SMALL DIVISOR AND DISCRETE SHOCK PROFILE

SHIH-HSIEN YU

Abstract. We consider a discrete travelling wave solution of a finite difference approximation to a shock wave. The existence of a travelling wave solution is the key to understand the finite difference approximations to hyperbolic conservation laws. Particularly, the structures of discrete travelling wave solutions play the main role in analyzing the error of a finite difference approximation. The structure of a discrete travelling wave solution contains much richer wave phenomenon than the hyperbolic conservation laws itself. It is due to the coupling of shock waves and mesh points of the finite difference approximation. This coupling results in a small divisor problem. We will give the analysis about the coupling of shock waves and mesh points.

1. Introduction

Consider a system of hyperbolic conservation laws

$$\begin{aligned} &u_t + f(u)_x = 0,\ u \in \mathbf{R}^n, \\ &f'(u)\ r_i(u) = \lambda_i(u)\ r_i(u),\ l_i(u)\ f'(u) = \lambda_i(u) l_i(u) \text{ for } i = 1, \cdots, n \\ &\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \end{aligned} \tag{1.1}$$

Furthermore, each characteristic field $\lambda_i(u)$ is either genuinely nonlinear

$$\nabla_{r_i(u)}\lambda_i(u) \neq 0 \text{ for } u \in \mathbf{R}^n \tag{g.nl}$$

or linearly degenerated

$$\nabla_{r_i(u)}\lambda_i(u) \equiv 0 \text{ for } u \in \mathbf{R}^n. \tag{l.dg}$$

A shock wave solution (u_-, u_+) of (1.1) is a two-valued weak solution

$$u(x,t) = \begin{cases} u_- \text{ for } x < st, \\ u_+ \text{ for } x > st, \end{cases}$$

satisfying the Lax's entropy condition, see [3]

$$\begin{cases} \lambda_i(u_-) > s > \lambda_i(u_+), \\ \lambda_{i+1}(u_+) > s > \lambda_{i-1}(u_-). \end{cases} \tag{E}$$

where s is the shock speed given by the Rankine-Hugoniot condition

$$s(u_- - u_+) = f(u_-) - f(u_+). \tag{R-H}$$

Finite difference approximations are often used to approximate the solution of (1.1). Most of the schemes used for computing (1.1) are conservative finite difference scheme. A conservative finite difference scheme for (1.1) is a sequence of functions $\{u_k\}_{k\in\mathbf{N}}$ to approximate $\{u(\cdot, k\Delta t)\}_{k\in\mathbf{N}}$, where $(\Delta x, \Delta t)$ is a pair of the space-time mesh sizes of a given conservative finite difference scheme:

$$
\begin{aligned}
&u_k(x) = \mathscr{L}^k[u_0](x),\\
&\mathscr{L}[v](x) \equiv v(x) - \frac{\Delta t}{\Delta x}\frac{\{F[v](x+\frac{\Delta x}{2}) - F[v](x-\frac{\Delta x}{2})\}}{2},
\end{aligned}
\tag{1.2}
$$

where $F[v]$ is the numerical flux which is a scheme dependent functional, and $u_0(x)$ is the initial value of (1.1). A **consistence condition** for a numerical flux to be consistent with the flux of (1.1) is

$$
F[\vec{v}] = f(\vec{v}) \text{ for all constant vector-valued function } \vec{v}. \tag{C}
$$

For well-posedness of a numerical scheme, the **CFL** condition is necessary by imposing $\Delta t/\Delta x$ to satisfy

$$
\frac{\Delta t}{\Delta x}\sup_{1\le i\le n}|\lambda_i(u)| < 1 \text{ for all } u \text{ under consideration.} \tag{CFL}
$$

In this paper, we assume the scheme is **dissipative** in the following sense

$$
\left|\left.\frac{d\mathscr{L}[u+\epsilon\, e^{-i\xi\frac{x}{\Delta x}}]}{d\epsilon}\right|_{\epsilon=0}\right| < 1 - C(u)\;|\xi|^2 \text{ for some } C(u)>0 \text{ and } \xi\in[-\pi,\pi]. \tag{D}
$$

A **discrete shock profile** $\phi(\xi)$ connecting (u_-, u_+) is a **travelling wave solution** of a finite difference scheme satisfying

$$
\begin{cases}
u_k(x) \equiv \phi\left(\frac{x-sk\Delta t}{\Delta x}\right),\\
\lim_{\xi\to\pm\infty}\phi(\xi) = u_\pm.
\end{cases}
\tag{T}
$$

A discrete shock profile is a continuum function whose shape is invariant under the numerical iterations; and its structure can be described in terms of the grid points of the scheme.

