^n, \end{array}\right. \tag{2.4}$$

then the initial value problem (1.4)–(1.5) *has a unique solution (η, ϕ_0, ϕ_1) on the time interval $[0, T_1]$. Moreover, the solution satisfies the uniform bound:*

$$\left\{\begin{array}{l} \|\eta(t)\|_m + \delta\|\eta(t)\|_{m+1} + \|\nabla\phi_0(t)\|_m + \delta\|\nabla\phi_0(t)\|_{m+1} \\ \qquad\qquad + \|\phi_1(t)\|_{m-1} + \delta\|\phi_1(t)\|_m + \delta^2\|\phi_1(t)\|_{m+1} + \delta^3\|\phi_1(t)\|_{m+2} \\ \qquad\qquad + \|\partial_t\eta(t)\|_{m-1} + \delta\|\partial_t\eta(t)\|_m + \|\partial_t\phi_0(t)\|_m + \delta\|\partial_t\phi_0(t)\|_{m+1} \\ \qquad\qquad + \delta\|\partial_t\phi_1(t)\|_{m-1} + \delta^2\|\partial_t\phi_1(t)\|_m + \delta^3\|\partial_t\phi_1(t)\|_{m+1} \le C, \\ 1+\eta(x,t) \ge c_0/2, \quad a(x,t) \ge 1/2 \qquad \textit{for} \quad x \in \mathbf{R}^n,\ 0 \le t \le T_1, \end{array}\right. \tag{2.5}$$

where a is defined in terms of the solution by (2.3).

Remark 2.1 In the above theorem the constant δ_1 is small. We can reduce the restriction $0 < \delta \le \delta_1$ to, for example, $0 < \delta \le 1$, if we impose the sign condition $a(x,0) \ge c_0$ on the initial data. However, we are interested in the shallow water approximation, that is, the asymptotic behavior of the solution as $\delta \to +0$ so that the condition $0 < \delta \le \delta_1$ is not an essential restriction.

Next, we proceed to show that the water wave equations (1.6) are consistent at order $O(\delta^6)$ with the Isobe–Kakinuma model (1.4). To this end we need to relate the dependent variables (η, ϕ_0, ϕ_1) for (1.4) and (η, ϕ) for (1.6). In view of the facts that ϕ is the trace of the velocity potential Φ on the water surface and that ϕ_0 and ϕ_1 appear in the approximation (1.2), these variables are related by the formula

$$\phi = \phi_0 + \delta^2 H^2 \phi_1 \tag{2.6}$$

in the nondimensional variables.

Theorem 2.2 *In addition to hypothesis of Theorem* 2.1 *we assume that $m > n/2+6$. Let (η, ϕ_0, ϕ_1) be the solution obtained in Theorem* 2.1 *and define ϕ by* (2.6)*. Then, (η, ϕ) satisfy the water wave equations with errors of order $O(\delta^6)$, that is,*

$$\left\{\begin{array}{l} \partial_t\eta - \Lambda(\eta,\delta)\phi = \delta^6 r_1, \\ \partial_t\phi + \eta + \dfrac{1}{2}|\nabla\phi|^2 - \delta^2\dfrac{(\Lambda(\eta,\delta)\phi + \nabla\eta\cdot\nabla\phi)^2}{2(1+\delta^2|\nabla\eta|^2)} = \delta^6 r_2. \end{array}\right. \tag{2.7}$$

Here, (r_1, r_2) satisfy the uniform bound:

$$\|r_1(t)\|_{m-7} + \|r_2(t)\|_{m-5} \le C \qquad \textit{for} \quad 0 \le t \le T_1, \tag{2.8}$$

where C is a positive constant independent of $\delta \in (0, \delta_1]$.

The above theorem concerns the approximation of the equations. Next, we will be concerned with the approximation of the solution to give a rigorous justification of the Isobe–Kakinuma model. Here we recall the existence theorem for the initial value problem to the full water wave equations (1.6)–(1.7) obtained by T. Iguchi [3]. Similar results are obtained by B. Alvarez-Samaniego and D. Lannes [1] and D. Lannes [12].

Theorem 2.3 *Let $M_0, c_0 > 0$ and $m > n/2+1$. There exist a time $T_2 > 0$ and constants $C, \delta_2 > 0$ such that for any $\delta \in (0, \delta_2]$ if the initial data $(\eta_{(0)}, \phi_{(0)})$ satisfy*

$$\begin{cases} \|\eta_{(0)}\|_{m+3+1/2} + \|\nabla\phi_{(0)}\|_{m+3} \le M_0, \\ 1+\eta_{(0)}(x) \ge c_0 \qquad \text{for} \quad x \in \mathbf{R}^n, \end{cases}$$

then the initial value problem (1.6)–(1.7) *has a unique solution (η, ϕ) on the time interval $[0, T_2]$. Moreover, the solution satisfies the uniform bound:*

$$\begin{cases} \|\eta(t)\|_{m+3} + \|\nabla\phi(t)\|_{m+2} + \|\partial_t\eta(t)\|_{m+2} + \|\partial_t\phi(t)\|_{m+2} \le C, \\ 1+\eta(x,t) \ge c_0/2, \qquad \text{for} \quad x \in \mathbf{R}^n,\ 0 \le t \le T_2. \end{cases}$$

Remark 2.2 In the above theorem the constant δ_2 is small. As in the case of Theorem 2.1 we can reduce the restriction $0 < \delta \le \delta_2$ to $0 \le \delta \le 1$, if we impose the sign condition $a^{\mathrm{WW}}(x,0) \ge c_0$ on the initial data, where $a^{\mathrm{WW}} = 1 + \delta^2\partial_t Z + \delta^2 \boldsymbol{v}\cdot\nabla Z$ with

$$\begin{cases} Z = (1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda(\eta,\delta)\phi + \nabla\eta\cdot\nabla\phi), \\ \boldsymbol{v} = \nabla\phi - \delta^2 Z\nabla\eta. \end{cases}$$

In order that the solution to the Isobe–Kakinuma model (1.4)–(1.5) approximates the solution to the full water wave problem (1.6)–(1.7), we need to prepare the initial data $\phi_{0(0)}$ and $\phi_{1(0)}$ for the Isobe–Kakinuma model appropriately in terms of the initial data $\eta_{(0)}$ and $\phi_{(0)}$ for the water wave problem. In view of the necessary condition (2.1) or (2.2) and the relation (2.6), the initial data have to satisfy

$$\begin{cases} \frac{1}{2}\Delta\phi_{0(0)} + \frac{1}{10}\delta^2(1+\eta_{(0)})^2\Delta\phi_{1(0)} + \phi_{1(0)} = 0, \\ \phi_{0(0)} + \delta^2(1+\eta_{(0)})^2\phi_{1(0)} = \phi_{(0)}. \end{cases} \tag{2.9}$$

As we will see in Section 7 (see also Lemma 5.2 and Remark 5.1), given the initial data $(\eta_{(0)}, \phi_{(0)})$, these equations determine uniquely the initial data $(\phi_{0(0)}, \phi_{1(0)})$. The next theorem gives a rigorous justification of the Isobe–Kakinuma model for the full water wave problem as a higher order shallow water approximation.

Theorem 2.4 *Let $M_0, c_0 > 0$ and m be an integer such that $m > n/2+1$, and put $T_* = \min\{T_1, T_2\}$ and $\delta_* = \min\{\delta_1, \delta_2\}$, where these constants $T_1, T_2, \delta_1, \delta_2$ are those in Theorems* 2.1 *and* 2.3. *Suppose that $0 < \delta \le \delta_*$ and the initial data $(\eta_{(0)}, \phi_{(0)})$ satisfy*

$$\begin{cases} \|\eta_{(0)}\|_{m+11} + \|\nabla\phi_{(0)}\|_{m+10} \le M_0, \\ 1+\eta_{(0)}(x) \ge c_0 \qquad \text{for} \quad x \in \mathbf{R}^n. \end{cases} \tag{2.10}$$

Then, (2.9) *determines uniquely the initial data $(\phi_{0(0)}, \phi_{1(0)})$. Let $(\eta^{\mathrm{WW}}, \phi^{\mathrm{WW}})$ be the solution to the initial value problem* (1.6)–(1.7) *obtained in Theorem* 2.3 *and $(\eta^{\mathrm{IK}}, \phi_0, \phi_1)$ the solution to the initial value problem* (1.4)–(1.5) *obtained in Theorem* 2.1, *and define ϕ^{IK} by* (2.6). *Then, for any $\delta \in (0, \delta_*]$ we have*

$$\|\eta^{\mathrm{WW}}(t) - \eta^{\mathrm{IK}}(t)\|_{m+2} + \|\nabla\phi^{\mathrm{WW}}(t) - \nabla\phi^{\mathrm{IK}}(t)\|_{m+1} \le C\delta^6 \qquad \text{for} \quad 0 \le t \le T_*, \tag{2.11}$$

where C is a positive constant independent of $\delta \in (0, \delta_*]$.

Remark 2.3 The error estimate (2.11) together with the Sobolev imbedding theorem implies the pointwise error estimate (1.10).

We will give the proof of Theorems 2.1, 2.2, and 2.4 in Sections 5, 6, and 7, respectively.

3 Strategy to obtain uniform estimates

The unique existence of the solution locally in time to the initial value problem for the Isobe–Kakinuma model (1.4)–(1.5) for each fixed $\delta > 0$ is established in Y. Murakami and T. Iguchi [17]. However, the energy method used in [17] does not give uniform estimates of the solution with respect to the small parameter δ. In order to obtain such estimates we have to make use of a good symmetric structure of the Isobe–Kakinuma model (1.4). In this section we treat linearized equations of the Isobe–Kakinuma model to give an idea for obtaining such estimates.

We note that $(\eta, \phi_0, \phi_1) = \mathbf{0}$ is the solution of the Isobe–Kakinuma model (1.4), which corresponds to the still water with flat water surface. The linearized equations of (1.4) around this trivial solution have the form

$$\begin{cases} \partial_t \eta + \Delta\phi_0 + \dfrac{1}{3}\delta^2\Delta\phi_1 = 0, \\ \partial_t \eta + \dfrac{1}{3}\Delta\phi_0 + \dfrac{1}{5}\delta^2\Delta\phi_1 - \dfrac{4}{3}\phi_1 = 0, \\ \partial_t\phi_0 + \delta^2\partial_t\phi_1 + \eta = 0. \end{cases} \tag{3.1}$$

Put $\boldsymbol{U} = (\eta, \phi_0, \phi_1)^{\mathrm{T}}$ and

$$S = \begin{pmatrix} 0 & 1 & \delta^2 \\ -1 & 0 & 0 \\ -\delta^2 & 0 & 0 \end{pmatrix}, \quad A_0(D) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\Delta & -\frac{1}{3}\delta^2\Delta \\ 0 & -\frac{1}{3}\delta^2\Delta & -\frac{1}{5}\delta^4\Delta + \frac{4}{3}\delta^2 \end{pmatrix}.$$

Then, the system of equations (3.1) can be written in the matrix form as

$$S\partial_t\boldsymbol{U} + A_0(D)\boldsymbol{U} = \mathbf{0}. \tag{3.2}$$

It is easy to see that S is skew-symmetric and $A_0(D)$ is symmetric in $L^2(\mathbf{R}^n)$. Moreover, we can easily show the following lemma, which guarantees the positivity of $A_0(D)$.

