Electrorheological Fluids: Modeling and Mathematical Theory

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Abstract: Electrorheological fluids are materials which dramatically change their mechanical properties in dependence on an applied electric field. We derive a system of equations which capture this behaviour. For this system we discuss existence, regularity and uniqueness of weak solutions. Finally, a numerical analysis of a fully implicit time discretization is carried out.

1 Modeling

Electrorheological fluids are special viscous fluids, which are characterized by their ability to change dramatically their mechanical properties in dependence on an applied electric field. This property can be exploited in technological applications, as e.g. actuators, clutches, shock absorber and rehabilitation equipment to name a few. Great strides have been made to overcome the impediments of early electrorheological fluids, as the abrasive nature and the instability of the suspension and the enormous voltage requirements that are necessary for a significant change in the material properties, since the first observation of the behaviour of electrorheological fluids by Winslow [37]. The nowadays existing electrorheological fluids, which make the above mentioned devices possible, are the result of intensive efforts to manufacture materials without these impediments. Thus, electrorheological fluids are at the brink of having their potential being realized.

Electrorheological fluids can be modeled in many ways. One possibility consists in the investigation of the underlying microstructure to obtain a macroscopic description of the material (cf. Klingenberg, van Swol, Zukoski [15], Halsey, Toor [10], Bonnecaze, Brady [6], Parthasarathy, Klingenberg [24]). Another approach uses the frame-work of continuum mechanics and treats the electrorheological fluid as a homogenized single continuum (cf. Atkin, Shi, Bullogh [1], Rajagopal, Wineman [28], Wineman, Rajagopal [36]) or models it using the theory of mixture (cf. Rajagopal, Yalamanchili, Wineman [29]). A completely different perspective is provided by modeling based on direct numerical simulations taking into account the dynamics and interaction of particles (cf. Whittle [38], Bailey, Gillies, Heyes, Sutcliffe [2]). In all these models the electric field is treated as a constant parameter. Recently, Rajagopal, Růžička [30], [27] have developed a model which takes into account the complex interaction of the electro-magnetic fields and the moving liquid and thus, the electric field is treated as a variable, which has to be determined.

Let us briefly go through the main steps of this procedure (see Růžička [33] for missing details). The starting point are the general balance laws for mass, linear momentum, angular momentum, energy, the second law of thermodynamics in the form of the Clausius-Duhem inequality and Maxwell's equations in their Minkowskian formulation. The system has to be completed by choosing appropriate constitutive relations, reflecting the material properties. Moreover, we will require that both the constitutive relations and all balance laws are invariant under Galilean transformations. Thus we deal with the following system:

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0\,, \tag{1.1}$$

$$\rho\dot{\mathbf{v}} - \operatorname{div} \mathbf{T} = \rho\mathbf{f} + \mathbf{f}_e \tag{1.2}$$

$$\begin{aligned} \rho\dot{e} - k\Delta\theta = \mathbf{T}\cdot\mathbf{D} + \dot{\mathbf{P}}\cdot\boldsymbol{\mathcal{E}} + (\mathbf{P}\cdot\boldsymbol{\mathcal{E}}) \operatorname{div} \mathbf{v} \\ - \boldsymbol{\mathcal{M}}\cdot\dot{\mathbf{B}} + \boldsymbol{\mathcal{J}}\cdot\boldsymbol{\mathcal{E}} + \rho r \end{aligned} \tag{1.3}$$

$$\epsilon\big(\mathbf{P}\otimes\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{M}}\otimes\mathbf{B}\big) = \mathbf{0}\,, \tag{1.4}$$

$$(\mathbf{T} + \pi\mathbf{I})\cdot\mathbf{D} + k\frac{|\nabla\,\theta|^2}{\theta} + \boldsymbol{\mathcal{J}}\cdot\boldsymbol{\mathcal{E}} \geq 0\,, \tag{1.5}$$

$$\operatorname{div} \mathbf{D}_e = q_e\,, \tag{1.6}$$

$$\operatorname{curl} \mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = \mathbf{0}\,, \tag{1.7}$$

$$\operatorname{div} \mathbf{B} = 0\,, \tag{1.8}$$

$$\operatorname{curl} \mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{D}_e}{\partial t} + \frac{1}{c}\big(\boldsymbol{\mathcal{J}} + q_e\mathbf{v}\big)\,, \tag{1.9}$$

$$\frac{\partial q_e}{\partial t} + \operatorname{div}\,\big(\boldsymbol{\mathcal{J}} + q_e\mathbf{v}\big) = 0\,, \tag{1.10}$$

where $\mathbf{f}_e$ is given by

$$\begin{aligned} \mathbf{f}_e = q_e\,\boldsymbol{\mathcal{E}} + \frac{1}{c}\boldsymbol{\mathcal{J}}\times\mathbf{B} + \frac{1}{c}\big(\dot{\mathbf{P}} + (\operatorname{div}\,\mathbf{v})\mathbf{P}\big)\times\mathbf{B} + \frac{1}{c}\mathbf{v}\times\big([\nabla\mathbf{B}]\mathbf{P}\big) \\ + [\nabla\,\mathbf{B}]^T\boldsymbol{\mathcal{M}} + [\nabla\,\mathbf{E}]\mathbf{P}\,, \end{aligned} \tag{1.11}$$

where the thermodynamic pressure π is defined through

$$\pi \equiv \rho^2\frac{\partial\psi}{\partial\rho}\,, \tag{1.12}$$

and where constitutive relations for $e, \psi, \mathbf{T}, \boldsymbol{\mathcal{M}}, \mathbf{P}, \boldsymbol{\mathcal{J}}$ depending on $\rho, \mathbf{D}, \boldsymbol{\mathcal{E}}, \mathbf{B}, \theta$ have to be specified. Here we used the notation that $\mathbf{v}$ is the velocity, ρ the density, $\mathbf{T}$ the Cauchy stress tensor, $\mathbf{f}$ the external mechanical body force, $\mathbf{D}$ the symmetric part of the velocity gradient, e the internal energy, r the heat source density, θ the absolute temperature, $\mathbf{f}_e$ the electro-magnetic body force, q_e the electric charge density, $\mathbf{E}$ the electric field, $\boldsymbol{\mathcal{E}}$ the electromotive intensity, $\boldsymbol{\mathcal{J}}$ the conduction current, $\mathbf{P}$ the electric polarization, $\mathbf{B}$ the magnetic induction, $\boldsymbol{\mathcal{M}}$ the magnetization in the co-moving frame and $\mathbf{D}_e$ the electric displacement.