There is an important condition on the **CFL speed** $\frac{\Delta t}{\Delta x}s$ for a discrete shock profile. The CFL speed is assumed to be a Diophantine number. Diophantine number is an irrational number which can not be well approximated by rational numbers in the following sense.

Definition 1.1. *α is a Diophantine number of degree $\mu > 0$:*
There exists $\beta > 0$ such that

$$
|\alpha - \frac{p}{q}| > \frac{\beta}{|q|^\mu} \textit{ for all } p,\ q\in\mathbf{Z}.
$$

Theorem 1.2. *Let* $\boldsymbol{u}_0 \in \mathbf{R}^n$ *satisfying* $\lambda_i(\boldsymbol{u}_0) = s$ *and* $\frac{\Delta t}{\Delta x}s$ *is a Diophantine number with degree greater than two. Suppose that the conservative finite difference scheme* $\mathscr{L}$ *is dissipative and CFL condition satisfied. Then, there exists* $\epsilon > 0$ *such that for any shock* (u_-, u_+) *with speed* s *and with that* $\|u_- - \boldsymbol{u}_0\| + \|u_+ - \boldsymbol{u}_0\| < \epsilon$ *there is a discrete shock profile* ϕ *connecting* (u_-, u_+).

It is an interesting problem to analyze the error between the finite difference approximation and the solution of (1.1). When the solution of (1.1) is smooth, due to (D) the first variation of $\mathscr{L}$ at u is L^2-stable. Through a linear L^2-stability theorem, one can show that the error remains $O(1)\Delta x$, see [5]. When $u(x,t)$ contains a discontinuity, the errors won't converge even though the mesh sizes tend to zero. There is an error of order $O(1)$ concentrated around the shock wave. This structure can be realized as the presence of a discrete shock layer. Its shape is invariant under numerical iterations. Discrete shock profiles could serve as an inner solution for constructing an approximate solution to the finite difference approximation, see [1].

2. Primary Approximation to the Travelling Wave Solution

A finite difference scheme can be defined in terms of grid points and the ratio of the time to space mesh size. With the ratio $\frac{\Delta t}{\Delta x} = \lambda$ fixed, we can rescale the space-time grid sizes $(\Delta x, \Delta t) = (1, \lambda)$. Under this rescaling the scheme is given in terms of grid points. Then, the equation for $\phi(x)$ in (T) becomes

$$\begin{aligned} &\phi(x - s\lambda) - \mathscr{L}[\phi](x) = 0, \\ &\lim_{x\to\pm\infty} \phi(x) = u_\pm. \end{aligned} \tag{2.1}$$

For simplicity of our presentation, we may assume the scheme $\mathscr{L}[v](x)$ is

$$\mathscr{L}[v](x) = \frac{v(x+1) + v(x) + v(x-1)}{3} - \lambda\frac{f(v(x+1)) - f(v(x-1))}{2}$$

as well as the CFL speed is

$$\lambda s = \frac{\sqrt{2}}{4}.$$

Due to [2], a finite difference scheme approximates the solution of $u_t + f(u)_x = (O(1)\Delta x\; u_x)_x$ rather than the solution of $u_t + f(u)_x = 0$. This is the effect of a numerical viscosity. So, we use the travelling solution of $u_t + f(u)_x = (O(1)\Delta x\; u_x)_x$ to construct a primary approximation $U(x)$ to (2.1) as follows, see [4]:

$$\begin{cases} -s\lambda\hat{U}_x + \lambda f(\hat{U})_x - \frac{1}{2}\left(\frac{2}{3} - (s\lambda)^2\right)\hat{U}_{xx} = 0, \\ \lim_{x\to\pm\infty}\hat{U}(x) = u_\pm, \end{cases}$$

$$U(x) \equiv \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \hat{U}(\xi)\, d\xi.$$