Lemma 3.1 *Let $E(\boldsymbol{U}) = \frac{1}{2}(A_0(D)\boldsymbol{U}, \boldsymbol{U})_{L^2}$. Then, we have*

$$E(\boldsymbol{U}) = \frac{1}{2}\int_{\mathbf{R}^n} \eta(x)^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbf{R}^n}\mathrm{d}x\int_0^1 |\nabla_X^\delta(\phi_0(x) + \delta^2 z^2\phi_1(x))|^2\,\mathrm{d}z, \tag{3.3}$$

where $\nabla_X^\delta = (\nabla, \delta^{-1}\partial_z)$. Moreover, $E(\boldsymbol{U})$ is equivalent to

$$\widetilde{E}(\boldsymbol{U}) = \|\eta\|^2 + \|\nabla\phi_0\|^2 + \delta^2\|\phi_1\|^2 + \delta^4\|\nabla\phi_1\|^2$$

uniformly with respect to δ.

Let $\boldsymbol{U}$ be a smooth solution to (3.2). Taking the Euclidean inner product of (3.2) with $\partial_t\boldsymbol{U}$, we obtain $A_0(D)\boldsymbol{U}\cdot\partial_t\boldsymbol{U} = 0$ because S is skew-symmetric. Integrating this with respect to x on $\mathbf{R}^n$ we see that $\frac{\mathrm{d}}{\mathrm{d}t}E(\boldsymbol{U}(t)) = 0$. Therefore, $E(\boldsymbol{U})$ is a conserved quantity for (3.2). In fact, $E(\boldsymbol{U})$ is the physical energy function: the first term in the right-hand side of (3.3) is the potential energy due to the gravity and the second one the kinetic energy. However, (3.2) is not standard form of partial differential equations because S is a singular matrix. In the standard theory of positive systems of partial differential equations, the system whose energy function is given by the quadratic form associated to the positive operator $A_0(D)$ has the form

$$A_0(D)\partial_t\boldsymbol{U} + A_1(D)\boldsymbol{U} = \mathbf{0} \tag{3.4}$$

with a skew-symmetric operator $A_1(D)$ in $L^2(\mathbf{R}^n)$. Therefore, we may have a temptation to transform (3.2) into (3.4). Thus, again let $\boldsymbol{U} = (\eta, \phi_0, \phi_1)^{\mathrm{T}}$ be a smooth solution to (3.2), which is equivalent to (3.1), and we will derive a system of the form (3.4). By eliminating the time derivative $\partial_t \eta$ from the first two equations in (3.1), we obtain the necessary condition

$$\frac{1}{2}\Delta\phi_0 + \frac{1}{10}\delta^2\Delta\phi_1 + \phi_1 = 0$$

for the existence of the solution. We differentiate this with respect to the time t. The resulting equation together with the third equation in (3.1) can be written in the matrix form

$$\begin{pmatrix} 1 & \delta^2 \\ \frac{1}{2}\Delta & \frac{1}{10}\delta^2\Delta + 1 \end{pmatrix} \partial_t \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} + \begin{pmatrix} \eta \\ 0 \end{pmatrix} = \mathbf{0}.$$

We note that the coefficient matrix operator is invertible, so that this implies

$$\begin{pmatrix} -\Delta & -\frac{1}{3}\delta^2\Delta \\ -\frac{1}{3}\delta^2\Delta & -\frac{1}{5}\delta^4\Delta + \frac{4}{3}\delta^2 \end{pmatrix} \partial_t \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} - c_{IK}(D)^2\Delta \begin{pmatrix} \eta \\ \delta^2\eta \end{pmatrix} = \mathbf{0}, \tag{3.5}$$

where $c_{IK}(D)^2 = (1 - \frac{2}{5}\delta^2\Delta)^{-1}(1 - \frac{1}{15}\delta^2\Delta)$. We note that the symbol of the operator $c_{IK}(D)$ is the phase speed of the plane wave for the linearized Isobe–Kakinuma model. This is an evolution equation for (ϕ_0, ϕ_1). We proceed to derive an evolution equation for η. Let $\alpha_j = \alpha_j(D)$ $(j = 1, 2)$ be Fourier multipliers satisfying $\alpha_1 + \alpha_2 = 1$ to be determined later. Applying $\alpha_1(D)$ and $\alpha_2(D)$ to the first and the second equations in (3.1), respectively, we obtain

$$\partial_t\eta + \left(\Delta\alpha_1 + \frac{1}{3}\Delta\alpha_2\right)\phi_0 + \left(\frac{1}{3}\delta^2\Delta\alpha_1 + \left(\frac{1}{5}\delta^2\Delta - \frac{4}{3}\right)\alpha_2\right)\phi_1 = 0.$$

This and (3.5) constitute a system of the form (3.4) with

$$A_1(D) = \begin{pmatrix} 0 & \Delta\alpha_1 + \frac{1}{3}\Delta\alpha_2 & \frac{1}{3}\delta^2\Delta\alpha_1 + \left(\frac{1}{5}\delta^2\Delta - \frac{4}{3}\right)\alpha_2 \\ -c_{IK}(D)^2\Delta & 0 & 0 \\ -\delta^2 c_{IK}(D)^2\Delta & 0 & 0 \end{pmatrix}.$$

In order that this matrix operator is skew-symmetric in $L^2(\mathbf{R}^n)$, the operators α_1 and α_2 have to satisfy

$$\begin{pmatrix} 1 & \frac{1}{3} \\ \frac{1}{3}\delta^2\Delta & \frac{1}{5}\delta^2\Delta - \frac{4}{3} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = c_{IK}(D)^2 \begin{pmatrix} 1 \\ \delta^2\Delta \end{pmatrix},$$

which yields

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = c_{IK}(D)^2 \left(1 - \frac{1}{15}\delta^2\Delta\right)^{-1} \begin{pmatrix} 1 + \frac{1}{10}\delta^2\Delta \\ -\frac{1}{2}\delta^2\Delta \end{pmatrix}.$$

We note that this choice of α_1 and α_2 implies the relation $\alpha_1 + \alpha_2 = 1$. Therefore, we have transformed (3.2) into (3.4) with $A_1(D) = c_{IK}(D)^2\Delta S$.

However, in this case (3.4) is not a system of partial differential equations because $c_{IK}(D)$ contains a nonlocal operator $(1 - \frac{2}{5}\delta^2\Delta)^{-1}$. Nevertheless, it follows from (3.4) that

$$\left(1 - \frac{2}{5}\delta^2\Delta\right)A_0(D)\partial_t\boldsymbol{U} + \left(1 - \frac{1}{15}\delta^2\Delta\right)\Delta S\boldsymbol{U} = \mathbf{0}. \tag{3.6}$$

This is a positive symmetric system of partial differential equations. The corresponding energy function $E_1(\boldsymbol{U})$ is the quadratic form associated with the positive operator $\left(1 - \frac{2}{5}\delta^2\Delta\right)A_0(D)$, that is,

$$E_1(\boldsymbol{U}) = E(\boldsymbol{U}) + \frac{2}{5}\delta^2 \sum_{j=1}^{n} E(\partial_j\boldsymbol{U}),$$

which is equivalent to

$$\widetilde{E}_1(\boldsymbol{U}) = \|\eta\|^2 + \delta^2\|\nabla\eta\|^2 + \|\nabla\phi_0\|^2 + \delta^2\|\Delta\phi_0\|^2 + \delta^2\|\phi_1\|^2 + \delta^4\|\nabla\phi_1\|^2 + \delta^6\|\Delta\phi_1\|^2$$

uniformly with respect to δ. Since $(1-\frac{1}{15}\delta^2\Delta)\Delta S$ is skew-symmetric in $L^2(\mathbf{R}^n)$, $E_1(\boldsymbol{U})$ is also a conserved quantity for (3.1). By using this energy function, we can obtain a uniform bound of the solution to (3.1).

In view of the above argument, our strategy to obtain uniform estimate of the solution to the nonlinear problem is to derive a nonlinear version of the symmetric system (3.6). We will carry out it in the next section.

4 Transformation of the system

Let $\boldsymbol{U} = (\eta, \phi_0, \phi_1)^{\mathrm{T}}$ be a solution to the Isobe–Kakinuma model (1.4) throughout this section. We introduce second order differential operators $L_{11} = L_{11}(H)$, $L_{12} = L_{12}(H)$, and $L_{22} = L_{22}(H)$ depending on the depth of the water $H = 1 + \eta$ by

$$(4.1)\qquad \begin{cases} L_{11}\psi = -\nabla\cdot(H\nabla\psi), \\ L_{12}\psi = -\nabla\cdot\left(\dfrac{1}{3}H^3\nabla\psi\right), \\ L_{22}\psi = -\delta^2\nabla\cdot\left(\dfrac{1}{5}H^5\nabla\psi\right) + \dfrac{4}{3}H^3\psi. \end{cases}$$

Then, we see that these operators are symmetric in $L^2(\mathbf{R}^n)$ and that the Isobe–Kakinuma model (1.4) and the relation (2.1) can be written as

$$(4.2)\qquad \begin{cases} \partial_t\eta - L_{11}\phi_0 - \delta^2 L_{12}\phi_1 = 0, \\ H^2\partial_t\eta - L_{12}\phi_0 - L_{22}\phi_1 = 0, \\ \partial_t\phi_0 + \delta^2 H^2\partial_t\phi_1 + F_1 = 0 \end{cases}$$

and

$$(4.3)\qquad H^2(L_{11}\phi_0 + \delta^2 L_{12}\phi_1) = L_{12}\phi_0 + L_{22}\phi_1,$$

respectively, where

$$(4.4)\qquad F_1 = \eta + \frac{1}{2}|\nabla\phi_0|^2 + \delta^2 H^2\nabla\phi_0\cdot\nabla\phi_1 + \frac{1}{2}\delta^4 H^4|\nabla\phi_1|^2 + 2\delta^2 H^2(\phi_1)^2.$$

We differentiate (4.3) (equivalently, (2.2)) with respect to t. Then, the resulting equation and the third equation in (4.2) form the system

$$(4.5)\qquad \begin{cases} \partial_t\phi_0 + \delta^2 H^2\partial_t\phi_1 = -F_1, \\ H^2(L_{11}\partial_t\phi_0 + \delta^2 L_{12}\partial_t\phi_1) = L_{12}\partial_t\phi_0 + L_{22}\partial_t\phi_1 + F_2, \end{cases}$$

where

$$(4.6)\qquad F_2 = \frac{4}{15}\delta^2 H^4(\partial_t\eta)\Delta\phi_1.$$

Differentiating the equations in (4.5) with respect to t once more we obtain

$$
\begin{cases} \partial_t^2\phi_0 + \delta^2 H^2\partial_t^2\phi_1 = -F_3, \\ H^2(L_{11}\partial_t^2\phi_0 + \delta^2 L_{12}\partial_t^2\phi_1) = L_{12}\partial_t^2\phi_0 + L_{22}\partial_t^2\phi_1 + F_4, \end{cases} \tag{4.7}
$$

where

$$
\begin{cases} F_3 = \partial_t F_1 + 2\delta^2 H(\partial_t\eta)(\partial_t\phi_1), \\ F_4 = \dfrac{2}{15}\delta^2 H^2[\partial_t^2, H^2]\Delta\phi_1. \end{cases} \tag{4.8}
$$

These systems (4.5) and (4.7) are used to obtain estimates of time derivatives $(\partial_t\phi_0, \partial_t\phi_1)$ and $(\partial_t^2\phi_0, \partial_t^2\phi_1)$, respectively.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index satisfying $|\alpha| \leq m$. We proceed to derive an evolution equation for $\partial^\alpha \boldsymbol{U}$, which is a nonlinear version of the symmetric system (3.6). Applying ∂^α to (4.5) we obtain

$$
\begin{pmatrix} 1 & \delta^2 H^2 \\ \frac{1}{2}\Delta & \frac{1}{10}\delta^2 H^2\Delta + 1 \end{pmatrix} \partial_t \begin{pmatrix} \partial^\alpha\phi_0 \\ \partial^\alpha\phi_1 \end{pmatrix} = \begin{pmatrix} F_5 \\ F_6 \end{pmatrix}, \tag{4.9}
$$

where

$$
\begin{cases} F_5 = -\delta^2[\partial^\alpha, H^2]\partial_t\phi_1 - \partial^\alpha F_1, \\ F_6 = -\dfrac{1}{10}\delta^2[\partial^\alpha, H^2]\Delta\partial_t\phi_1 - \dfrac{1}{5}\delta^2\partial^\alpha(H(\partial_t\eta)\Delta\phi_1). \end{cases} \tag{4.10}
$$