The above system covers much more general situations than the flow of electrorheological fluids. Thus we shall simplify it by specifying some constitutive relations that reflect the observed behaviour of electrorheological fluids and then carry out a dimensional analysis and a subsequent approximation which restricts the validity of the resulting system to special but typical situations. Firstly, we assume that the stress tensor does not depend on the magnetic induction $\mathbf{B}$, i.e.

$$\mathbf{T} = \hat{\mathbf{T}}(\rho, \mathbf{D}, \boldsymbol{\mathcal{E}}, \theta)\,. \tag{1.13}$$

This assumption is based on the experimental evidence that on applying a magnetic field to a electrorheological fluid no effect is detectable. Secondly, we shall assume that the fluid is non-conducting, i.e. (cf. Grot [9] Section 2.2)

$$\boldsymbol{\mathcal{J}} \equiv \mathbf{0}\,, \tag{1.14}$$

and thirdly, as we are dealing with a dielectric (cf. Grot [9] Section 2.2)

$$\boldsymbol{\mathcal{M}} \equiv \mathbf{0}\,. \tag{1.15}$$

Moreover we shall neglect terms which are of secondary importance for the behaviour of the electrorheological fluid. In order to detect these terms we introduce the following non-dimensional quantities:

$$\begin{aligned} &\bar{\mathbf{E}} = \frac{\mathbf{E}}{E_0}\,, \quad \bar{\mathbf{B}} = \frac{\mathbf{B}}{B_0}\,, \quad \bar{q}_e = \frac{q_e}{q_0}\,, \quad \bar{\mathbf{v}} = \frac{\mathbf{v}}{V_0}\,, \quad \bar{\mathbf{x}} = \frac{\mathbf{x}}{L_0}\,, \\ &\bar{t} = \frac{t}{T_0}\,, \quad \bar{\mathbf{P}} = \frac{\mathbf{P}}{E_0}\,, \quad \bar{\rho} = \frac{\rho}{\rho_0}\,, \quad \bar{\mathbf{f}} = \frac{\mathbf{f}}{f_0}\,, \quad \bar{\theta} = \frac{\theta}{\theta_0}\,, \end{aligned} \tag{1.16}$$

where the quantities with the suffix zero are appropriate representative quantities. The Reynolds number Re is defined through $Re = \frac{\rho_0\, L_0\, V_0}{\mu_0}$, where ρ_0 and μ_0 are the density and viscosity of the fluid in the absence of an electric field. We shall be interested in problems wherein the Reynolds number Re lies between 1 and 10^2 and where

$$\begin{aligned} &E_0 \sim 10^{-2} - 10^2 \ \text{ statvolts cm}^{-1}\,, \\ &L_0 \sim 10^{-1} - 1 \ \text{ cm}\,, \\ &T_0 \sim 10^{-4} - 1 \ \text{ sec}\,. \end{aligned} \tag{1.17}$$

Finally, we introduce a small non-dimensional number ε through

$$\varepsilon \equiv 10^{-3} \tag{1.18}$$

and make the following assumptions concerning the role of the magnetic induction[1] and the amount of free charges

$$\frac{E_0}{B_0}\frac{L_0}{c\,T_0} = O(1)\,, \tag{1.19}$$

$$\frac{q_0\,L_0}{E_0} = O(\varepsilon^4)\,. \tag{1.20}$$

From (1.17) and the fact that the Reynolds number ranges in between 1 and 10^2 we can also conclude that

$$\frac{L_0}{c\,T_0} = O(\varepsilon^2) - O(\varepsilon^4)\,, \tag{1.21}$$

$$\frac{V_0}{c} = O(\varepsilon^3) - O(\varepsilon^4)\,. \tag{1.22}$$

We will approximate the system (1.1)–(1.10), written in non-dimensional quantities, by neglecting terms of order ε^3 and higher, while retaining terms up to order ε^2. The result of this procedure reads

$$\dot\rho + \rho\,\mathrm{div}\;\mathbf{v} = 0\,, \tag{1.23}$$

$$\rho\dot{\mathbf{v}} - \mathrm{div}\;\mathbf{T} = \rho\mathbf{f} + [\nabla\,\mathbf{E}]\mathbf{P}\,, \tag{1.24}$$

$$c_v\rho\dot\theta - k\Delta\theta + \theta\big(\frac{\partial\mathbf{P}}{\partial\theta}\cdot\dot{\mathbf{E}} - \frac{\partial\pi}{\partial\theta}\,\mathrm{tr}\;\mathbf{D}\big) = \big(\mathbf{T}+\pi\mathbf{I}\big)\cdot\mathbf{D} + \rho r\,, \tag{1.25}$$

$$(\mathbf{T}+\pi\mathbf{I})\cdot\mathbf{D} + k\frac{|\nabla\,\theta|^2}{\theta} \geq 0\,, \tag{1.26}$$

$$\mathrm{div}\,(\mathbf{E}+\mathbf{P}) = 0\,, \tag{1.27}$$

$$\mathrm{curl}\;\mathbf{E} = \mathbf{0}\,, \tag{1.28}$$

$$\mathrm{div}\;\mathbf{B} = 0\,, \tag{1.29}$$

$$\mathrm{curl}\;\mathbf{B} + \frac{1}{c}\,\mathrm{curl}\,(\mathbf{v}\times\mathbf{P}) = \frac{1}{c}\frac{\partial\mathbf{E}+\mathbf{P}}{\partial t}\,, \tag{1.30}$$

$$\dot q_e + q_e\,\mathrm{div}\;\mathbf{v} = 0\,, \tag{1.31}$$

where $\mathbf{P}$, c_v and π are functions of $\rho,\theta,\mathbf{E}$ and $\mathbf{T} = \hat{\mathbf{T}}(\rho,\theta,\mathbf{D},\mathbf{E})$. Note that (1.4) is trivially satisfied. In order to come to the final system which we want to investigate in the following sections we restrict ourselves to incompressible, isothermal flows, i.e. $\mathrm{div}\;\mathbf{v} = 0$ and $\theta = \mathrm{const.}$ and we have to chose constitutive relations for the polarization and the stress tensor. The following choice is motivated by the fact that it is capable to explain many experimental observations (cf. Rajagopal, Růžička [27])

$$\mathbf{P} = \chi^E\,\mathbf{E} \tag{1.32}$$

$$\mathbf{T} = \pi\mathbf{I} + \mathbf{S}\,, \tag{1.33}$$

[1] The assumption (1.19) means that the magnetic induction is only induced by oscillations of the electric field.

where

$$\mathbf{S} = \alpha_{21}\big((1+|\mathbf{D}|^2)^{\frac{p-1}{2}} - 1\big)\mathbf{E}\otimes\mathbf{E} + (\alpha_{31}+\alpha_{33}|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{\frac{p-2}{2}}\mathbf{D} \\ + \alpha_{51}(1+|\mathbf{D}|^2)^{\frac{p-2}{2}}(\mathbf{DE}\otimes\mathbf{E}+\mathbf{E}\otimes\mathbf{DE})\,, \tag{1.34}$$

and where $p = p(|\mathbf{E}|^2)$ is a C^1-function such that

$$1 < p_\infty \le p(|\mathbf{E}|^2) \le p_0 < \infty\,. \tag{1.35}$$

If a electrorheological fluids satisfies the above constitutive relations (1.32)–(1.35) we call it shear dependent electrorheological fluid.