Substitute $U(x)$ into (2.1) to obtain that

$$U(x-\lambda s)-\mathscr{L}[U](x)=\Delta_0 O(1)\ \epsilon^3\ e^{-O(1)\epsilon|x|}, \tag{2.2}$$

where $\epsilon \equiv \|u_- - u_+\|$ and $O(1)$ is a positive bounded function independent of ϵ, and the center difference operator Δ_0 is given by

$$\begin{cases} \Delta_0 g(x) \equiv [g(x+\frac{1}{2})-g(x-\frac{1}{2})], \\ \Delta_- g(x) \equiv [g(x)-g(x-1)], \\ \Delta_+ g(x) \equiv [g(x+1)-g(x)]. \end{cases}$$

Now we consider ϕ as a perturbation of U. Let

$$v(x) \equiv \phi(x)-U(x),$$
$$w(x) \equiv \sum_{-\infty}^{0} v(x+i).$$

The equations for v and w are

$$v(x-\lambda s)-\boldsymbol{L}[v](x)=\Delta_0\left(N[v]-O(1)\ \epsilon^3\ e^{-O(1)\ \epsilon|x|}\right), \tag{2.3}$$

$$w(x-\lambda s)-\bar{\boldsymbol{L}}[w](x)=\frac{N[v](x+\frac{1}{2})+N[v](x-\frac{1}{2})}{2}+O(1)\ \epsilon^3\ e^{-\epsilon|x|}, \tag{2.4}$$

where

$$\boldsymbol{L}[v] \equiv \left.\frac{d\mathscr{L}[U+\epsilon\ v]}{d\epsilon}\right|_{\epsilon=0},$$
$$\bar{\boldsymbol{L}}[w](x) \equiv w(x)-\frac{\lambda}{2}\frac{d}{d\epsilon}\left.\left[F[U+\epsilon\Delta_- w](x+\frac{1}{2})+F[U+\epsilon\Delta_- w](x-\frac{1}{2})\right]\right|_{\epsilon=0},$$
$$N[v] \equiv F[U+v]-F[U]-\left.\frac{dF[U+\epsilon v]}{d\epsilon}v\right|_{\epsilon=0}.$$

We consider the diagonalization of (2.4):

$$w(x) \equiv \sum_{j=1}^{n} w^j(x)\ r_j(U(x)),$$

$$w^j(x-\lambda s)=\bar{\boldsymbol{L}}_j[w^j](x)+O(1)\left[\epsilon^2\sum_{k=1}^{n}|\lambda_j(U)-s|\ \epsilon^2\ w^k\ +O(1)\ \epsilon\right]e^{-O(1)\ \epsilon|x|} + O(1)N[v]. \tag{2.5}$$

Here, the term $\epsilon^2\sum_{k=1}^{n}|\lambda_j(U)-s|\ \epsilon^2\ w^k\ e^{-O(1)\ \epsilon|x|}$ is due to the linear coupling of a system of equations.

In the rest of the paper, we devote to solve (2.5). We will separate this problem into two parts. One is for the waves crossing shock wave. It requires a combination of parabolic type time asymptotic analysis and Fourier analysis. The other is for waves in the same family as the shock wave. The analysis for this essentially is a parabolic type time asymptotic analysis with strong stability condition due to the presence of a shock wave.

3. Waves Crossing Shock Waves

We need to consider the following model problem

$$
\begin{aligned}
W(x-\lambda s) &= \bar{\boldsymbol{L}}_j[W](x)+\mathscr{S}(x) \text{ for } j\neq i,\\
|\mathscr{S}(x)| &\le (1+\epsilon|x|)^{-\alpha} \text{ for } \alpha\ge 5,\\
|\partial_x^j \mathscr{S}(x)| &\le \epsilon^{2j/3}(1+\epsilon|x|)^{-\alpha} \text{ for } j=0,\cdots,\alpha,\\
|\partial_x^j \mathscr{S}(x)| &\le \epsilon^{2j/3}(1+\epsilon|x|)^{-\alpha+3} \text{ for } j=\alpha+1,\cdots,2\alpha+3.
\end{aligned}
\tag{3.1}
$$