Here, we need to extract principal terms in F_5. In view of (4.4) we write $F_1 = F_1(\boldsymbol{U})$ with $\boldsymbol{U} = (\eta, \phi_0, \phi_1)^{\mathrm{T}}$. We define a by (2.3) and $\boldsymbol{u}$ by

$$
\boldsymbol{u} = \nabla\phi_0 + \delta^2 H^2\nabla\phi_1, \tag{4.11}
$$

which is the horizontal component of the velocity on the water surface. Since the Fréchet derivative of $F_1(\boldsymbol{U})$ with respect to $\boldsymbol{U}$ is given by

$$
\begin{aligned} D_{\boldsymbol{U}} F_1(\boldsymbol{U})[\zeta, \psi_0, \psi_1] = {}& (1 + 2\delta^2 H\nabla\phi_0\cdot\nabla\phi_1 + 2\delta^4 H^3|\nabla\phi_1|^2 + 4\delta^2 H(\phi_1)^2)\zeta \\ & + \boldsymbol{u}\cdot\nabla\psi_0 + \delta^2 H^2\boldsymbol{u}\cdot\nabla\psi_1 + 4\delta^2 H^2\phi_1\psi_1, \end{aligned} \tag{4.12}
$$

we have

$$
F_5 = -a\partial^\alpha\eta - \boldsymbol{u}\cdot\nabla\partial^\alpha\phi_0 - \delta^2 H^2\boldsymbol{u}\cdot\nabla\partial^\alpha\phi_1 + F_7, \tag{4.13}
$$

where

$$
F_7 = -\delta^2([\partial^\alpha, H^2] - 2H(\partial^\alpha H))\partial_t\phi_1 - (\partial^\alpha F_1(\boldsymbol{U}) - D_{\boldsymbol{U}}F_1(\boldsymbol{U})[\partial^\alpha\boldsymbol{U}]) - 4\delta^2 H^2\phi_1\partial^\alpha\phi_1. \tag{4.14}
$$

Now, we apply the matrix operator

$$
\begin{pmatrix} -\Delta(H\,\cdot) & -\Delta(\frac{1}{3}\delta^2 H^3\,\cdot) \\ -\Delta(\frac{1}{3}\delta^2 H^3\,\cdot) & -\Delta(\frac{1}{5}\delta^4 H^5\,\cdot) + \frac{4}{3}\delta^2 H^3 \end{pmatrix} \begin{pmatrix} \frac{1}{10}\delta^2 H^2\Delta + 1 & -\delta^2 H^2 \\ -\frac{1}{2}\Delta & 1 \end{pmatrix}
$$

to (4.9). In view of the identities

$$\begin{pmatrix} \frac{1}{10}\delta^2 H^2\Delta + 1 & -\delta^2 H^2 \\ -\frac{1}{2}\Delta & 1 \end{pmatrix} \begin{pmatrix} 1 & \delta^2 H^2 \\ \frac{1}{2}\Delta & \frac{1}{10}\delta^2 H^2\Delta + 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(1 - \frac{2}{5}\delta^2 H^2\Delta\right) + \begin{pmatrix} 0 & \frac{1}{10}\delta^4 H^2 \\ 0 & -\frac{1}{2}\delta^2 \end{pmatrix} [\Delta, H^2],$$

$$\begin{pmatrix} -\Delta(H\,\cdot\,) & -\Delta(\frac{1}{3}\delta^2 H^3\,\cdot\,) \\ -\Delta(\frac{1}{3}\delta^2 H^3\,\cdot\,) & -\Delta(\frac{1}{5}\delta^4 H^5\,\cdot\,) + \frac{4}{3}\delta^2 H^3 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{10}\delta^4 H^2 \\ 0 & -\frac{1}{2}\delta^2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \Delta(\frac{1}{15}\delta^4 H^3\,\cdot\,) \\ 0 & \Delta(\frac{1}{15}\delta^6 H^5\,\cdot\,) - \frac{2}{3}\delta^4 H^3 \end{pmatrix},$$

$$\begin{pmatrix} -\Delta(H\,\cdot\,) & -\Delta(\frac{1}{3}\delta^2 H^3\,\cdot\,) \\ -\Delta(\frac{1}{3}\delta^2 H^3\,\cdot\,) & -\Delta(\frac{1}{5}\delta^4 H^5\,\cdot\,) + \frac{4}{3}\delta^2 H^3 \end{pmatrix} \begin{pmatrix} \frac{1}{10}\delta^2 H^2\Delta + 1 & -\delta^2 H^2 \\ -\frac{1}{2}\Delta & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \Delta(\frac{1}{15}\delta^2 H^3\Delta\,\cdot\,) - \Delta(H\,\cdot\,) & 0 \\ \Delta(\frac{1}{15}\delta^4 H^5\Delta\,\cdot\,) - \Delta(\delta^2 H^3\,\cdot\,) & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Delta(\frac{2}{3}\delta^2 H^3\,\cdot\,) \\ \frac{2}{3}\delta^2[\Delta, H^3] & \Delta(\frac{2}{15}\delta^4 H^5\,\cdot\,) + \frac{4}{3}\delta^2 H^3 \end{pmatrix},$$

$$\begin{pmatrix} -\Delta(H\,\cdot\,) & -\Delta(\frac{1}{3}\delta^2 H^3\,\cdot\,) \\ -\Delta(\frac{1}{3}\delta^2 H^3\,\cdot\,) & -\Delta(\frac{1}{5}\delta^4 H^5\,\cdot\,) + \frac{4}{3}\delta^2 H^3 \end{pmatrix} \left(1 - \frac{2}{5}\delta^2 H^2\Delta\right)$$
$$= \mathscr{A}_{22}^{(0)} + \begin{pmatrix} -\nabla\cdot((\nabla\eta)\,\cdot\,) & -\nabla\cdot(\delta^2 H^2(\nabla\eta)\,\cdot\,) \\ -\nabla\cdot(\delta^2 H^2(\nabla\eta)\,\cdot\,) & -\nabla\cdot(\delta^4 H^4(\nabla\eta)\,\cdot\,) + \frac{8}{3}\delta^4 H^4\nabla\eta\cdot\nabla \end{pmatrix},$$

$$\begin{pmatrix} \Delta(\frac{1}{15}\delta^2 H^3\Delta\,\cdot\,) - \Delta(H\,\cdot\,) \\ \Delta(\frac{1}{15}\delta^4 H^5\Delta\,\cdot\,) - \Delta(\delta^2 H^3\,\cdot\,) \end{pmatrix} (\boldsymbol{u}\cdot\nabla\partial^\alpha\phi_0 + \delta^2 H^2\boldsymbol{u}\cdot\nabla\partial^\alpha\phi_1)$$
$$= \mathscr{A}_{22}^{(1)} \begin{pmatrix} \partial^\alpha\phi_0 \\ \partial^\alpha\phi_1 \end{pmatrix} + \begin{pmatrix} \Delta(\frac{1}{15}\delta^2 H^3[\Delta,\boldsymbol{u}]\cdot\nabla\partial^\alpha\phi_0 + \frac{1}{15}\delta^4 H^3[\Delta, H^2\boldsymbol{u}]\cdot\nabla\partial^\alpha\phi_1) \\ \Delta(\frac{1}{15}\delta^4 H^5[\Delta,\boldsymbol{u}]\cdot\nabla\partial^\alpha\phi_0 + \frac{1}{15}\delta^6 H^5[\Delta, H^2\boldsymbol{u}]\cdot\nabla\partial^\alpha\phi_1) \end{pmatrix}$$
$$- \begin{pmatrix} \nabla\cdot([\nabla, H(\boldsymbol{u}\cdot\nabla)]\partial^\alpha\phi_0 + \delta^2[\nabla, H^3(\boldsymbol{u}\cdot\nabla)]\partial^\alpha\phi_1) \\ \nabla\cdot(\delta^2[\nabla, H^3(\boldsymbol{u}\cdot\nabla)]\partial^\alpha\phi_0 + \delta^4[\nabla, H^5(\boldsymbol{u}\cdot\nabla)]\partial^\alpha\phi_1) \end{pmatrix},$$

where

(4.15)
$$\mathscr{A}_{22}^{(0)} = \begin{pmatrix} \Delta(\frac{2}{5}\delta^2 H^3\Delta\,\cdot\,) & \Delta(\frac{2}{15}\delta^4 H^5\Delta\,\cdot\,) \\ \Delta(\frac{2}{15}\delta^4 H^5\Delta\,\cdot\,) & \Delta(\frac{2}{25}\delta^6 H^7\Delta\,\cdot\,) \end{pmatrix}$$
$$- \begin{pmatrix} \nabla\cdot(H\nabla\,\cdot\,) & \nabla\cdot(\frac{1}{3}\delta^2 H^3\nabla\,\cdot\,) \\ \nabla\cdot(\frac{1}{3}\delta^2 H^3\nabla\,\cdot\,) & \nabla\cdot(\frac{11}{15}\delta^4 H^5\nabla\,\cdot\,) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{4}{3}\delta^2 H^3 \end{pmatrix},$$

(4.16)
$$\mathscr{A}_{22}^{(1)} = \begin{pmatrix} \Delta(\frac{1}{15}\delta^2 H^3(\boldsymbol{u}\cdot\nabla)\Delta\,\cdot\,) & \Delta(\frac{1}{15}\delta^4 H^5(\boldsymbol{u}\cdot\nabla)\Delta\,\cdot\,) \\ \Delta(\frac{1}{15}\delta^4 H^5(\boldsymbol{u}\cdot\nabla)\Delta\,\cdot\,) & \Delta(\frac{1}{15}\delta^6 H^7(\boldsymbol{u}\cdot\nabla)\Delta\,\cdot\,) \end{pmatrix}$$
$$- \begin{pmatrix} \nabla\cdot(H(\boldsymbol{u}\cdot\nabla)\nabla\,\cdot\,) & \nabla\cdot(\delta^2 H^3(\boldsymbol{u}\cdot\nabla)\nabla\,\cdot\,) \\ \nabla\cdot(\delta^2 H^3(\boldsymbol{u}\cdot\nabla)\nabla\,\cdot\,) & \nabla\cdot(\delta^4 H^5(\boldsymbol{u}\cdot\nabla)\nabla\,\cdot\,) \end{pmatrix},$$

we obtain

$$\mathscr{A}_{22}^{(0)}\partial_t\begin{pmatrix}\partial^\alpha\phi_0\\ \partial^\alpha\phi_1\end{pmatrix}+\mathscr{A}_{22}^{(1)}\begin{pmatrix}\partial^\alpha\phi_0\\ \partial^\alpha\phi_1\end{pmatrix}+\mathscr{A}_{21}^{(1)}\partial^\alpha\eta=\begin{pmatrix}\delta\Delta G_{1,\alpha}+\nabla\cdot\boldsymbol{G}_{3,\alpha}\\ \delta^3\Delta G_{2,\alpha}+\delta^2\nabla\cdot\boldsymbol{G}_{4,\alpha}+\delta G_{5,\alpha}\end{pmatrix},$$