2 Flows of Shear Dependent Electrorheological Fluids

In the previous section we have shown that the isothermal flow of an incompressible shear dependent electrorheological fluid is governed by the following system[2]

$$\begin{aligned} \operatorname{div}\,\mathbf{E} &= 0\,,\\ \operatorname{curl}\,\mathbf{E} &= \mathbf{0}\,, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \frac{\partial\mathbf{v}}{\partial t} - \operatorname{div}\,\mathbf{S} + [\nabla\mathbf{v}]\mathbf{v} + \nabla\pi &= \mathbf{f} + \chi^E[\nabla\mathbf{E}]\mathbf{E}\,,\\ \operatorname{div}\,\mathbf{v} &= 0\,, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \operatorname{div}\,\mathbf{B} &= 0\,,\\ \operatorname{curl}\,\mathbf{B} + \frac{\chi^E}{c}\operatorname{curl}(\mathbf{v}\times\mathbf{E}) &= \frac{1+\chi^E}{c}\frac{\partial\mathbf{E}}{\partial t}\,, \end{aligned} \tag{2.3}$$

$$\frac{\partial q_e}{\partial t} + \operatorname{div}(q_e\mathbf{v}) = 0\,, \tag{2.4}$$

$$\mathbf{S}\cdot\mathbf{D} + \rho r = 0\,, \tag{2.5}$$

where $\mathbf{S}$ is given by (1.34), (1.35). We require that the constant coefficients α_{ij} and the function p are such that the operator induced by $-\operatorname{div}\,\mathbf{S}(\mathbf{D},\mathbf{E})$ is coercive, i.e.

$$\mathbf{S}(\mathbf{D},\mathbf{E})\cdot\mathbf{D} \ge c_0(1+|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{\frac{p(|\mathbf{E}|^2)-2}{2}}|\mathbf{D}|^2 \tag{2.6}$$

[2]Here and in the following we use the notation $[\nabla\mathbf{u}]\mathbf{w} = \big(w_j\frac{\partial u_i}{\partial x_j}\big)_{i=1,2,3}$, where the summation convention over repeated indices is used. Moreover, we have divided equation (1.24) by the constant density ρ_0 and adapted the notation appropriately.

holds for all $\mathbf{D} \in X := \{\mathbf{D} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}, \operatorname{tr}\mathbf{D} = 0\}$, and uniformly monotone, i.e.

$$\frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial D_{kl}} B_{ij} B_{kl} \geq c_1 (1 + |\mathbf{E}|^2)(1 + |\mathbf{D}|^2)^{\frac{p(|\mathbf{E}|^2)-2}{2}} |\mathbf{B}|^2 \tag{2.7}$$

is satisfied for all $\mathbf{B}, \mathbf{D} \in X$.[3] The system (2.1)–(2.5) is separated, so we can first solve the quasi-static Maxwell's equations (2.1) for the electric field and then seek for the velocity field by solving (2.2). Knowing $\mathbf{E}$ and $\mathbf{v}$ we can solve (2.3), (2.4) and (2.5). Note that equation (2.5) has to be interpreted as an equation for the heat source r. It was already pointed out in the previous section that the magnetic induction $\mathbf{B}$ is of secondary importance, which is reflected by the structure of the above system. Moreover, in this investigation of electrorheological fluids we are mainly interested in the electric field $\mathbf{E}$ and the velocity field $\mathbf{v}$, and not in the free charges q_e and the heat source r. Therefore we shall only consider (2.1) and (2.2), to which appropriate initial and boundary conditions should be added. The quasi-static Maxwell's equations (2.1) are widely studied in the literature (cf. the overview article Milani, Picard [22]) and thus we will investigate in this paper the system (2.2) only, in which $\mathbf{E}$ is assumed to be any given vector field, having certain regularity properties. Moreover, for simplicity we shall complete (2.2) by space periodic boundary conditions and an initial condition $\mathbf{v}_0$.

Before we state our results, we shall introduce some notation. Let $\Omega = (0, L)^3$ be a cube of given length L, and $T > 0$ a given length of the time interval $I = (0, T)$. We denote by $(L^q(\Omega), \|.\|_q)$ and $(W^{k,q}(\Omega), \|.\|_{k,q})$, $q \in [1, \infty], k \in \mathbb{N}$, the usual Lebesgue and Sobolev spaces of periodic functions with mean value zero. The space of divergence free smooth functions is denoted by $\mathcal{V}$. The closure of $\mathcal{V}$ in the $\|.\|_2$-norm and the $\|\nabla.\|_q$-norm, resp. is labeled H and V_q, resp. We use the notation $L^q(I, X(\Omega))$ for Bochner spaces with values in some function space over Ω. We make also use of the discrete counterparts, $l^p(I, X)$, for a given net $\{t_{m+1}\}_{m=-1}^{M}$ in an interval $I = [0, t_{M+1}]$, and a constant time-step $k = t_{m+1} - t_m$. Then, $l^p(I, X)$ is the space of functions $\{\phi^{m+1}\}_{m=-1}^{M}$ with bounded norm $\left(k \sum_{m=-1}^{M} \|\phi^{m+1}\|_X^p\right)^{1/p}$. For the case $p = \infty$, functions $\{\phi^{m+1}\}_{m=-1}^{M}$ need to satisfy the bound $\max_{0\leq m\leq M} \|\phi^{m+1}\|_X < \infty$. We also need Lebesgue and Sobolev spaces with variable exponents, which are denoted by $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, respectively. For given $p(x) \in L^\infty(\Omega), 1 < p_\infty \leq p(x) \leq p_0 < \infty$, we define the modular

$$|f|_{p(x)} \equiv \int_\Omega |f(x)|^{p(x)}\, dx\,,$$

which can be used to define a norm on the generalized Lebesgue space

$$L^{p(x)}(\Omega) = \{f \in L^1(\Omega)\,; \quad |\lambda f|_{p(x)} < \infty \text{ for some } \lambda > 0\}\,.$$

[3]Conditions for α_{ij} and p that ensure the validity of (2.6) and (2.7) can be found in Růžička [33] Chapter 1. Note, that the coercivity is related to the Clausius-Duhem inequality, while the monotonicity is an additional mathematical requirement.

Generalized Sobolev spaces are defined analogously. We refer to Kováčik, Rákosník [16] for a detailed treatment of these spaces.