This problem is a variable coefficient problem

$$W(x-\lambda s)=W(x)+A(x)(W(x+1)-W(x))+B(x)(W(x)-W(x-1))+\mathscr{S}(x)$$

with $A(x)-B(x)-\lambda s\neq 0$ for all $x\in\mathbf{R}$.
The coefficients $A(x)$ and $B(x)$ have the asymptotic structures

$$
\begin{aligned}
|A(x)-A_+| &\le O(1)\,\epsilon\, e^{-O(1)\,\epsilon\,|x|} \text{ for } x>0,\\
|A(x)-A_-| &\le O(1)\,\epsilon\, e^{-O(1)\,\epsilon\,|x|} \text{ for } x<0,\\
|B(x)-B_+| &\le O(1)\,\epsilon\, e^{-O(1)\,\epsilon\,|x|} \text{ for } x>0,\\
|B(x)-B_-| &\le O(1)\,\epsilon\, e^{-O(1)\,\epsilon\,|x|} \text{ for } x<0,
\end{aligned}
$$

where

$$
\begin{cases}
A_\pm \equiv \lim_{x\to\pm\infty} A(x),\\
B_\pm \equiv \lim_{x\to\pm\infty} B(x).
\end{cases}
$$

We will consider two auxiliary asymptotic problems to establish the solution of (3.1):

$$
\begin{aligned}
&W^\pm(x-\lambda s)=\bar{\boldsymbol{L}}_j^\pm[W^\pm]+\mathscr{S}(x),\\
&\bar{\boldsymbol{L}}_j^\pm[W^\pm](x)\equiv W^\pm(x)+A_\pm\,(W^\pm(x+1)-W^\pm(x))+B_\pm\,(W^\pm(x)-W^\pm(x-1)).
\end{aligned}
\tag{3.2}
$$

The problems in (3.2) are constant coefficient problems. The solutions can be obtained through time limits of solutions of parabolic equations.
We proceed to construct W^+. Consider the evolution equation

$$
\begin{cases}
V^{k+1}(\xi)=\bar{\boldsymbol{L}}_j^+[V^k](\xi)+\mathscr{S}(\xi-\lambda sk),\\
V^0(\xi)\equiv 0.
\end{cases}
\tag{3.3}
$$

By Duhamel's principle,

$$V^k(x)=\sum_{j=1}^{k}\int_R k_j^+(x-y,k-j)\mathscr{S}(y-\lambda sj)\,dy,$$

one can show that $\lim_{k\to\infty} V^k$ exists, where $k^+j(x-y, k-j)$ is the green function of the problem (3.2), see [4].

Remark 3.1. Here, one can treat the function $k^+(x-y, k-j)$ as a random walk process on lattice points. It is a formal sum of delta functions as follows

$$k^+(x-y, k-j) = \sum_{z\in\mathbf{Z}} K^+(x-z, k-j)\ \delta(y-x+z),$$

$$K^+(\xi,\sigma) \le O(1)\frac{e^{-\frac{(\xi-(A^+-B^+)\sigma)^2}{D\sigma}}}{\sqrt{\sigma}}, \tag{3.4}$$

where $D = 2(A^+ + B^+ - (A^+ - B^+)^2)$ which can be identified as the coefficient of the numerical viscosity, too.

■

Consider the limit function

$$W^+(x) \equiv \lim_{k\to\infty} V^k(x+\lambda sk),$$

$$\|W^+\|_\infty = O(1)\frac{\|\mathscr{S}\|_{L_1}}{|A_+ - B_+ - \lambda s|}.$$

The limit function $W^+(x)$ solves (3.2); and it also yields that

$$\|\partial_x^j W^+\|_\infty = O(1)\frac{\|\partial_x^j \mathscr{S}\|_{L_1}}{|A_+ - B_+ - \lambda s|} \text{ for } j \ge 0. \tag{3.5}$$