where

$$\mathscr{A}_{21}^{(1)}=\begin{pmatrix}\Delta(\frac{1}{15}\delta^2H^3\Delta(a\,\cdot\,))-\Delta(Ha\,\cdot\,)\\ \Delta(\frac{1}{15}\delta^4H^5\Delta(a\,\cdot\,))-\Delta(\delta^2H^3a\,\cdot\,)\end{pmatrix}\tag{4.17}$$

and

$$\begin{cases} G_{1,\alpha}=\dfrac{1}{15}\delta^3H^3[\Delta,H^2]\partial_t\partial^\alpha\phi_1+\dfrac{1}{15}\delta H^3\Delta F_7+\dfrac{2}{3}\delta H^3F_6\\ \qquad\quad -\dfrac{1}{15}\delta H^3[\Delta,\boldsymbol{u}]\cdot\nabla\partial^\alpha\phi_0-\dfrac{1}{15}\delta^3H^3[\Delta,H^2\boldsymbol{u}]\cdot\nabla\partial^\alpha\phi_1,\\ G_{2,\alpha}=H^2G_{1,\alpha}-\dfrac{8}{15}\delta H^5F_6,\\ \boldsymbol{G}_{3,\alpha}=(\nabla\eta)(\partial_t\partial^\alpha\phi_0+\delta^2H^2\partial_t\partial^\alpha\phi_1)-\nabla(HF_7)\\ \qquad\quad +[\nabla,H(\boldsymbol{u}\cdot\nabla)]\partial^\alpha\phi_0+\delta^2[\nabla,H^3(\boldsymbol{u}\cdot\nabla)]\partial^\alpha\phi_1,\\ \boldsymbol{G}_{4,\alpha}=H^2(\nabla\eta)(\partial_t\partial^\alpha\phi_0+\delta^2H^2\partial_t\partial^\alpha\phi_1)-\nabla(H^3F_7)\\ \qquad\quad +[\nabla,H^3(\boldsymbol{u}\cdot\nabla)]\partial^\alpha\phi_0+\delta^2[\nabla,H^5(\boldsymbol{u}\cdot\nabla)]\partial^\alpha\phi_1,\\ G_{5,\alpha}=-\dfrac{8}{3}\delta^3H^4\nabla\eta\cdot\nabla\partial_t\partial^\alpha\phi_1-\dfrac{2}{3}\delta^3H^3[\Delta,H^2]\partial_t\partial^\alpha\phi_1+\dfrac{2}{3}\delta[\Delta,H^3]F_5+\dfrac{4}{3}\delta H^3F_6. \end{cases}\tag{4.18}$$

As we will see later, $\mathscr{A}_{22}^{(0)}$ is positive and $\mathscr{A}_{22}^{(1)}$ is skew-symmetric in $L^2(\mathbf{R}^n)$ modulo lower order terms. This is the evolution equation for $(\partial^\alpha\phi_0,\partial^\alpha\phi_1)$.

We proceed to derive an evolution equation for $\partial^\alpha\eta$. In order to obtain the equation which has a good symmetry we need to note that

$$(\mathscr{A}_{21}^{(1)})^*=\big(a\Delta(\tfrac{1}{15}\delta^2H^3\Delta\,\cdot\,)-aH\Delta,\ a\Delta(\tfrac{1}{15}\delta^4H^5\Delta\,\cdot\,)-\delta^2aH^3\Delta\big),$$

where P^* denotes the adjoint operator of P in $L^2(\mathbf{R}^n)$. Applying ∂^α to the first and the second equations in (1.4) we obtain

$$\begin{cases} \partial_t\partial^\alpha\eta+\boldsymbol{u}\cdot\nabla\partial^\alpha\eta+H\Delta\partial^\alpha\phi_0+\dfrac{1}{3}\delta^2H^3\Delta\partial^\alpha\phi_1=F_8,\\ \partial_t\partial^\alpha\eta+\boldsymbol{u}\cdot\nabla\partial^\alpha\eta+\dfrac{1}{3}H\Delta\partial^\alpha\phi_0+\dfrac{1}{5}\delta^2H^3\Delta\partial^\alpha\phi_1-\dfrac{4}{3}H\partial^\alpha\phi_1=F_9, \end{cases}\tag{4.19}$$

where

$$\begin{cases} F_8=-[\partial^\alpha,\boldsymbol{u}]\cdot\nabla\eta-[\partial^\alpha,H]\Delta\phi_0-\dfrac{1}{3}\delta^2[\partial^\alpha,H^3]\Delta\phi_1,\\ F_9=-[\partial^\alpha,\boldsymbol{u}]\cdot\nabla\eta-\dfrac{1}{3}[\partial^\alpha,H]\Delta\phi_0-\dfrac{1}{5}\delta^2[\partial^\alpha,H^3]\Delta\phi_1+\dfrac{4}{3}[\partial^\alpha,H]\phi_1. \end{cases}\tag{4.20}$$

Applying the operators $a+\frac{1}{10}\delta^2\Delta(aH^2\,\cdot\,)$ and $-\frac{1}{2}\delta^2\Delta(aH^2\,\cdot\,)$ to the first and the second equations in (4.19), respectively, adding the resulting equations, and using the equality $\Delta(af)=a\Delta f-(\Delta a)f+2\nabla\cdot((\nabla a)f)$, we obtain

$$\mathscr{A}_{11}^{(0)}\partial_t\partial^\alpha\eta+\mathscr{A}_{11}^{(1)}\partial^\alpha\eta+\mathscr{A}_{12}^{(1)}\begin{pmatrix}\partial^\alpha\phi_0\\ \partial^\alpha\phi_1\end{pmatrix}=\delta\nabla\cdot\boldsymbol{G}_{6,\alpha}+G_{7,\alpha},$$

where

$$
(4.21) \qquad \begin{cases} \mathscr{A}_{11}^{(0)} = a - \nabla \cdot \left(\frac{2}{5} \delta^2 a H^2 \nabla \cdot \right), \\ \mathscr{A}_{11}^{(1)} = a(\boldsymbol{u} \cdot \nabla) - \nabla \cdot \left(\frac{2}{5} \delta^2 a H^2 (\boldsymbol{u} \cdot \nabla) \nabla \cdot \right), \\ \mathscr{A}_{12}^{(1)} = -(\mathscr{A}_{21}^{(1)})^* \end{cases}
$$

and

$$
(4.22) \qquad \begin{cases} \boldsymbol{G}_{6,\alpha} = \frac{2}{5} \delta [\nabla, a H^2] \partial_t \partial^\alpha \eta + \frac{2}{5} \delta [\nabla, a H^2 (\boldsymbol{u} \cdot \nabla)] \partial^\alpha \eta \\ \qquad + \frac{2}{15} (\nabla a)(\delta H^3 \Delta \partial^\alpha \phi_0 + \delta^3 H^5 \Delta \partial^\alpha \phi_1) + \frac{1}{10} \delta \nabla (a H^2 F_8) - \frac{1}{2} \delta \nabla (a H^2 F_9), \\ G_{7,\alpha} = -\frac{1}{15} (\Delta a)(\delta^2 H^3 \Delta \partial^\alpha \phi_0 + \delta^4 H^5 \Delta \partial^\alpha \phi_1) - \frac{2}{3} \delta^2 [\Delta, a H^3] \partial^\alpha \phi_1 + a F_8. \end{cases}
$$

This is the evolution equation for $\partial^\alpha \eta$.

To summarize we derived the evolution equations for $\partial^\alpha \boldsymbol{U}$:

$$
(4.23) \qquad \mathscr{A}^{(0)} \partial_t \partial^\alpha \boldsymbol{U} + \mathscr{A}^{(1)} \partial^\alpha \boldsymbol{U} = \boldsymbol{G}_\alpha,
$$

where

$$
(4.24) \qquad \mathscr{A}^{(0)} = \begin{pmatrix} \mathscr{A}_{11}^{(0)} & 0 \\ 0 & \mathscr{A}_{22}^{(0)} \end{pmatrix}, \quad \mathscr{A}^{(1)} = \begin{pmatrix} \mathscr{A}_{11}^{(1)} & \mathscr{A}_{12}^{(1)} \\ \mathscr{A}_{21}^{(1)} & \mathscr{A}_{22}^{(1)} \end{pmatrix},
$$

and

$$
(4.25) \qquad \boldsymbol{G}_\alpha = \begin{pmatrix} \delta \nabla \cdot \boldsymbol{G}_{6,\alpha} + G_{7,\alpha} \\ \delta \Delta G_{1,\alpha} + \nabla \cdot \boldsymbol{G}_{3,\alpha} \\ \delta^3 \Delta G_{2,\alpha} + \delta^2 \nabla \cdot \boldsymbol{G}_{4,\alpha} + \delta G_{5,\alpha} \end{pmatrix}.
$$

Using these equations we will derive uniform bounds of the solution in the next section.

5 Uniform estimates

In this section we will prove Theorem 2.1. Since the existence theorem has already been established in [17], it is sufficient to give a priori estimates of the solution. In the following of this paper we assume that $0 < \delta \le 1$.

In view of the equations (4.5) and (4.7) for the time derivatives and (2.9) for the initial data, we consider the following elliptic partial differential equations for (ψ_0, ψ_1):

$$
(5.1) \qquad \begin{cases} \psi_0 + \delta^2 H^2 \psi_1 = f_1, \\ H^2 (L_{11} \psi_0 + \delta^2 L_{12} \psi_1) = L_{12} \psi_0 + L_{22} \psi_1 + f_2 + \nabla \cdot \boldsymbol{f}_3, \end{cases}
$$

where $H = 1 + \eta$ and the operators L_{11}, L_{12}, L_{22} are those in (4.1). It follows from the first equation in (5.1) that $\psi_0 = f_1 - \delta^2 H^2 \psi_1$. Plugging this into the second equation in (5.1) to eliminate ψ_0, we obtain

$$
(5.2) \qquad L_1 \psi_1 = -\nabla \cdot \left(\frac{2}{3} H^3 \nabla f_1 + \boldsymbol{f}_3 \right) + 2 H^2 \nabla \eta \cdot \nabla f_1 - f_2,
$$

where $L_1 = L_1(H)$ is a second order differential operator defined by

$$L_1\psi_1 = \delta^2(H^2L_{11} - L_{12})(H^2\psi_1) + (L_{22} - \delta^2H^2L_{12})\psi_1. \tag{5.3}$$

We note that the operator L_1 is symmetric in $L^2(\mathbf{R}^2)$. As was shown in [17] that the operator L_1 is positive in $L^2(\mathbf{R}^n)$. More precisely, we have the following lemma.