For given $\mathbf{E} \in L^\infty(I, W^{1,\infty}(\Omega))$ we consider the system (2.2), where $\mathbf{S}$ is given by (1.34), (1.35) and satisfies (2.6), (2.7), on the time-space cylinder $Q_T = I \times \Omega$ together with an initial condition

$$\mathbf{v}(0) = \mathbf{v}_0 \qquad \text{in } \Omega\,. \tag{2.8}$$

Then we have

Theorem 2.9. *Let $\Omega = (0, L)^3$ be a given cube and assume that $T > 0, \mathbf{v}_0 \in V_2$, $\mathbf{E} \in L^\infty(I, W^{1,\infty}(\Omega))$, and $\mathbf{f} \in L^r(Q_T)$, $r = \max(p'_\infty, 2)$, are given.*

(i) *Whenever*

$$9/5 < p_\infty \le p(|\mathbf{E}|^2) \le p_0 < p_\infty + 1 \tag{2.10}$$

there exists a solution $\mathbf{v}$ of the problem (2.2), (2.8) such that

$$\begin{aligned} \mathbf{v} &\in L^\infty(I, H) \cap L^{p_\infty}(I, V_{p_\infty})\,, \\ \mathbf{D}(\mathbf{v}) &\in L^{p(|\mathbf{E}|^2)}(Q_T)\,, \end{aligned} \tag{2.11}$$

which satisfies (2.2) in the weak sense, i.e. for almost all $t \in I$ and all $\boldsymbol{\varphi} \in \mathcal{V}$ we have

$$\begin{aligned} &\langle \frac{\partial \mathbf{v}}{\partial t}(t), \boldsymbol{\varphi} \rangle_{W^{3,2}(\Omega)\cap V_2} + \int_\Omega \mathbf{S}(\mathbf{D}(\mathbf{v}(t)), \mathbf{E}(t)) \cdot \mathbf{D}(\boldsymbol{\varphi})\, dx \\ &+ \int_\Omega [\nabla \mathbf{v}(t)]\mathbf{v}(t) \cdot \boldsymbol{\varphi}\, dx = \int_\Omega \mathbf{f}(t) \cdot \boldsymbol{\varphi}\, dx - \chi^E \int_\Omega \mathbf{E}(t) \otimes \mathbf{E}(t) \cdot \mathbf{D}(\boldsymbol{\varphi})\, dx\,. \end{aligned} \tag{2.12}$$

(ii) *Moreover, if*

$$11/5 < p_\infty \le p(|\mathbf{E}|^2) \le p_0 < p_\infty + 4/3 \tag{2.13}$$

there exists a unique solution of the problem (2.2), (2.8) with the additional property.

$$\mathbf{v} \in L^\infty(I, V_2) \cap L^2(I, W^{2,2}(\Omega)) \cap L^{p_\infty}(I, V_{3p_\infty})\,. \tag{2.14}$$

The main problem in the proof of the previous theorem consists in the identification of the limit

$$\lim_{N\to\infty} \int_0^T \int_\Omega \mathbf{S}(\mathbf{D}(\mathbf{v}^N), \mathbf{E}) \cdot \mathbf{D}(\boldsymbol{\varphi})\, dx\, dt\,, \tag{2.15}$$

where $\mathbf{v}^N$ is some approximate solution of (2.2). The method used here is based on Vitali's convergence theorem and the almost everywhere convergence of $\mathbf{D}(\mathbf{v}^N)$. This basic idea was initiated by Nečas [23] and developed in Málek, Nečas, Růžička

[18], [19], Bellout, Bloom, Nečas [5], Málek, Nečas, Rokyta, Růžička [17] to handle situations, when monotonicity methods fail to identify the above limit. Theorem 2.9 contains the results in [18] as a special case (put $p = \text{const.}, \mathbf{E} \equiv 0$) and thus shows that the basic idea is widely applicable. It is worth noticing that unsteady problems for electrorheological fluids cannot be treated with the help of monotonicity methods even for large p_∞ due to the non-standard growth of the governing system. Besides the results of the author [33] it seems that Theorem 2.9 is the only result for parabolic systems with non-standard growth and a nonlinear right-hand side. The case of Dirichlet boundary conditions instead of space periodic boundary conditions is much more involved. We refer the reader to Růžička [33], [35], where this case is treated in detail.

Let us briefly indicate the main steps of the proof of Theorem 2.9, for full details we refer to Růžička [34]. We use the Galerkin method with a basis consisting of eigenfunctions $\boldsymbol{\omega}^r$ of the Stokes operator. Let $\mathbf{v}^N \equiv \sum_{r=1}^N c_r^N(t)\boldsymbol{\omega}^r$ be the Galerkin approximation, which solves, for $r = 1, \dots, N$,

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \mathbf{v}^N \cdot \boldsymbol{\omega}^r \, dx + \int_\Omega \mathbf{S}(\mathbf{D}(\mathbf{v}^N), \mathbf{E}) \cdot \mathbf{D}(\boldsymbol{\omega}^r)\, dx + \int_\Omega [\nabla \mathbf{v}^N]\mathbf{v}^N \cdot \boldsymbol{\omega}^r \, dx \\
&= \int_\Omega \mathbf{f} \cdot \boldsymbol{\omega}^r \, dx - \chi^E \int_\Omega \mathbf{E} \otimes \mathbf{E} \cdot \mathbf{D}(\boldsymbol{\omega}^r)\, dx\,, \qquad (2.16)
\end{aligned}
$$

which initial condition $\mathbf{v}^N(0) = P_N(\mathbf{v}_0)$, where P_N is the orthogonal continuous projector of H onto the linear hull of the first N eigenvectors $\boldsymbol{\omega}^r, r = 1, \dots, N$. Using in (2.16) $\mathbf{v}^N$ as a test function one easily derives the energy estimate

$$
\sup_{t\in I} \|\mathbf{v}^N(t)\|_2^2 + \int_0^T \int_\Omega |\mathbf{D}(\mathbf{v}^N)|^{p(|\mathbf{E}|^2)} + |\nabla \mathbf{v}^N|^{p_\infty} \, dx\, dt \le c\,, \qquad (2.17)
$$

where the constant c depends on $\mathbf{f}, \mathbf{v}_0, \mathbf{E}$, but not on N. This estimate implies that

$$
\begin{aligned}
\mathbf{v}^N &\text{ is bounded in } L^\infty(I, H) \cap L^{p_\infty}(I, V_{p_\infty})\,, \\
\mathbf{D}(\mathbf{v}^N) &\text{ is bounded in } L^{p(|\mathbf{E}|^2)}(Q_T)\,.
\end{aligned} \qquad (2.18)
$$

This information together with an appropriate estimate of $\frac{\partial \mathbf{v}^N}{\partial t}$, which will be proved later on, is sufficient to pass to the limit as $N \to \infty$ in all terms in (2.16) except the elliptic nonlinearity. In order to identify the limiting element also for this term (cf. (2.15)) we need some additional information, which we shall derive next. Due to the space periodic boundary conditions we can use $-\Delta \mathbf{v}^N$ as a test function in (2.16), which gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{v}^N\|_2^2 - \int_\Omega \mathbf{S}(\mathbf{D}(\mathbf{v}^N), \mathbf{E}) \cdot \mathbf{D}(\Delta \mathbf{v}^N)\, dx \qquad (2.19) \\
&= \int_\Omega [\nabla \mathbf{v}^N]\mathbf{v}^N \cdot \Delta \mathbf{v}^N \, dx - \int_\Omega \mathbf{f} \cdot \Delta \mathbf{v}^N \, dx + \chi^{\mathbf{E}} \int_\Omega \mathbf{E} \otimes \mathbf{E} \cdot \mathbf{D}(\Delta \mathbf{v}^N)\, dx\,.
\end{aligned}
$$