This time asymptotic parabolic approach gives the existence of $W^+(x)$ in the $\|\cdot\|_\infty$ sense. In order to apply the linear analysis to the original nonlinear problem we still need the far field structure of $\partial_x^i W^+$ with $i \ge 1$. We use Fourier analysis to study the far field structures of $\partial_x^j W^+$. Take the Fourier transformation of (3.2) to obtain that

$$\widehat{\partial_x^j W^+}(\xi) = \frac{(i\xi)\ \widehat{\partial_x^{j-1}\mathscr{S}}(\xi)}{e^{-i\lambda s\xi} - (A_+e^{i\xi} + (1 - A_+ + B_+) - B_+e^{-i\xi})}. \tag{3.6}$$

The denominator could approach to zero very fast, if one does not impose any condition on λs. In order to resolve this small divisor problem, one can impose a Diophantine condition on λs. In our case, $\lambda s = \sqrt{2}/4$ is a Diophantine number of degree 2. See [4], this Diophantine property yields that there exists $C > 0$ such that

$$\|e^{-i\lambda s\xi} - (A_+e^{i\xi} + (1 - A_+ + B_+) - B_+e^{-i\xi})\| > C\frac{1}{|\xi|^2} \text{ for } |\xi| \ge 1.$$

Under this property, this small divisor can be canceled by the regularity of the source term $\mathscr{S}$. We have that

$$|\partial_x^{2j} W^+(x)| \le O(1)\ \epsilon^{4j/3}\frac{(1+\epsilon|x|)^{-\alpha+j}}{\epsilon^{4/3}} \text{ for } j = 1, \cdots, \alpha \tag{3.7}$$

$$(3.8)\qquad |\partial_x^{2j+1} W^+(x)| \le O(1)\ \epsilon^{(4j+2)/3} \frac{(1+\epsilon|x|)^{-\alpha+j+\frac{1}{2}}}{\epsilon^{4/3}} \text{ for } j \le \alpha-1.$$

The estimates (3.7) and (3.8) can be applied to W^-, too. For the purpose to indicate the dependence of the source term $\mathscr{S}$, we can write W^+ and W^- as follows

$$W^+ = \boldsymbol{S}_j^+[\mathscr{S}],\ \ W^- = \boldsymbol{S}_j^-[\mathscr{S}].$$

The above two expressions $\boldsymbol{S}_j^\pm$ give the solution operators for (3.2).
By these two solution operators $\boldsymbol{S}_j^\pm$, we can construct the solution operator of (3.1) through the following iterations:

$$\begin{aligned}
&\chi_+(x) \equiv \begin{cases} 1 \text{ if } x \ge \epsilon^{-1}, \\ 0 \text{ if} x \le -\epsilon^{-1}, \end{cases} \\
&|\chi_+'(x)| = O(1)\ \epsilon\ e^{-\epsilon|x|} \text{ for } x \in [-\epsilon^{-1}, \epsilon^{-1}], \\
&\chi_- = 1 - \chi_+, \\
&W_1(x) \equiv \chi_+\ \boldsymbol{S}_j^+[\mathscr{S}] + \chi_-\ \boldsymbol{S}_j^-[\mathscr{S}], \\
&E_1 \equiv W_1(x-\lambda s) - \bar{\boldsymbol{L}}_j[W_1](x) - \mathscr{S}(x), \\
&W_{i+1} \equiv -\left\{\chi_+\ \boldsymbol{S}_j^+[E_i] + \chi_-\ \boldsymbol{S}_j^-[E_i]\right\} \text{ for } i \ge 1, \\
&E_{i+1} \equiv W_{i+1}(x-\lambda s) - \bar{\boldsymbol{L}}_j[W_{i+1}](x) + E_i(x) \text{ for } i \ge 1.
\end{aligned}$$