Lemma 5.1 *Suppose that $H(x) \geq c_0 > 0$. There exists a positive constant $C = C(c_0)$ depending only on c_0 such that we have*

$$(L_1\psi_1, \psi_1)_{L^2} \geq C^{-1}(\|\psi_1\|^2 + \delta^2\|\nabla\psi_1\|^2).$$

Proof. We can prove the above estimate in exactly the same way as in [17]. For the sake of completeness, we sketch the proof. By direct calculation we have

$$\begin{aligned}
&(L_{11}\psi_0 + \delta^2L_{12}\psi_1, \psi_0)_{L^2} + (\delta^2L_{12}\psi_0 + \delta^2L_{22}\psi_1, \psi_1)_{L^2} \\
&= \int_{\mathbf{R}^n} \mathrm{d}x \int_0^{H(x)} \big(|\nabla\psi_0(x) + \delta^2z^2\nabla\psi_1(x)|^2 + (2\delta z\psi_1(x))^2\big)\,\mathrm{d}z \\
&\geq C^{-1}(\|\nabla\psi_0\|^2 + \delta^2\|\psi_1\|^2 + \delta^4\|\nabla\psi_1\|^2) \\
&\geq C^{-1}(\delta^2\|\psi_1\|^2 + \delta^4\|\nabla\psi_1\|^2).
\end{aligned}$$

Therefore, by the definition (5.3) of the operator L_1 we see that

$$\begin{aligned}
(L_1\psi_1, \psi_1)_{L^2} &= (L_{11}(-\delta H^2\psi_1) + \delta^2L_{12}(\delta^{-1}\psi_1), (-\delta H^2\psi_1))_{L^2} \\
&\quad + (\delta^2L_{12}(-\delta H^2\psi_1) + \delta^2L_{22}(\delta^{-1}\psi_1), \delta^{-1}\psi_1)_{L^2} \\
&\geq C^{-1}(\|\psi_1\|^2 + \delta^2\|\nabla\psi_1\|^2).
\end{aligned}$$

This gives the desired estimate. □

Once we obtain this type of estimate, we can easily show the unique existence of the solution to (5.2) so that to (5.1) in an appropriate Sobolev space by using the standard theory of elliptic partial differential equations. Concerning uniform estimates of the solution with respect to δ we have the following lemma.

Lemma 5.2 *Let $M_0, c_0 > 0$ and m be an integer such that $m > n/2 + 1$. There exists a positive constant C such that if η and $\delta \in (0, 1]$ satisfy*

$$\|\eta\|_m + \delta\|\eta\|_{m+1} \leq M_0, \quad c_0 \leq H(x) = 1 + \eta(x) \quad \textit{for} \quad x \in \mathbf{R}^n,$$

then for any $\nabla f_1, f_2, \boldsymbol{f}_3 \in H^l$ with $0 \leq l \leq m$, (5.1) has a unique solution (ψ_0, ψ_1) satisfying

$$\|\nabla\psi_0\|_l^2 + \delta^2\|\psi_1\|_l^2 + \delta^4\|\nabla\psi_1\|_l^2 \leq C(\|\nabla f_1\|_l^2 + \|\boldsymbol{f}_3\|_l^2 + \delta^2\|f_2\|_l^2). \tag{5.4}$$

If in addition $f_1 \in H^{l+1}$, then we have

$$\|\psi_0\|_l^2 \leq C(\|f_1\|_l^2 + \delta^2\|\nabla f_1\|_l^2 + \delta^2\|\boldsymbol{f}_3\|_l^2 + \delta^4\|f_2\|_l^2). \tag{5.5}$$

Remark 5.1 (1) We can reduce the hypothesis $f_2 \in H^l$ to $f_2 \in H^{l-1}$. In this case, the term $\|f_2\|_l^2$ in (5.4) and (5.5) should be replaced with $\delta^{-2}\|f_2\|_{l-1}^2$.

(2) This lemma guarantees that (2.9) determines uniquely the initial data $(\eta_{(0)}, \phi_{0(0)}, \phi_{1(0)})$ for the Isobe–Kakinuma model from the initial data $(\eta_{(0)}, \phi_{(0)})$ for the full water wave problem.

Proof. Throughout the proof we use the same symbol C to denote positive constants depending only on (M_0, c_0, m) and independent of δ. It follows from (5.2) and Lemma 5.1 that

$$\delta^2\|\psi_1\|^2 + \delta^4\|\nabla\psi_1\|^2 \le C(\|\nabla f_1\|^2 + \|\boldsymbol{f}_3\|^2 + \delta^2\|f_2\|^2), \tag{5.6}$$

which together with the first equation in (5.1) implies (5.4) in the case $l = 0$.

Let $l \ge 1$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index satisfying $|\alpha| \le l$. Applying ∂^α to (5.1) we obtain

$$\begin{cases} \partial^\alpha\psi_0 + \delta^2 H^2\partial^\alpha\psi_1 = \partial^\alpha f_1 - \delta^2[\partial^\alpha, H^2]\psi_1, \\ H^2(L_{11}\partial^\alpha\psi_0 + \delta^2 L_{12}\partial^\alpha\psi_1) = L_{12}\partial^\alpha\psi_0 + L_{22}\partial^\alpha\psi_1 \\ \qquad\qquad + \partial^\alpha f_2 - \dfrac{2}{15}\delta^2[\partial^\alpha, \nabla(H^2)]\cdot\nabla\psi_1 + \nabla\cdot\left(\partial^\alpha\boldsymbol{f}_3 + \dfrac{2}{15}\delta^2[\partial^\alpha, H^2]\nabla\psi_1\right). \end{cases}$$

We apply the estimate obtained just above to $(\partial^\alpha\psi_0, \partial^\alpha\psi_1)$ and obtain

$$\|\nabla\psi_0\|_l^2 + \delta^2\|\psi_1\|_l^2 + \delta^4\|\nabla\psi_1\|_l^2 \le C(\|\nabla f_1\|_l^2 + \|\boldsymbol{f}_3\|_l^2 + \delta^2\|f_2\|_l^2 + I), \tag{5.7}$$

where

$$I = \sum_{|\alpha|\le l}(\delta^4\|\nabla[\partial^\alpha, H^2]\psi_1\|^2 + \delta^4\|[\partial^\alpha, H^2]\nabla\psi_1\|^2 + \delta^6\|[\partial^\alpha, \nabla(H^2)]\cdot\nabla\psi_1\|^2).$$

Now, in view of the hypothesis $m > n/2+1$ and $1 \le l \le m$ we can use the standard commutator estimate $\|[\partial^\alpha, u]v\| \lesssim \|\nabla u\|_{m-1}\|v\|_{l-1}$. Since $\nabla[\partial^\alpha, H^2]\psi_1 = [\partial^\alpha, H^2]\nabla\psi_1 + [\partial^\alpha, \nabla(H^2)]\psi_1$, we obtain

$$\begin{aligned} I &\le C(\delta^4\|\nabla(H^2)\|_{m-1}^2\|\nabla\psi_1\|_{l-1}^2 + \delta^4\|\nabla(H^2)\|_m^2\|\psi_1\|_{l-1}^2 + \delta^6\|\nabla(H^2)\|_m^2\|\nabla\psi_1\|_{l-1}^2) \\ &\le C(\delta^4\|\psi_1\|_l^2 + \delta^2\|\psi_1\|_{l-1}^2 + \delta^4\|\nabla\psi_1\|_{l-1}^2) \\ &\le \epsilon(\delta^2\|\psi_1\|_l^2 + \delta^4\|\nabla\psi_1\|_l^2) + C_\epsilon\delta^2\|\psi_1\|^2 \end{aligned}$$

for any $\epsilon > 0$. Plugging this into (5.7), taking $\epsilon > 0$ sufficiently small, and using (5.6) we obtain (5.4), which together with the first equation in (5.1) implies (5.5). □

Now, we are ready to give a proof of Theorem 2.1. Let $\boldsymbol{U} = (\eta, \phi_0, \phi_1)^{\mathrm{T}}$ be a solution of the Isobe–Kakinuma model (1.4). In view of (4.23) we define a basic energy function $\mathscr{E}(\boldsymbol{U}) = (\mathscr{A}^{(0)}\boldsymbol{U}, \boldsymbol{U})_{L^2}$. It is easy to see that

$$\begin{aligned} \mathscr{E}(\boldsymbol{U}) &= (a\eta, \eta)_{L^2} + \frac{2}{5}\delta^2(aH^2\nabla\eta, \nabla\eta)_{L^2} + (H\nabla\phi_0, \nabla\phi_0)_{L^2} \\ &\quad + \frac{2}{3}\delta^2(H^3\nabla\phi_0, \nabla\phi_1)_{L^2} + \frac{1}{5}\delta^4(H^5\nabla\phi_1, \nabla\phi_1)_{L^2} + \frac{4}{3}\delta^2(H^3\phi_1, \phi_1)_{L^2} \\ &\quad + \frac{2}{5}\delta^2\biggl\{(H^3\Delta\phi_0, \Delta\phi_0)_{L^2} + \frac{2}{3}\delta^2(H^5\Delta\phi_0, \Delta\phi_1)_{L^2} + \frac{1}{5}\delta^4(H^7\Delta\phi_1, \Delta\phi_1)_{L^2} \\ &\qquad + \frac{4}{3}\delta^2(H^5\nabla\phi_1, \nabla\phi_1)_{L^2}\biggr\}, \end{aligned} \tag{5.8}$$

where $H = 1 + \eta$ and a is the function defined by (2.3). As is usual, a higher order energy function is defined by

$$\mathscr{E}_m(t) = \sum_{|\alpha|\le m}\mathscr{E}(\partial^\alpha\boldsymbol{U}(t)).$$

Here we remind that m is assumed to satisfy $m > n/2+1$. In view of (5.8) we see that this energy function $\mathscr{E}_m(t)$ is equivalent to

$$
\begin{aligned}
E_m(t) = \|\eta(t)\|_m^2 + \delta^2\|\eta(t)\|_{m+1}^2 + \|\nabla\phi_0(t)\|_m^2 + \delta^2\|\nabla\phi_0(t)\|_{m+1}^2 \\
+ \delta^2\|\phi_1(t)\|_m^2 + \delta^4\|\phi_1(t)\|_{m+1}^2 + \delta^6\|\phi_1(t)\|_{m+2}^2
\end{aligned}
$$

uniformly with respect to $\delta \in (0,1]$ under the positivity and the boundedness of H and a. More precisely, we have the following. Suppose that the solution $\boldsymbol{U} = (\eta, \phi_0, \phi_1)^{\mathrm{T}}$ satisfies

$$
E_m(t) \leq M_1, \quad \frac{1}{2}c_0 \leq H(x,t) \leq 2C_0, \quad \frac{1}{2} \leq a(x,t) \leq \frac{3}{2} \tag{5.9}
$$

for $x \in \mathbf{R}^n$, $0 \leq t \leq T_1$, and $0 < \delta \leq \delta_1$, where the constants c_0 and C_0 are determined from the initial datum $\eta_{(0)}$ by $c_0 \leq H(x,0) \leq C_0$ and the constants M_1, T_1, and δ_1 will be determined later. In the following we simply write the constants depending only on (c_0, C_0, m) and on (c_0, C_0, m, M_1) by C_1 and C_2, respectively, which may change from line to line. Then, it holds that

$$
C_1^{-1}E_m(t) \leq \mathscr{E}_m(t) \leq C_1 E_m(t) \tag{5.10}
$$

for $0 \leq t \leq T_1$ and $0 < \delta \leq 1$. It follows from (4.23) that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_m(t) = \sum_{|\alpha|\leq m} \{([\partial_t, \mathscr{A}^{(0)}]\partial^\alpha \boldsymbol{U}, \partial^\alpha \boldsymbol{U})_{L^2} + 2(\boldsymbol{G}_\alpha, \partial^\alpha \boldsymbol{U})_{L^2} - 2(\mathscr{A}^{(1)}\partial^\alpha \boldsymbol{U}, \partial^\alpha \boldsymbol{U})_{L^2}\}.
$$

Here it is easy to see that

$$
\begin{cases}
|([\partial_t, \mathscr{A}^{(0)}]\partial^\alpha \boldsymbol{U}, \partial^\alpha \boldsymbol{U})_{L^2}| \leq C_1|(\partial_t\eta, \partial_t a)|_\infty \mathscr{E}_m(t), \\
|(\boldsymbol{G}_\alpha, \partial^\alpha \boldsymbol{U})_{L^2}| \leq C_1\mathscr{E}_m(t) + \|(G_{1,\alpha}, G_{2,\alpha}, \boldsymbol{G}_{3,\alpha}, \boldsymbol{G}_{4,\alpha}, G_{5,\alpha}, \boldsymbol{G}_{6,\alpha}, G_{7,\alpha})\|^2.
\end{cases}
$$