One easily computes

$$
\begin{aligned}
-\int_\Omega \mathbf{S}(\mathbf{D}(\mathbf{v}^N),\mathbf{E})\cdot\mathbf{D}(\Delta\mathbf{v}^N)\,dx &= \int_\Omega \frac{\partial S_{ij}(\mathbf{D}(\mathbf{v}^N),\mathbf{E})}{\partial E_k}\nabla E_k D_{ij}(\nabla\mathbf{v}^N)\,dx \\
&+\int_\Omega \frac{\partial S_{ij}(\mathbf{D}(\mathbf{v}^N),\mathbf{E})}{\partial D_{kl}} D_{kl}(\nabla\mathbf{v}^N)D_{ij}(\nabla\mathbf{v}^N)\,dx\,,
\end{aligned} \tag{2.20}
$$

$$
\int_\Omega [\nabla\mathbf{v}^N]\mathbf{v}^N\cdot\Delta\mathbf{v}^N\,dx = -\int_\Omega \frac{\partial v_j^N}{\partial x_k}\frac{\partial v_i^N}{\partial x_j}\frac{\partial v_i^N}{\partial x_k}\,dx\,, \tag{2.21}
$$

$$
\chi^E\int_\Omega \mathbf{E}\otimes\mathbf{E}\cdot\mathbf{D}(\Delta\mathbf{v}^N)\,dx = -2\,\chi^E\int_\Omega E_i\frac{\partial E_j}{\partial x_k}D_{ij}\big(\frac{\partial\mathbf{v}^N}{\partial x_k}\big)\,dx\,. \tag{2.22}
$$

From (2.7) follows that the second term on the right-hand side of (2.20) is bounded from below by

$$
c_1\int_\Omega (1+|\mathbf{D}(\mathbf{v}^N(t))|^2)^{\frac{p(|\mathbf{E}(t)|^2)-2}{2}}|\mathbf{D}(\nabla\mathbf{v}^N(t))|^2\,dx =: c_1\mathcal{I}_p(\mathbf{v}^N(t))\,.
$$

The term $\mathcal{I}_p(\mathbf{v}(t))$ plays an important role for which we have the following lower bounds.

Lemma 2.23. *There are constants depending only on Ω and p such that*

$$
1+\|\nabla^2\mathbf{v}\|^{p_\infty}_{\frac{3p_\infty}{p_\infty+1}} \le c\,(1+\mathcal{I}_p(\mathbf{v})) \qquad \textit{if } p_\infty\in(1,2)\,, \tag{2.24}
$$

$$
\|\nabla^2\mathbf{v}\|_2^2 \le c\,\mathcal{I}_p(\mathbf{v}) \qquad \textit{if } p_\infty\ge 2\,. \tag{2.25}
$$

Moreover, there is a constant $c=c(\Omega,p,\mathbf{E},q)$ such that

$$
\|(1+|\mathbf{D}(\mathbf{v})|^2)^{\frac{p(|\mathbf{E}|^2)}{2}}\|_3 \le c\,\mathcal{I}_p(\mathbf{v}) + c\,\|(1+|\mathbf{D}(\mathbf{v})|^2)^{\frac{p(|\mathbf{E}|^2)}{2}}\|_s^s\,. \tag{2.26}
$$

From (1.34) one easily computes that for arbitrary $\alpha>0$

$$
\Big|\frac{\partial\mathbf{S}(\mathbf{D},\mathbf{E})}{\partial E_k}\Big| \le c(\alpha)|\mathbf{E}|(1+|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{\frac{p(|\mathbf{E}|^2)-1}{2}}(1+|\mathbf{D}|^2)^\alpha\,,
$$

and thus we can bound the first term on the right-hand side of (2.20) by

$$
\begin{aligned}
&c(\mathbf{E})\int_\Omega\{(1+|\mathbf{D}(\mathbf{v}^N)|^2)^{\frac{p(|\mathbf{E}|^2)-2}{2}}|\mathbf{D}(\nabla\mathbf{v}^N)|^2\}^{\frac12}(1+|\mathbf{D}(\mathbf{v}^N)|^2)^{\frac{p(|\mathbf{E}|^2)s}{2}\frac12}\,dx \\
&\le \frac{c_1}{4}\mathcal{I}_p(\mathbf{v}^N(t)) + c(\mathbf{E})\|(1+|\mathbf{D}(\mathbf{v})^N|^2)^{\frac{p(|\mathbf{E}|^2)}{2}}\|_s^s\,,
\end{aligned} \tag{2.27}
$$

where $s>1$ will be chosen later. The right-hand side of (2.22) is bounded by

$$
\begin{aligned}
&\frac{c_1}{4}\mathcal{I}_p(\mathbf{v}^N(t)) + c\int_\Omega |\nabla\mathbf{E}|^2(1+|\mathbf{D}(\mathbf{v}^N)|^2)^{\frac{2-p(|\mathbf{E}|^2)}{2}}\,dx \\
&\le \frac{c_1}{4}\mathcal{I}_p(\mathbf{v}^N(t)) + c(\mathbf{E})\int_\Omega (1+|\mathbf{D}(\mathbf{v}^N)|^2)^{\frac{p(|\mathbf{E}|^2)}{2}}\,dx\,,
\end{aligned} \tag{2.28}
$$

while the second term on the right-hand side of (2.19) can be estimated by (cf. the definition of $\mathcal{I}_p(\mathbf{v}(t))$ and $\|\nabla^2\mathbf{v}\|_{r'} \le c\|\mathbf{D}(\nabla\mathbf{v})\|_{r'}$)

$$\begin{aligned} \|\mathbf{f}\|_r \|\nabla^2\mathbf{v}^N\|_{r'} &\le \|\mathbf{f}\|_r\, \mathcal{I}_p(\mathbf{v}^N(t))^{\frac{1}{2}} \|(1+|\mathbf{D}(\mathbf{v}^N)|^2)^{\frac{p(|\mathbf{E}|^2)}{2}}\|_1^{\frac{2-r'}{2r'}} \\ &\le \frac{c_1}{4}\mathcal{I}_p(\mathbf{v}^N(t)) + c\,\|\mathbf{f}\|_r^r + c\,\|(1+|\mathbf{D}(\mathbf{v})^N|^2)^{\frac{p(|\mathbf{E}|^2)}{2}}\|_1 \,. \end{aligned} \tag{2.29}$$