Since $\epsilon \ll 1$, the sequence W_j is a geometric sequence in $\|\cdot\|_\infty$. The function $W(x) \equiv \sum_{i=1}^\infty W_i(x)$ solves

$$(3.9)\qquad \begin{aligned}
&W(x-\lambda s) - \bar{\boldsymbol{L}}_j[W](x) = \mathscr{S}(x), \\
&\|W\|_\infty = O(1)\|\mathscr{S}\|_{L_1}, \\
&|\partial_x^{2j} W(x)| \le O(1)\ \epsilon^{4j/3} \frac{(1+\epsilon|x|)^{-\alpha+j}}{\epsilon^{4/3}} \text{ for } j = 1, \cdots, \alpha \\
&|\partial_x^{2j+1} W(x)| \le O(1)\ \epsilon^{(4j+2)/3} \frac{(1+\epsilon|x|)^{-\alpha+j+\frac{1}{2}}}{\epsilon^{4/3}} \text{ for } j \le \alpha-1.
\end{aligned}$$

We write $W(x)$ in terms of the solution operator of (3.1) $\boldsymbol{S}_j$ as follows

$$W \equiv \boldsymbol{S}_j[\mathscr{S}].$$

4. Waves in Shock Family

For waves in the same family as shock wave, we will consider the following model equation

$$(4.1)\qquad \begin{aligned}
&W(x-\lambda s) = \bar{\boldsymbol{L}}_i[W](x) + \mathscr{S}(x), \\
&|\mathscr{S}(x)| \le (1+\epsilon|x|)^{-\alpha} \text{ for } \alpha \ge 5, \\
&|\partial_x^j \mathscr{S}(x)| \le \epsilon^{2j/3}(1+\epsilon|x|)^{-\alpha} \text{ for } j = 0, \cdots, \alpha, \\
&|\partial_x^j \mathscr{S}(x)| \le \epsilon^{2j/3}(1+\epsilon|x|)^{-\alpha+3} \text{ for } j = \alpha+1, \cdots, 2\alpha+3.
\end{aligned}$$

In this compressive field, all the linear waves propagate into the shock front. Thus, by a parabolic type analysis the far field structures of solutions can be directly related to the far field structures in the source term $\mathscr{S}(x)$.

We consider the following time evolution equation

$$\begin{aligned} &W^{k+1}(x-\lambda s)=\bar{\boldsymbol{L}}_i[W^k](x)+\mathscr{S}(x),\\ &W^0(x)\equiv 0. \end{aligned}$$

The solution, W, of (4.1) can be related to the time limit

$$W(x)\equiv\lim_{k\to\infty}W^k(x).$$

This limit function has the same algebraic far field structure as that of $\mathscr{S}(x)$. This fact is due to that the field is compressive. We will explain this in the rest of this section.

Consider $V^k(x)\equiv W^k(x)-W^{k-1}(x)$. The equation for V^k is

$$\begin{aligned} V^{k+1}(x-\lambda s)&=\bar{\boldsymbol{L}}_i[V^k](x),\\ V^0(x)&=\mathscr{S}(x). \end{aligned}\tag{4.2}$$

The behavior of $V^k(x)$ essentially resembles to the solution $Y(x,k)$ of

$$Y_t+(\lambda_i(U)-s)Y_x-\frac{1}{2}\left(\frac{2}{3}-(s\lambda)^2\right)Y_{xx}=0.$$

From this parabolic equation, we can construct an approximate green function of $g_i(x,k;y,j)$ of (4.2) in the following sense, see [4]:

(4.3)

$$V^k(x)=\int_R g_i(x,t;y,0)\mathscr{S}(y)\,dy+\sum_{j=0}^{k}\int_R g_i(x,t;y,j)\,e^{-O(1)\,\epsilon|y|}\,\epsilon^2\left(\frac{1}{\sqrt{k-j}}+\epsilon\right)\,V^j(y)dy,$$

where $g_i(x,k;y,j)$ is a formal sum of delta functions

$$g_i(x,k;y,j)=\sum_{z\in\mathbf{Z}}H(x,k;x+(k-j)\lambda s+z,j)\;\delta(y-x-(k-j)\lambda s-z),\tag{4.4}$$