By the definition (4.24) (see also (4.16) and (4.21)) and integration by parts, we see that

$$
\begin{aligned}
(\mathscr{A}^{(1)}\partial^\alpha \boldsymbol{U}, \partial^\alpha \boldsymbol{U})_{L^2} &= (\mathscr{A}_{11}^{(1)}\partial^\alpha\eta, \partial^\alpha\eta)_{L^2} + (\mathscr{A}_{22}^{(1)}\begin{pmatrix} \partial^\alpha\phi_0 \\ \partial^\alpha\phi_1 \end{pmatrix}, \begin{pmatrix} \partial^\alpha\phi_0 \\ \partial^\alpha\phi_1 \end{pmatrix})_{L^2} \\
&= -\frac{1}{2}((\nabla\cdot(a\boldsymbol{u}))\partial^\alpha\eta, \partial^\alpha\eta)_{L^2} + \frac{1}{5}\delta^2((\nabla\cdot(aH^2\boldsymbol{u}))\nabla\partial^\alpha\eta, \nabla\partial^\alpha\eta)_{L^2} \\
&\quad - \frac{1}{2}((\nabla\cdot(H\boldsymbol{u}))\nabla\partial^\alpha\phi_0, \nabla\partial^\alpha\phi_0)_{L^2} - \delta^2((\nabla\cdot(H^3\boldsymbol{u}))\nabla\partial^\alpha\phi_0, \nabla\partial^\alpha\phi_1)_{L^2} \\
&\quad - \frac{1}{2}\delta^4((\nabla\cdot(H^5\boldsymbol{u}))\nabla\partial^\alpha\phi_1, \nabla\partial^\alpha\phi_1)_{L^2} - \frac{1}{30}\delta^2((\nabla\cdot(H^3\boldsymbol{u}))\Delta\partial^\alpha\phi_0, \Delta\partial^\alpha\phi_0)_{L^2} \\
&\quad - \frac{1}{15}\delta^4((\nabla\cdot(H^5\boldsymbol{u}))\Delta\partial^\alpha\phi_0, \Delta\partial^\alpha\phi_1)_{L^2} - \frac{1}{30}\delta^6((\nabla\cdot(H^7\boldsymbol{u}))\Delta\partial^\alpha\phi_1, \Delta\partial^\alpha\phi_1)_{L^2}
\end{aligned}
$$

so that

$$
|(\mathscr{A}^{(1)}\partial^\alpha \boldsymbol{U}, \partial^\alpha \boldsymbol{U})_{L^2}| \leq C_1(1 + |(\nabla a, \nabla\eta)|_\infty)\|\boldsymbol{u}\|_{W^{1,\infty}}\mathscr{E}_m(t).
$$

Therefore, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_m(t) &\leq C_1(1 + |(\partial_t\eta, \partial_t a)|_\infty + (1 + |(\nabla\eta, \nabla a)|_\infty)\|\boldsymbol{u}\|_{W^{1,\infty}})\mathscr{E}_m(t) \\
&\quad + \sum_{|\alpha|\leq m}\|(G_{1,\alpha}, G_{2,\alpha}, \boldsymbol{G}_{3,\alpha}, \boldsymbol{G}_{4,\alpha}, G_{5,\alpha}, \boldsymbol{G}_{6,\alpha}, G_{7,\alpha})\|^2.
\end{aligned} \tag{5.11}
$$

We proceed to estimate the terms in the right-hand side of the above inequality under the condition (5.9). In the following, we use the standard calculus inequalities

$$\begin{cases} \|uv\|_l \lesssim |u|_\infty \|v\|_l + |v|_\infty \|u\|_l, \\ \|\nabla F(u)\|_l \lesssim C(|u|_\infty) \|\nabla u\|_l \end{cases}$$

and the Sobolev imbedding theorem $|u|_\infty \lesssim \|u\|_{m-1}$ without any comment. It follows from the first equation in (1.4), the necessary condition (2.2), and (4.11) that

$$\|\partial_t \eta\|_{m-1}^2 + \delta^2 \|\partial_t \eta\|_m^2 + \|\phi_1\|_{m-1}^2 + \|\boldsymbol{u}\|_m^2 + \delta^2 \|\boldsymbol{u}\|_{m+1}^2 \leq C_2 \mathscr{E}_m(t). \tag{5.12}$$

This together with the definitions (4.4) and (4.6) of F_1 and F_2 implies

$$\|F_1\|_m^2 + \delta^2 \|F_1\|_{m+1}^2 + \|F_2\|_{m-1}^2 + \delta^2 \|F_2\|_m^2 \leq C_2 \mathscr{E}_m(t).$$

Therefore, applying Lemma 5.2 with $l = m$ and $l = m - 1$ to the equation (4.5) for $(\partial_t \phi_0, \partial_t \phi_1)$ we obtain

$$\|\partial_t \phi_0\|_m^2 + \delta^2 \|\partial_t \phi_0\|_{m+1}^2 + \delta^2 \|\partial_t \phi_1\|_{m-1}^2 + \delta^4 \|\partial_t \phi_1\|_m^2 + \delta^6 \|\partial_t \phi_1\|_{m+1}^2 \leq C_2 \mathscr{E}_m(t). \tag{5.13}$$

Differentiating the first equation in (1.4) with respect to t we have

$$\partial_t^2 \eta = -\nabla \cdot \left(H \nabla \partial_t \phi_0 + \frac{1}{3} \delta^2 H^3 \nabla \partial_t \phi_1 + (\partial_t \eta) \boldsymbol{u} \right)$$

so that

$$\delta^2 \|\partial_t^2 \eta\|_{m-1}^2 \leq C_2 \mathscr{E}_m(t). \tag{5.14}$$

In view of (4.12) we have

$$\partial_t F_1 = (1 + 2\delta^2 H \boldsymbol{u} \cdot \nabla \phi_1 + 4\delta^2 H (\phi_1)^2) \partial_t \eta + \boldsymbol{u} \cdot \nabla \partial_t \phi_0 + \delta^2 H^2 \boldsymbol{u} \cdot \nabla \partial_t \phi_1 + 4\delta^2 H^2 \phi_1 \partial_t \phi_1,$$

which together with the definition (4.8) of F_3 and F_4 implies

$$\delta^2 \|F_3\|_m^2 + \delta^2 \|F_4\|_{m-1} \leq C_2 \mathscr{E}_m(t).$$

Therefore, applying Lemma 5.2 with $l = m - 1$ to the equation (4.7) for $(\partial_t^2 \phi_0, \partial_t^2 \phi_1)$ we obtain

$$\delta^4 \|\partial_t^2 \phi_1\|_{m-1}^2 + \delta^6 \|\partial_t^2 \phi_1\|_m^2 \leq C_2 \mathscr{E}_m(t). \tag{5.15}$$

Then, in view of the definition (2.3) of a we get

$$\delta^{-2} \|a - 1\|_{m-1}^2 + \|\partial_t a\|_{m-1}^2 + \|\nabla a\|_{m-1}^2 + \delta^2 \|\nabla a\|_m^2 \leq C_2 \mathscr{E}_m(t). \tag{5.16}$$

It follows from (5.9), (5.12), and (5.16) that

$$\delta^{-1} |a - 1|_\infty + |(\partial_t \eta, \partial_t a)|_\infty + \|(\eta, a, \boldsymbol{u})\|_{W^{1,\infty}} + \delta \|(\eta, a, \boldsymbol{u})\|_{W^{2,\infty}} \leq C_2. \tag{5.17}$$

By the definitions (4.10), (4.14), and (4.20) of $F_5, F_6, \ldots, F_9$, we also have

$$\|(F_5, F_8, F_9)\|^2 + \|F_7\|_1^2 + \delta^2 \big(\|F_6\|^2 + \|(F_5, F_8, F_9)\|_1^2 + \|F_7\|_2^2 \big) \leq C_2 \mathscr{E}_m(t). \tag{5.18}$$

Here, in the estimation of F_7 we used the fact that F_1 is a polynomial of $(\eta, \nabla\phi_0, \delta^2\nabla\phi_1, \delta\phi_1)$ with coefficients independent of δ and the calculus inequality

$$\|\partial^\alpha F(u) - D_u F(u)[\partial^\alpha u]\|_l \le C(\|u\|_{W^{1,\infty}})\|u\|_{|\alpha|+l-1}.$$

Therefore, by the definitions (4.18) and (4.22) of $G_{j,\alpha}$ we obtain

$$\|(G_{1,\alpha}, G_{2,\alpha}, \boldsymbol{G}_{3,\alpha}, \boldsymbol{G}_{4,\alpha}, G_{5,\alpha}, \boldsymbol{G}_{6,\alpha}, G_{7,\alpha})\|^2 \le C_2 \mathscr{E}_m(t).$$

Using this and (5.17) to (5.11) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_m(t) \le C_2 \mathscr{E}_m(t),$$

which together with Gronwall's inequality and (5.10) yields $E_m(t) \le C_1 e^{C_2 t} M_0^2$, where M_0 is the constant in the assumption (2.4) on the initial data.

To summarize, we have derived the estimates

$$E_m(t) \le C_1 e^{C_2 t} M_0^2, \quad |\eta(x,t) - \eta_{(0)}(x)| \le C_2 t, \quad |a(x,t) - 1| \le \delta C_2.$$

Taking these into account we define the constants M_1, T_1, and δ_1 by $M_1 = 2C_1 M_0^2$, $T_1 = C_2^{-1}\min\{\log 2, c_0/2, C_0\}$, and $\delta_1 = (2C_2)^{-1}$, respectively. Then, as in the usual way we can show that the solution actually satisfies (5.9), which together with (5.12) and (5.13) yields the uniform estimate (2.5) of the solution. This completes the proof of Theorem 2.1.

6 Consistency of the Isobe–Kakinuma model

In this section we will prove Theorem 2.2. Let (η, ϕ_0, ϕ_1) be a solution of the Isobe–Kakinuma model (1.4) satisfying the uniform bound (2.5), and define ϕ by the relation (2.6), that is, $\phi = \phi_0 + \delta^2 H^2 \phi_1$ with $H = 1 + \eta$. Then, it holds that

$$\|\nabla\phi(t)\|_m \le C \qquad \text{for} \quad 0 \le t \le T_1 \tag{6.1}$$

with a constant C independent of δ.