From (2.19)–(2.22), (2.26)–(2.29) we therefore deduce using Young's inequality for the last term in (2.29)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(1+\|\nabla\mathbf{v}^N(t)\|_2^2) + \frac{c_1}{4}\mathcal{I}_{p(|\mathbf{E}(t)|^2)}(\mathbf{v}^N(t)) \\ &\le c(1+\|\nabla\mathbf{v}^N(t)\|_3^3) + c(\mathbf{E})\|(1+|\mathbf{D}(\mathbf{v}^N(t))|^2)^{\frac{p(|\mathbf{E}|^2)}{2}}\|_s^s + c\,\|\mathbf{f}(t)\|_r^r \,. \end{aligned} \tag{2.30}$$

In order to bound the right-hand side of (2.30) we now use all available norms for gradients of $\mathbf{v}$, i.e. the L^2-norm of $\nabla\mathbf{v}$ appearing on the left-hand side of (2.30), the $L^{p(|\mathbf{E}|^2)}$-norm of $\mathbf{D}(\mathbf{v})$, for which we have an apriori estimate (2.17) and the lower bounds of $\mathcal{I}_{p(|\mathbf{E}(t)|^2)}(\mathbf{v}(t))$ from Lemma 2.23. For example, we use the interpolation inequalities

$$\begin{aligned} \|g\|_3 &\le \|g\|_{p_\infty}^{\frac{p_\infty-1}{2}} \|g\|_{3p_\infty}^{\frac{3-p_\infty}{2}} \,, \\ \|g\|_3 &\le \|g\|_2^{\frac{2(p_\infty-1)}{3p_\infty-2}} \|g\|_{3p_\infty}^{\frac{p_\infty}{3p_\infty-2}} \,, \end{aligned} \tag{2.31}$$

to bound the first term on the right-hand side of (2.30). The third term is treated similarly, however one must be more careful and make also use of the following inequalities, valid for all $q \in [1,\infty)$

$$c\,\|\nabla\mathbf{v}(t)\|_{qp_\infty}^{p_\infty} \le \|(1+|\mathbf{D}(\mathbf{v}(t))|^2)^{\frac{p(|\mathbf{E}|^2)}{2}}\|_q \le c\big(1+\|\nabla\mathbf{v}(t)\|_{qp_0}^{p_0}\big)\,, \tag{2.32}$$

together with inequality (2.26). This is a lengthy procedure where several sub-cases must be considered. We do not go into details here, but refer to Růžička [34] for all details. The final result of this argumentation is the following inequality

$$\begin{aligned} &\frac{d}{dt}(1+\|\nabla\mathbf{v}^N(t)\|_2^2) + \mathcal{I}_{p(|\mathbf{E}(t)|^2)}(\mathbf{v}^N(t)) + \|(1+|\mathbf{D}(\mathbf{v}^N(t))|^2)^{\frac{p(|\mathbf{E}|^2)}{2}}\|_3 \\ &\le c\big(1+\|\mathbf{f}\|_r^r + \|(1+|\mathbf{D}(\mathbf{v}^N(t))|^2)^{\frac{p(|\mathbf{E}|^2)}{2}}\|_1\big(1+\|\nabla\mathbf{v}^N(t)\|_2^2)^\lambda\big)\,, \end{aligned} \tag{2.33}$$

where λ in the case $p_\infty \ge 3$, $p_0 < p_\infty + 4/3$ is given by

$$\lambda = 2\frac{(s-1)(3p_\infty - sp_0)}{(3-s)(4-3sp_0+3p_\infty)}\,. \tag{2.34}$$

with appropriate $s > 1$, while in the case $5/3 < p_\infty \leq 3$ we have

$$\lambda = \frac{3 - p_\infty}{3p_\infty - 5}\,. \tag{2.35}$$

Dividing (2.33) by $(1 + \|\nabla \mathbf{v}(t)\|_2^2)^\lambda$ and integrating over $(0,t)$ yields

$$\begin{aligned} &\frac{1}{1-\lambda}(1 + \|\nabla \mathbf{v}^N(t)\|_2^2)^{1-\lambda} \qquad\qquad (2.36)\\ &+ \int_0^t \big(\mathcal{I}_{p(|\mathbf{E}(\tau)|^2)}(\mathbf{v}^N(\tau)) + \|(1 + |\mathbf{D}(\mathbf{v}(^N\tau))|^2)^{\frac{p(|\mathbf{E}|^2)}{2}}\|_3\big)(1 + \|\nabla \mathbf{v}^N(\tau)\|_2^2)^{-\lambda}\, d\tau \;\leq c\,. \end{aligned}$$

The first term in (2.36) gives only an information if $\lambda \leq 1$, which is the case when either $p_\infty \geq 3$, $p_0 < p_\infty + 4/3$, since $\lambda \to 0$ as $s \to 1$ or $11/5 \leq p_\infty \leq 3$. These are exactly the requirements which appear in the second part of the Theorem 2.9. In this case we take the supremum over $t \in (0,T)$ and obtain that

$$\mathbf{v}^N \text{ is bounded in } L^\infty(I, W^{1,2}(\Omega)) \tag{2.37}$$

and consequently also that (cf. (2.25), (2.32))

$$\mathbf{v}^N \text{ is bounded in } L^2(I, W^{2,2}(\Omega)) \cap L^{p_\infty}(I, V_{3p_\infty})\,. \tag{2.38}$$

In the case that $\lambda > 1$, i.e. $p_\infty < 11/5$, the first term in (2.36) is negative, but it can be moved to the right-hand side and estimated there by $(\lambda - 1)^{-1}$. In this case we obtain from (2.36) that

$$\int_0^T \mathcal{I}_{p(|\mathbf{E}(t)|^2)}(\mathbf{v}^N(t))(1 + \|\nabla \mathbf{v}^N(t)\|_2^2)^{-\lambda}\, dt \leq c(\mathbf{v}_0, \mathbf{f}, \mathbf{E})\,. \tag{2.39}$$

Similarly as in Málek, Nečas, Rokyta, Růžička [17] we can derive from this estimate that for certain $q > 1, r < p_\infty$, and $\sigma > 0$

$$\mathbf{v}^N \text{ is bounded in } L^r(I, W^{1+\sigma,q}(\Omega))\,. \tag{2.40}$$

Now, it remains to derive an estimate for $\frac{\partial \mathbf{v}^N}{\partial t}$. This estimate is obtained from the Galerkin system (2.16), where the continuity of the projection P_N in $L^2(\Omega)$ resp. $W^{3,2}(\Omega) \cap V_2$ is used. In fact, we have for all $\boldsymbol{\varphi} \in W^{3,2}(\Omega) \cap V_2$

$$\begin{aligned} &|\int_\Omega \frac{\partial \mathbf{v}^N}{\partial t} \cdot \boldsymbol{\varphi}\, dx| = |\int_\Omega \frac{\partial \mathbf{v}^N}{\partial t} \cdot P_N \boldsymbol{\varphi}\, dx| \qquad\qquad (2.41)\\ &\leq |\int_\Omega [\nabla \mathbf{v}^N]\mathbf{v}^N \cdot P_N \boldsymbol{\varphi}\, dx| + \int_\Omega |\mathbf{S}(\mathbf{D}(\mathbf{v}^N), \mathbf{E}) \cdot \mathbf{D}(P_N \boldsymbol{\varphi})|\, dx \\ &\quad + \int_\Omega |\mathbf{f} \cdot P_N \boldsymbol{\varphi}|\, dx + |\chi^E| \int_\Omega |\mathbf{E} \otimes \mathbf{E} \cdot \mathbf{D}(P_N \boldsymbol{\varphi})|\, dx = I_1 + \ldots + I_4\,. \end{aligned}$$