and where

$$H(x,k;\xi,j)=O(1)\begin{cases}\dfrac{e^{\frac{-(x-\xi-\lambda_+(k-j))^2}{4(k-j)}}}{\sqrt{k-j}} & \text{for } x>0,\ \xi>0,\\[2ex] \dfrac{e^{-|O(1)\epsilon x|}\,e^{\frac{-(x-\xi-\lambda_-(k-j))^2}{4(k-j)}}}{\sqrt{k-j}} & \text{for } x>0,\ \xi<0,\\[2ex] \dfrac{e^{\frac{-(x-\xi-\lambda_-(k-j))^2}{4(k-j)}}}{\sqrt{k-j}} & \text{for } x<0,\ \xi<0,\\[2ex] \dfrac{e^{-|O(1)\epsilon x|}\,e^{\frac{-(x-\xi-\lambda_-(k-j))^2}{4(k-j)}}}{\sqrt{k-j}} & \text{for } x<0,\ \xi>0.\end{cases}$$

$$\lambda_\pm\equiv\lim_{\xi\to\pm\infty}\lambda_i(\xi).$$

The representation (4.3) and the structure of the approximate green $g_i(x,k;y,j)$ in (4.4) yield that $V^k(x)$:

$$|V^k(x)| \le O(1)\,\frac{1}{(1+\epsilon(|x|+\epsilon\,k))^\alpha}, \tag{4.5}$$

and

$$|V^k(x)-V^k(x-1)| \le O(1)\,\frac{\epsilon\,e^{-O(1)\epsilon|x|}}{(1+\epsilon(|x|+\epsilon\,k))^\alpha}+O(1)\,\frac{\epsilon^{2/3}}{(1+\epsilon(|x|+\epsilon\,k))^\alpha}. \tag{4.6}$$

Hence $\|V^k\|_\infty$ decays fast enough to yield the convergence of $\sum_{k=1}^\infty V^k$. It results in

$$|W(x)| = |\lim_{k\to\infty} W^k(x)| \le O(1)\frac{\epsilon^{-2}}{(1+\epsilon|x|)^{\alpha-1}}, \tag{4.7}$$

$$|\partial_x^j W(x)| = |\lim_{k\to\infty} W^k(x)-W^k(x-1)| \le O(1)\frac{\epsilon^{-2+4j/3}}{(1+\epsilon|x|)^{\alpha-1}} \text{ for } j=1,\cdots,2\alpha+3. \tag{4.8}$$

We also write W as follows

$$W = \boldsymbol{S}_i[\mathscr{S}].$$

Here, $\boldsymbol{S}_i$ is the solution operator of (4.1).

5. Nonlinear Problem

We return to the nonlinear problem (2.5). We shall express the nonlinear term $N[v]$ in terms of its coordinate

$$N[v] = \sum_{j=1} N^j[v]\; r_j(U).$$

The nonlinear term in (2.5) essentially is a quadratic nonlinearity, that is,

$$\lim_{\|v\|\to 0}\left\|\frac{N^j[v]}{v^2}\right\| < \infty \text{ for } j=1,\cdots,n.$$

The vector-valued function v is related to w^j

$$v(x) = \sum_{j=1}^n [w^j(x)-w^j(x-1)]\; r_j(U(x,t)) + O(1)\;\|w\|\;\epsilon^2\; e^{-O(1)\;\epsilon|x|}.$$

We can reformulate (2.5) as follows

$$w^j(x-\lambda s) - \bar{\boldsymbol{L}}_j[w^j](x) = N^j[v] + O(1)\;(\;\epsilon^3+\epsilon^2\;|\lambda_j(U)-s|\;\;\|w\|\;)\;\;e^{-O(1)\;\epsilon|x|}. \tag{5.1}$$

This gives the representation

$$\begin{aligned} w^j(x) &= \boldsymbol{S}_j[\mathscr{S}_j[w]] + \boldsymbol{S}_j[O(1)\epsilon^3 e^{-O(1)\epsilon|x|}], \\ \mathscr{S}_j[w] &\equiv N^j[v] + O(1)\;\epsilon^2\;|\lambda_j(U)-s|\;\;\|w\|\;e^{-O(1)\;\epsilon|x|}. \end{aligned} \tag{5.2}$$