We begin with deriving equations for (η, ϕ) with errors of order $O(\delta^6)$. To this end we need to express ϕ_0 and ϕ_1 in terms of ϕ and η. Plugging $\phi_0 = \phi - \delta^2 H^2 \phi_1$ into the necessary condition (2.2) we obtain in turn that

$$\begin{cases} \phi_1 = -\dfrac{1}{2}\Delta\phi + \delta^2 R_1, \\ \phi_1 = -\dfrac{1}{2}\Delta\phi + \delta^2\left\{-\dfrac{1}{4}\Delta(H^2\Delta\phi) + \dfrac{1}{20}H^2\Delta^2\phi\right\} + \delta^4 R_2, \end{cases} \tag{6.2}$$

where

$$\begin{cases} R_1 = \dfrac{1}{2}\Delta(H^2\phi_1) - \dfrac{1}{10}H^2\Delta\phi_1, \\ R_2 = \dfrac{1}{2}\Delta(H^2 R_1) - \dfrac{1}{10}H^2\Delta R_1. \end{cases} \tag{6.3}$$

Plugging $\phi_0 = \phi - \delta^2 H^2 \phi_1$ into the first equation in (1.4) and using (6.2) we obtain

$$(6.4) \quad \begin{cases} \partial_t \eta = -\nabla \cdot (H\nabla\phi) + \delta^2 R_3, \\ \partial_t \eta = -\nabla \cdot (H\nabla\phi) - \dfrac{1}{3}\delta^2 \Delta(H^3\Delta\phi) + \delta^4 R_4, \\ \partial_t \eta = -\nabla \cdot (H\nabla\phi) - \dfrac{1}{3}\delta^2 \Delta(H^3\Delta\phi) \\ \qquad - \delta^4 \Big\{ \dfrac{1}{6}\Delta(H^3\Delta(H^2\Delta\phi)) - \dfrac{1}{30}\Delta(H^5\Delta^2\phi) \Big\} + \delta^6 R_5, \end{cases}$$

where

$$(6.5) \quad R_3 = \frac{2}{3}\Delta(H^3\phi_1), \quad R_4 = \frac{2}{3}\Delta(H^3 R_1), \quad R_5 = \frac{2}{3}\Delta(H^3 R_2).$$

We note that the last equation in (6.4) can be written in the symmetrical form as

$$(6.6) \quad \begin{aligned} &\partial_t \eta + \nabla \cdot (H\nabla\phi) + \frac{1}{3}\delta^2 \Delta(H^3\Delta\phi) \\ &\quad + \delta^4 \Big\{ \frac{1}{15}\Delta(H^3\Delta(H^2\Delta\phi)) + \frac{1}{15}\Delta(H^2\Delta(H^3\Delta\phi)) - \frac{1}{5}\Delta(|\nabla\eta|^2 H^3\Delta\phi) \Big\} = \delta^6 R_5. \end{aligned}$$

Plugging $\phi_0 = \phi - \delta^2 H^2 \phi_1$ into the third equation in (1.4) we obtain

$$\partial_t \phi + \eta + \frac{1}{2}|\nabla\phi|^2 - 2\delta^2 H(\phi_1 \partial_t \eta + \phi_1 \nabla\eta \cdot \nabla\phi) + 2\delta^2 H^2 (1 + \delta^2 |\nabla\eta|^2)(\phi_1)^2 = 0,$$

which together with (6.2) and (6.4) yields

$$(6.7) \quad \partial_t \phi + \eta + \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2}\delta^2 H^2 (\Delta\phi)^2 - \delta^4 (\Delta\phi)\Big\{ \frac{1}{3} H\Delta(H^3\Delta\phi) - \frac{1}{2}H^2 |\nabla\eta|^2 \Delta\phi \Big\} = \delta^6 R_6,$$

where

$$(6.8) \quad \begin{aligned} R_6 = H\Big\{ &-(\Delta\phi)R_4 - \Big(\frac{1}{2}\Delta(H^2\Delta\phi) - \frac{1}{10}H^2\Delta^2\phi \Big) R_3 + 2(\partial_t \eta + \nabla\eta \cdot \nabla\phi) R_2 \Big\} \\ &+ H^2 \Big\{ (\Delta\phi) R_2 + \Big(\frac{1}{2}\Delta(H^2\Delta\phi) - \frac{1}{10}H^2\Delta^2\phi \Big) R_1 - 2\phi_1 R_2 + |\nabla\eta|^2 (\Delta\phi - 2\phi_1) R_1 \Big\}. \end{aligned}$$

(6.6) and (6.7) are the desired equations for (η, ϕ) with errors of order $O(\delta^6)$.

Next, we proceed to expand the full water wave equations with respect to δ with errors of order $O(\delta^6)$. To this end we need to expand the Dirichlet-to-Neumann map $\Lambda(\eta, \delta)$ with respect to δ. It is well-known that $\Lambda(\eta, \delta)$ can be expanded with respect to δ^2 as

$$(6.9) \quad \Lambda(\eta, \delta) = \Lambda^{(0)}(\eta) + \delta^2 \Lambda^{(1)}(\eta) + \delta^4 \Lambda^{(2)}(\eta) + \cdots .$$

The explicit forms of these linear operators $\Lambda^{(j)}(\eta)$ are given by the following lemma.

Lemma 6.1 *It holds that*

$$\begin{cases} \Lambda^{(0)}(\eta)\psi = -\nabla \cdot (H\nabla\psi), \\ \Lambda^{(1)}(\eta)\psi = -\dfrac{1}{3}\Delta(H^3\Delta\psi), \\ \Lambda^{(2)}(\eta)\psi = -\dfrac{1}{15}\Delta(H^3\Delta(H^2\Delta\psi)) - \dfrac{1}{15}\Delta(H^2\Delta(H^3\Delta\psi)) + \dfrac{1}{5}\Delta(|\nabla\eta|^2 H^3\Delta\psi). \end{cases}$$

Proof. These formulae can be derived by the method in T. Iguchi [3] and D. Lannes [12]. For the sake of completeness we sketch the proof by following [12]. Let Φ be the unique solution of the boundary value problem (1.9), define a diffeomorphism Θ from the strip $\mathbf{R}^n \times (-1, 0)$ to the water region by $\Theta(x, z) = (x, z + (z + 1)\eta(x))$, and set $\tilde{\Phi} = \Phi \circ \Theta$. Then, $\tilde{\Phi}$ satisfies the boundary value problem

$$
\begin{cases} H^{-1}\partial_z^2\tilde{\Phi} + \delta^2\nabla_X \cdot P\nabla_X\tilde{\Phi} = 0 & \text{in} \quad -1 < z < 0, \\ \tilde{\Phi} = \phi & \text{on} \quad z = 0, \\ \partial_z\tilde{\Phi} = 0 & \text{on} \quad z = -1, \end{cases} \tag{6.10}
$$

where $H = 1 + \eta$, $\nabla_X = (\nabla, \partial_z)$, and

$$
P(x, z) = \begin{pmatrix} H(x)I_n & -(z + 1)\nabla\eta(x) \\ -(z + 1)(\nabla\eta(x))^{\mathrm{T}} & H^{-1}|\nabla\eta(x)|^2(z + 1)^2 \end{pmatrix}
$$

with the identity matrix I_n of size n. On the other hand, we define $\overline{\boldsymbol{V}}$ by

$$
\begin{aligned} \overline{\boldsymbol{V}}(x) &= \frac{1}{H(x)} \int_{-1}^{\eta(x)} \nabla\Phi(x, z)\,\mathrm{d}z \\ &= \int_{-1}^{0} \{\nabla\tilde{\Phi}(x, z) - (z + 1)H(x)^{-1}(\nabla\eta(x))\partial_z\tilde{\Phi}(x, z)\}\,\mathrm{d}z, \end{aligned}
$$

which is the vertical average of the horizontal component of the velocity field. Then, it holds that $\Lambda(\eta, \delta)\phi = -\nabla \cdot (H\overline{\boldsymbol{V}})$. Now, expanding $\tilde{\Phi}$ and $\overline{\boldsymbol{V}}$ with respect to δ^2 as

$$
\begin{cases} \tilde{\Phi} = \tilde{\Phi}_0 + \delta^2\tilde{\Phi}_1 + \delta^4\tilde{\Phi}_2 + \cdots, \\ \overline{\boldsymbol{V}} = \overline{\boldsymbol{V}}_0 + \delta^2\overline{\boldsymbol{V}}_1 + \delta^4\overline{\boldsymbol{V}}_2 + \cdots, \end{cases}
$$

we have

$$
\begin{cases} \Lambda^{(j)}(\eta)\phi = -\nabla \cdot (H\overline{\boldsymbol{V}}_j), \\ \overline{\boldsymbol{V}}_j(x) = \displaystyle\int_{-1}^{0} \{\nabla\tilde{\Phi}_j(x, z) - (z + 1)H(x)^{-1}(\nabla\eta(x))\partial_z\tilde{\Phi}_j(x, z)\}\,\mathrm{d}z. \end{cases} \tag{6.11}
$$

Plugging the above expansion of $\tilde{\Phi}$ into (6.10) we see that $\tilde{\Phi}_0(x, z) = \phi(x)$ and

$$
\begin{cases} H^{-1}\partial_z^2\tilde{\Phi}_j = -\nabla_X \cdot P\nabla_X\tilde{\Phi}_{j-1} & \text{in} \quad -1 < z < 0, \\ \tilde{\Phi}_j = 0 & \text{on} \quad z = 0, \\ \partial_z\tilde{\Phi}_j = 0 & \text{on} \quad z = -1 \end{cases}
$$

for $j = 1, 2, \ldots$. It is not difficult to solve this boundary value problem and we obtain

$$
\begin{cases} \tilde{\Phi}_1(x, z) = \left(-\frac{1}{2}(z + 1)^2 + \frac{1}{2}\right)H(x)^2\Delta\phi(x), \\ \tilde{\Phi}_2(x, z) = \left(\frac{1}{8}(z + 1)^4 - \frac{1}{4}(z + 1)^2 + \frac{1}{8}\right)H(x)^2\Delta(H(x)^2\Delta\phi(x)) \\ \qquad + \left(-\frac{1}{12}(z + 1)^4 + \frac{1}{12}\right)H(x)\Delta(H(x)^3\Delta\phi(x)) \\ \qquad + \left(\frac{1}{4}(z + 1)^4 - \frac{1}{4}\right)H(x)^2|\nabla\eta(x)|^2\Delta\phi(x). \end{cases}
$$

Plugging these into (6.11) we obtain the desired formulae. □

By the formulae in this lemma we can rewrite (6.6) as

$$\partial_t\eta - \Lambda^{(0)}(\eta) - \delta^2\Lambda^{(1)}(\eta) - \delta^4\Lambda^{(2)}(\eta) = \delta^6 R_5. \tag{6.12}$$

We define remainder terms R_7, R_8, R_9 of the expansion (6.9) by

$$\begin{cases} \Lambda(\eta,\delta)\phi = \Lambda^{(0)}(\eta)\phi + \delta^2 R_7, \\ \Lambda(\eta,\delta)\phi = \Lambda^{(0)}(\eta)\phi + \delta^2\Lambda^{(1)}(\eta)\phi + \delta^4 R_8, \\ \Lambda(\eta,\delta)\phi = \Lambda^{(0)}(\eta)\phi + \delta^2\Lambda^{(1)}(\eta)\phi + \delta^4\Lambda^{(2)}(\eta)\phi + \delta^6 R_9. \end{cases} \tag{6.13}$$