The terms I_1, I_3, I_4 can be handled easily, so that we will concentrate on the term I_2 only. Since we have different informations in the cases $9/5 < p_\infty < 11/5$ and $11/5 \le p_\infty$ we estimate the crucial term differently. In the first case we have

$$\begin{aligned}\int_0^T I_2\,dt &\le c(\mathbf{E})\int_0^T\int_\Omega (1+|\mathbf{D}(\mathbf{v}^N)|)^{p_0-1}|\nabla P_N\boldsymbol{\varphi}|\,dx\,dt\\ &\le c\big(1+\|\mathbf{D}(\mathbf{v}^N)\|^{p_0-1}_{L^{p_\infty}(Q_T)}\big)\|\boldsymbol{\varphi}\|_{L^{\frac{p_\infty}{p_\infty+1-p_0}}(I,W^{3,2}(\Omega))}\end{aligned}\tag{2.42}$$

provided that

$$p_0 < p_\infty + 1\,,\tag{2.43}$$

which gives the upper bound in the first part of the theorem. From the above we conclude that

$$\frac{\partial \mathbf{v}^N}{\partial t}\ \text{is bounded in}\ L^s(I,(W^{3,2}(\Omega)\cap V_2)^*)\,,\tag{2.44}$$

provided

$$s = \frac{p_\infty}{p_0-1}\,,\qquad \text{and}\qquad p_0 < p_\infty + 1\,.\tag{2.45}$$

In the case $11/5 \le p_\infty$ we derive from (2.37), (2.38) via the interpolation of $L^{p_\infty+4/3}(\Omega)$ between $L^2(\Omega)$ and $L^{3p_\infty}(\Omega)$ that

$$\nabla\mathbf{v}^N\ \text{is bounded in}\ L^{\frac{3p_\infty+4}{3}}(Q_T)\,.\tag{2.46}$$

Now we estimate the crucial term as

$$\begin{aligned}\int_0^T I_2\,dt &\le c(\mathbf{E})\int_0^T\int_\Omega (1+|\mathbf{D}(\mathbf{v}^N)|)^{p_0-1}|\nabla P_N\boldsymbol{\varphi}|\,dx\,dt\\ &\le c\big(1+\|\mathbf{D}(\mathbf{v}^N)\|^{p_0-1}_{L^{\frac{3p_\infty+4}{3}}(Q_T)}\big)\|\boldsymbol{\varphi}\|_{L^{\frac{3p_\infty+4}{3(p_\infty-p_0)+7}}(I,W^{3,2}(\Omega))}\,,\end{aligned}\tag{2.47}$$

where we used Hölder's inequality with $\delta = \frac{3p_\infty+4}{3(p_0-1)}$. Note, that $\delta > 1$ for $p_0 < p_\infty+4/3$, which is again the upper bound in the second part of Theorem 2.9. Therefore, we obtain again that (2.44) holds with

$$s = \frac{3p_\infty+4}{3(p_0-1)}\,,\qquad \text{and}\qquad p_0 < p_\infty + 4/3\,.\tag{2.48}$$

Now we have all estimates which we need at our disposal and we can pass to the limit in (2.16) as $N\to\infty$. More precisely, our sequence of Galerkin solutions $\mathbf{v}^N$ fulfills (2.18), (2.44), (2.45) resp. (2.48). Moreover, we have that (2.40) is satisfied for $p_\infty \in (9/5, 11/5)$ and that (2.37), (2.38) and (2.46) hold for $p_\infty \ge 11/5$. These informations imply that we can chose a subsequence, still labeled $\mathbf{v}^N$, such that

$$\begin{aligned} \mathbf{v}^N &\rightharpoonup \mathbf{v} && \text{weakly in } L^{p_\infty}(I, V_{p_\infty}), \\ \mathbf{v}^N &\rightharpoonup^* \mathbf{v} && \text{weakly}^* \text{ in } L^\infty(I, L^2(\Omega)), \\ \frac{\partial \mathbf{v}^N}{\partial t} &\rightharpoonup \frac{\partial \mathbf{v}}{\partial t} && \text{weakly in } L^s(I, (W^{3,2}(\Omega) \cap V_2)^*), \\ \mathbf{v}^N &\to \mathbf{v} && \text{strongly in } L^{p_\infty}(I, L^2(\Omega)), \end{aligned} \tag{2.49}$$

where the last line is a consequence of (2.18), (2.44) and the Aubin-Lions lemma. In the case $p_\infty \ge 11/5$ we additionally can ensure that

$$\begin{aligned} \mathbf{v}^N &\rightharpoonup \mathbf{v} && \text{weakly in } L^2(I, W^{2,2}(\Omega)) \cap L^{p_\infty}(I, V_{3p_\infty}), \\ \mathbf{v}^N &\rightharpoonup^* \mathbf{v} && \text{weakly}^* \text{ in } L^\infty(I, W^{1,2}(\Omega)). \end{aligned} \tag{2.50}$$

Finally, from (2.40) in the case $p_\infty \in (9/5, 11/5)$ and (2.37) in the case $p_\infty \ge 11/5$, the Aubin-Lions lemma and the compact embedding $W^{1+\sigma,q}(\Omega) \hookrightarrow\hookrightarrow W^{1,p_\infty}(\Omega)$ resp. $W^{2,2}(\Omega) \hookrightarrow\hookrightarrow W^{1,2}(\Omega)$ we deduce that

$$\nabla \mathbf{v}^N \to \nabla \mathbf{v} \qquad \text{strongly in } L^\alpha(Q_T),$$

where $\alpha = 2$ if $p_\infty \ge 11/5$ and $\alpha = r$ with $r < p_\infty$ chosen appropriately. This in turn implies that $\nabla \mathbf{v}^N \to \nabla \mathbf{v}$ almost everywhere in Q_T and thus

$$\mathbf{S}(\mathbf{D}(\mathbf{v}^N), \mathbf{E}) \to \mathbf{S}(\mathbf{D}(\mathbf{v}), \mathbf{E}) \qquad \text{a.e. in } Q_T. \tag{2.51}$$

The convergence indicated in (2.49) is sufficient for the limiting process $N \to \infty$ in all terms of the weak formulation of (2.2) except the term with the nonlinear extra stress tensor $\mathbf{S}$. This term can be handled with Vitali's convergence theorem, which is applicable due to (2.18) and (2.51) and the growth of $\mathbf{S}$ given by (1.34), (1.35). For more details we refer again to Málek, Nečas, Rokyta, Růžička [17] and Růžička [34]. The proof of Theorem 2.9 is finished.