Ansatz of $W^j \equiv \boldsymbol{S}_j[O(1)\epsilon^3 e^{-O(1)\epsilon|x|}]$.
There exists $C_0 > 0$ such that
1) $j \neq i$

$$\begin{cases} \|\partial_x^k W^j\|_\infty \le C_0 \epsilon^{2+k} \text{ for } k = 0, \cdots, 2\alpha+3, \\ |\partial_x^k (W^j(x) - W^j(x-1))| \le C_0\ \epsilon^{2+\frac{1}{2}+k}\ (1+\epsilon|x|)^{-\alpha} \text{ for } k = 0, \cdots, 2\alpha+3 \end{cases}$$

2) $j = i$

$$|\partial_x^k W^i(x)| \le C_0\ \epsilon^{1+k}\ e^{-O(1)\epsilon|x|} \text{ for } k = 0, \cdots, 2\alpha+3.$$

From this ansatz, we can construct the ansatz for w^j.
Ansatz for $w^j(x)$

1) $j \neq i$

$$\|\partial_x^k w^j\|_\infty \le 2C_0\ \epsilon^{2+\frac{2k}{3}} \text{ for } k = 0, \cdots, 2\alpha+3, \tag{5.3}$$

$$|\ \partial_x^k [w^j(x) - w^j(x-1)]\ | \le 2C_0\ \epsilon^{2+\frac{1}{2}+\frac{2k}{3}}\ (1+\epsilon|x|)^{-\alpha+\frac{k}{2}} \text{ for } k = 0, \cdots, 2\alpha, \tag{5.4}$$

2) $j = i$

$$|\partial_x^k (w^i(x) - w^i(x-1))| \le 2C_0 \epsilon^{\frac{7}{3}+\frac{2k}{3}}\ (1+|\epsilon x|)^{-\alpha+1} \text{ for } k = 0, \cdots, 2\alpha+3. \tag{5.5}$$

Under this ansatz, there exists C_1 such that

$$\mathscr{S}_j[w] \le C_1 C_0^2 \epsilon^{10/3} (1+\epsilon|x|)^{-2\alpha},$$

$$|\partial_x^j \boldsymbol{S}_j[w]| \le C_1 C_0^2 \epsilon^{\frac{10}{3}+\frac{2j}{3}} (1+\epsilon|x|)^{-2\alpha+\frac{j}{2}} \text{ for } j = 1, \cdots, 2\alpha+3.$$

Substitute this into (5.2); and use properties of the solution operators $\boldsymbol{S}_j$ given in (3.9), (4.7), and (4.8) to yield that

1) $j \neq i$

$$\|\partial_x^k w^j\|_\infty \le (1 + O(1)\ \epsilon\ C_1 C_0)\ C_0\ \epsilon^{2+\frac{2k}{3}} \text{ for } k = 0, \cdots, 2\alpha+3,$$

$$|\ \partial_x^k [w^j(x) - w^j(x-1)]\ | \le C_0\ (1 + O(1)\ \epsilon\ C_1 C_0)\ \epsilon^{2+\frac{1}{2}+\frac{2k}{3}}\ (1+\epsilon|x|)^{-\alpha+\frac{k}{2}} \text{ for } k = 0, \cdots, 2\alpha,$$

2) $j = i$

$$|\partial_x^k (w^i(x) - w^i(x-1))| \le C_0 (1 + O(1)\ \epsilon\ C_1 C_0)\ \epsilon^{\frac{7}{3}+\frac{2k}{3}}\ (1+|\epsilon x|)^{-\alpha+1} \text{ for } k = 0, \cdots, 2\alpha+3.$$

When ϵ is sufficiently small, the above three estimates show that the ansatz in (5.3), (5.4), and (5.5) are valid. This concludes the existence of a discrete shock profile connecting (u_-, u_+). Theorem 1.2 is proved.

Remark 5.1. When $u \in \mathbf{R}$, our proof can be applied to show the existence a discrete shock profile for a monotone scheme without the Diophantine assumption. For scalar equation, there is only one characteristic field. No other characteristic curve will cross shock. The problem of small divisor do not arise for scalar equation. Our analysis for wave in the shock family is enough to establish the existence of a discrete shock profile. ■

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Department of Mathematics, Graduate School of Science, Osaka University

E-mail address: `shyu@math.sci.osaka-u.ac.jp`