In view of the identities $(1+\delta^2|\nabla\eta|^2)^{-1} = 1 - \delta^2|\nabla\eta|^2 + \delta^4|\nabla\eta|^4(1+\delta^2|\nabla\eta|^2)^{-1}$ and

$$\begin{aligned} &(\Lambda(\eta,\delta)\phi + \nabla\eta\cdot\nabla\phi)^2 \\ &= (\Lambda^{(0)}(\eta)\phi + \nabla\eta\cdot\nabla\phi)^2 + \delta^2(\Lambda^{(0)}(\eta)\phi + \Lambda(\eta,\delta)\phi + 2\nabla\eta\cdot\nabla\phi)R_7 \\ &= (\Lambda^{(0)}(\eta)\phi + \nabla\eta\cdot\nabla\phi)^2 + 2\delta^2(\Lambda^{(0)}(\eta)\phi + \nabla\eta\cdot\nabla\phi)(\Lambda^{(1)}(\eta)\phi) \\ &\quad + \delta^4\{(\Lambda^{(0)}(\eta)\phi + \Lambda(\eta,\delta)\phi + 2\nabla\eta\cdot\nabla\phi)R_8 + (\Lambda^{(1)}(\eta)\phi)R_7\}, \end{aligned}$$

and Lemma 6.1, we have

$$\begin{aligned} &\frac{1}{2}\delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda(\eta,\delta)\phi + \nabla\eta\cdot\nabla\phi)^2 \\ &= \frac{1}{2}\delta^2(\Lambda^{(0)}(\eta)\phi + \nabla\eta\cdot\nabla\phi)^2 \\ &\quad + \delta^4\left\{(\Lambda^{(0)}(\eta)\phi + \nabla\eta\cdot\nabla\phi)(\Lambda^{(1)}(\eta)\phi) - \frac{1}{2}|\nabla\eta|^2(\Lambda^{(0)}(\eta)\phi + \nabla\eta\cdot\nabla\phi)^2\right\} + \delta^6 R_{10} \\ &= \frac{1}{2}\delta^2 H^2(\Delta\phi)^2 + \delta^4(\Delta\phi)\left\{\frac{1}{3}H\Delta(H^3\Delta\phi) - \frac{1}{2}H^2|\nabla\eta|^2\Delta\phi\right\} + \delta^6 R_{10}, \end{aligned}$$

where

$$\begin{aligned} R_{10} = &\frac{1}{2}|\nabla\eta|^4(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda(\eta,\delta)\phi + \nabla\eta\cdot\nabla\phi)^2 \\ &- \frac{1}{2}|\nabla\eta|^2(\Lambda^{(0)}(\eta)\phi + \Lambda(\eta,\delta)\phi + 2\nabla\eta\cdot\nabla\phi)R_7 \\ &+ \frac{1}{2}\{(\Lambda^{(0)}(\eta)\phi + \Lambda(\eta,\delta)\phi + 2\nabla\eta\cdot\nabla\phi)R_8 + (\Lambda^{(1)}(\eta)\phi)R_7\}. \end{aligned} \tag{6.14}$$

This together with (6.7), (6.12), and (6.13) yields

$$\begin{cases} \partial_t\eta - \Lambda(\eta,\delta)\phi = \delta^6 r_1, \\ \partial_t\phi + \eta + \dfrac{1}{2}|\nabla\phi|^2 - \delta^2\dfrac{(\Lambda(\eta,\delta)\phi + \nabla\eta\cdot\nabla\phi)^2}{2(1+\delta^2|\nabla\eta|^2)} = \delta^6 r_2, \end{cases}$$

where

$$r_1 = R_5 - R_9, \quad r_2 = R_6 - R_{10}. \tag{6.15}$$

This is the equation (2.7) in Theorem 2.2.

It remains to show the uniform bound (2.8). To this end we need estimations related to the Dirichlet-to-Neumann map $\Lambda(\eta,\delta)$. The following lemma is given in T. Iguchi [3].

Lemma 6.2 *Let $M_0, c_0 > 0$ and $l > n/2 + 1$. There exists a positive constant C such that if $\eta \in H^{l+1}$ and $\delta \in (0,1]$ satisfy $\|\eta\|_{l+1} \le M_0$ and $H(x) \ge c_0$ for $x \in \mathbf{R}^n$, then we have*

$$\|\Lambda(\eta,\delta)\phi\|_l \le C\|\nabla\phi\|_{l+1}.$$

In order to give systematically error estimates of the expansion (6.9) it would be better to follow the strategy given by D. Lannes [12]. The following lemma comes easily from the result in [12].

Lemma 6.3 *Let $M_0, c_0 > 0$ and suppose that K and l are nonnegative integers such that $l + 2K+1 > n/2$. There exists a positive constant C such that if $\eta \in H^{l+2K+3}$ and $\delta \in (0,1]$ satisfy $\|\eta\|_{l+2K+3} \le M_0$ and $H(x) \ge c_0$ for $x \in \mathbf{R}^n$, then we have*

$$\|\Lambda(\eta,\delta)\phi - \sum_{k=0}^{K} \Lambda^{(k)}(\eta)\phi\|_l \le C\delta^{2K+2}\|\nabla\phi\|_{l+2K+3}.$$

In what follows we denote constants depending on $p_1, p_2, \ldots$ by the same symbol $C(p_1, p_2, \ldots)$ which may change from line to line. Moreover, we may assume that $C(p_1, p_2, \ldots)$ a nondecreasing function of each variable p_j. Let l be a nonnegative integer and $t_0 > n/2$. From (6.3) and (6.5) it follows in turn that

$$\begin{cases} \|(R_1,R_3)\|_l \le C(\|(\eta,\phi_1)\|_{t_0}, \|(\eta,\phi_1)\|_{l+2}), \\ \|(R_2,R_4)\|_l \le C(\|(\eta,R_1)\|_{t_0}, \|(\eta,R_1)\|_{l+2}) \le C(\|(\eta,\phi_1)\|_{t_0+2}, \|(\eta,\phi_1)\|_{l+4}), \\ \|R_5\|_l \le C(\|(\eta,R_2)\|_{t_0}, \|(\eta,R_2)\|_{l+2}) \le C(\|(\eta,\phi_1)\|_{t_0+4}, \|(\eta,\phi_1)\|_{l+6}), \end{cases}$$

which together with (6.15), (6.13), and Lemma 6.3 yields

$$\|r_1\|_l \le C(\|(\eta,\phi_1)\|_{t_0+4}, \|\phi_1\|_{l+6}, \|(\eta,\nabla\phi)\|_{l+7}).$$

In view of (2.5) and (6.1) choosing $t_0 = m-5$ and $l = m-7$ we obtain the estimate for r_1 in (2.8). Similarly, we have

$$\|R_6\|_l \le C(\|(\eta,\phi_1)\|_{t_0+4}, \|\nabla\phi\|_{t_0+3}, \|\partial_t\eta\|_{t_0}, \|(\eta,\phi_1)\|_{l+4}, \|\nabla\phi\|_{l+3}, \|\partial_t\eta\|_l).$$

It follows from (6.13) and Lemmas 6.2 and 6.3 that

$$\|\Lambda(\eta,\delta)\phi\|_{l+4} + \|R_7\|_{l+2} + \|R_8\|_l \le C(\|\eta\|_{l+5})\|\nabla\phi\|_{l+5},$$

so that

$$\|r_2\|_l \le C(\|(\eta,\nabla\phi)\|_{l+5}, \|\phi_1\|_{l+4}, \|\partial_t\eta\|_l).$$

if $l > n/2$. By choosing $l = m-5$ we obtain the estimate for r_2 in (2.8). The proof of Theorem 2.2 is complete.

7 Rigorous justification of the Isobe–Kakinuma model

In this section we will prove Theorem 2.4. To this end we take advantage of the stability of the full water wave equations (1.6), which is given by the following theorem. Although the statement is not explicitly given in T. Iguchi [3], we can prove it in exactly the same way as the proof of Theorem 2.3, so that we omit the proof. See also D. Lannes [12].

Theorem 7.1 *In addition to hypothesis of Theorem* 2.3 *we assume that* $0 < \delta \le \delta_1$ *and that* $(\eta^{\rm app}, \phi^{\rm app})$ *satisfy the equations*

$$\begin{cases} \partial_t \eta^{\rm app} - \Lambda(\eta^{\rm app}, \delta)\phi^{\rm app} = f_1^{\rm err}, \\ \partial_t \phi^{\rm app} + \eta^{\rm app} + \dfrac{1}{2}|\nabla \phi^{\rm app}|^2 - \delta^2 \dfrac{(\Lambda(\eta^{\rm app}, \delta)\phi^{\rm app} + \nabla \eta^{\rm app} \cdot \nabla \phi^{\rm app})^2}{2(1 + \delta^2 |\nabla \eta^{\rm app}|^2)} = f_2^{\rm err}, \end{cases}$$

on a time interval $[0, T_1]$*, the initial condition* (1.7)*, and the uniform bound:*

$$\begin{cases} \|\eta^{\rm app}(t)\|_{m+3+1/2} + \|\nabla \phi^{\rm app}(t)\|_{m+3} \le M_1, \\ 1 + \eta^{\rm app}(x,t) \ge c_0/2 \qquad \text{for} \quad x \in \mathbf{R}^n, \ 0 \le t \le T_1. \end{cases}$$

Let $(\eta^{\rm WW}, \phi^{\rm WW})$ *be the solution obtained in Theorem* 2.3 *and put* $T_* = \min\{T_1, T_2\}$ *and* $\delta_* = \min\{\delta_1, \delta_2\}$*, where* T_2 *and* δ_2 *are the constants in Theorem* 2.3*. Then, we have*

$$\begin{aligned} &\sup_{0 \le t \le T_*} \big(\|\eta^{\rm WW}(t) - \eta^{\rm app}(t)\|_{m+2} + \|\nabla \phi^{\rm WW}(t) - \nabla \phi^{\rm app}(t)\|_{m+1}\big) \\ &\qquad\qquad \le C_2 \sup_{0 \le t \le T_*} (\|f_1^{\rm err}(t)\|_{m+2} + \|\Lambda_0(\delta)^{1/2} f_2^{\rm err}(t)\|_{m+2}), \end{aligned}$$

where $\Lambda_0(\delta) = \Lambda(0, \delta)$ *and* C_2 *is a positive constant independent of* $\delta \in (0, \delta_*]$*.*

Suppose that the hypotheses in Theorem 2.4 are satisfied for the initial data $(\eta_{(0)}, \phi_{(0)})$. By Lemma 5.2 with m replaced by $m+10$ we see that (2.9) determines uniquely the initial data $(\phi_{0(0)}, \phi_{1(0)})$ satisfying

$$\|\nabla \phi_{0(0)}\|_{m+10} + \delta \|\phi_{1(0)}\|_{m+10} + \delta^2 \|\phi_{1(0)}\|_{m+11} \le C_0$$

with a constant C_0 independent of δ. For these initial data $(\eta_{(0)}, \phi_{0(0)}, \phi_{1(0)})$ the conditions in Theorems 2.1 and 2.2 with m replaced by $m+9$ are satisfied. Therefore, the initial value problem (1.4)–(1.5) for the Isobe–Kakinuma model has a unique solution $(\eta^{\rm IK}, \phi_0, \phi_1)$ on the time interval $[0, T_1]$ independent of $\delta \in (0, \delta_1]$. Moreover, the solution satisfies the uniform bound (2.5) with m replaced by $m+9$. Put $\phi^{\rm IK} = \phi_0 + \delta^2(1+\eta^{\rm IK})^2 \phi_1$. Then, by Theorem 2.2 we see that $(\eta^{\rm IK}, \phi^{\rm IK})$ satisfies (2.7) with (r_1, r_2) satisfying

$$\|r_1(t)\|_{m+2} + \|r_2(t)\|_{m+4} \le C \qquad \text{for} \quad 0 \le t \le T_1,$$

where C is a constant independent of $\delta \in (0, \delta_1]$. Moreover, we have

$$\|\eta^{\rm IK}(t)\|_{m+9} + \|\nabla \phi^{\rm IK}(t)\|_{m+9} \le C \qquad \text{for} \quad 0 \le t \le T_1.$$

Therefore, we can apply Theorem 7.1 and obtain

$$\begin{aligned} &\sup_{0 \le t \le T_*} \big(\|\eta^{\rm WW}(t) - \eta^{\rm IK}(t)\|_{m+2} + \|\nabla \phi^{\rm WW}(t) - \nabla \phi^{\rm IK}(t)\|_{m+1}\big) \\ &\qquad \le C\delta^6 \sup_{0 \le t \le T_*} (\|r_1(t)\|_{m+2} + \|\Lambda_0(\delta)^{1/2} r_2(t)\|_{m+2}) \\ &\qquad \le C\delta^6 \sup_{0 \le t \le T_*} (\|r_1(t)\|_{m+2} + \|r_2(t)\|_{m+3}) \\ &\qquad \le C\delta^6, \end{aligned}$$

where we used the estimate $\|\Lambda_0(\delta)^{1/2}\psi\|_s \le \|\nabla \psi\|_s$. This completes the proof of Theorem 2.4.

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