The method outlined above does not only work in the unsteady case but also in steady situations. In fact, the arguments are considerably easier because we do not have to guard the time integrability. This implies that we obtain less restrictive bounds on p_∞. We do not go in any detail here but only formulate the result and refer again to Růžička [34] where the steps are outlined.

Theorem 2.52. *Let $\Omega = (0, L)^3$ be a given cube and assume that $\mathbf{E} \in W^{1,\infty}(\Omega)$ and $\mathbf{f} \in L^{q'}(\Omega)$, $q = \min(p_\infty, 2)$, are given. Then there exists a solution $\mathbf{v}$ of the steady problem (2.2) such that*

$$\begin{aligned} \mathbf{v} &\in W^{2,q}(\Omega) \cap V_{3p_\infty}, \\ \mathbf{D}(\mathbf{v}) &\in V_{p(|\mathbf{E}|^2)}, \end{aligned} \tag{2.53}$$

which satisfies the steady version of (2.12)*, whenever*

$$9/5 < p_\infty \le p(|\mathbf{E}|^2) \le p_0 < 3p_\infty + 1. \tag{2.54}$$

3 Numerical Analysis

Before one starts a numerical analysis of the full system governing the flow of electrorheological fluids one should have a profound understanding of a simpler situation, namely when the electric field $\mathbf{E}$ is constant. In this case we do not have to deal with a system with non-standard growth conditions, but now $p = \text{const.}$ and the system has p-structure. Moreover we shall for simplicity assume that the extra stress tensor is given by

$$\mathbf{S}(\mathbf{D}) = \mu_0(1 + |\mathbf{D}|^2)^{\frac{p-2}{2}}\mathbf{D}\,, \tag{3.1}$$

where $\mu_0 > 0$. This form is a prototype for a whole class of so-called fluids with shear dependent viscosities and we refer the reader to Málek, Rajagopal, Růžička [20] and Málek, Nečas, Rokyta, Růžička [17] for a detailed discussion of the modeling and the present mathematical theory. While the special case $p = 2$, i.e. the Navier-Stokes equations, is well studied in the literature (cf. Heywood, Rannacher [11], [12], [13], Girault, Raviart [8], Temam [31], Prohl [25]) there are only a few papers dealing with numerical problems related to fluids with shear dependent viscosities (cf. Bao, Barrett [3], Barrett, Liu [4], Málek, Turek [21], Hron, Málek, Turek [14]). A first rigorous error analysis for a fully implicit space-time discretization can be found in Prohl, Růžička [26].

Here we shall briefly present the main results and ideas for the time discretization only. More precisely, we shall derive an error estimate for the difference of solutions $\mathbf{v}$ of the continuous problem[4] $(\mathrm{NS})_p$

$$\begin{aligned} \mathbf{v}_t - \operatorname{div}\,\mathbf{S} + [\nabla\mathbf{v}]\mathbf{v} + \nabla\pi &= \mathbf{f}\,, \\ \operatorname{div}\,\mathbf{v} &= 0\,, \end{aligned} \tag{3.2}$$

and solutions $\mathbf{v}^{m+1}$ of the semi-discretization in time, which reads: Given a time-step $k > 0$, and a corresponding net $\{t_{m+1}\}_{m=-1}^{M}$. For $m \geq 0$ and $\mathbf{v}^m \in V_p$ given from the previous step, compute iterates $\{\mathbf{v}^{m+1}, \pi^{m+1}\} \in V_p \times L_p$ that solve

$$\begin{aligned} \frac{1}{k}\{\mathbf{v}^{m+1} - \mathbf{v}^m\} - \operatorname{div}\mathbf{S}\big(\mathbf{D}(\mathbf{v}^{m+1})\big) + [\nabla\mathbf{v}^{m+1}]\mathbf{v}^{m+1} + \nabla\pi^{m+1} &= \mathbf{f}^{m+1}, \\ \operatorname{div}\mathbf{v}^{m+1} &= 0\,. \end{aligned} \tag{3.3}$$

The extra stress tensor[5] is given in both systems by (3.1) and the systems are completed with space periodic boundary conditions and an initial condition $\mathbf{v}(0) = \mathbf{v}_0$. Moreover we restrict ourselves to the case $p \leq 2$ and $\Omega = (0, L)^3$. In this situation we have

[4]In this section we use the notation $\mathbf{v}_t$ for the time derivative $\frac{\partial\mathbf{v}}{\partial t}$.

[5]The analysis presented below can be easily extended to the case that the extra stress tensor is derivable from a potential with appropriate growth, monotonicity and coercivity properties (cf. Prohl, Růžička [26])

Theorem 3.4. *Let $\{\mathbf{v},\pi\}$ be a strong solution of the problem (NS)$_p$ for $p \in [\frac{3+\sqrt{29}}{5}, 2]$. Suppose that $\{\mathbf{v}^{m+1}, \pi^{m+1}\}$ are iterates from (3.3). Then, there exists a constant c that only depends on $\mathbf{u}_0, \mathbf{f}, \Omega, T^\star$ but not on the time-step k, such that the following statement is valid, provided the time-step is chosen sufficiently small, i.e., $k \le k_0(p, T^\star)$,*

$$\max_{0\le m\le M} \|\mathbf{v}(t_{m+1}) - \mathbf{v}^{m+1}\|_2^2 + k\sum_{m=0}^{M} \|\mathbf{v}(t_{m+1}) - \mathbf{v}^{m+1}\|_{1,p}^2 \le c\, k^{2\alpha(p)},$$

with

$$\alpha(p) = \frac{5p-6}{2p}\,.$$

Concerning the existence of strong solutions of the problem (NS)$_p$ we have

Proposition 3.5. *Let $\mathbf{v}_0 \in V_2 \cap W^{2,2}(\Omega)$, $\mathbf{f} \in L^r(Q_T)$, $\mathbf{f}_t \in L^2(Q_T)$, $r > p'$ be given. Then there exists a $T^* > 0$ such that a strong solution $\mathbf{v}$ of the problem (NS)$_p$ exists on $I = (0, T^*)$ whenever $p \ge 5/3$. This solution satisfies the estimates:*[6]

$$\text{(a)} \qquad \operatorname*{esssup}_{s\in I} \|\mathbf{v}(s)\|_2^2 + \int_I \|\nabla\mathbf{v}(s)\|_p^p \, ds \le c(\mathbf{f}, \mathbf{v}_0)$$

$$\text{(b)} \qquad \operatorname*{esssup}_{s\in I} \|\nabla\mathbf{v}(s)\|_2^2 + \int_I \mathcal{I}_p(\mathbf{v}(s))\, ds \le c(\mathbf{f}, \mathbf{v}_0, T^*)$$

$$\text{(c)} \qquad \int_I \|\mathbf{v}_t(s)\|_2^2 \, ds + \operatorname*{esssup}_{s\in I} \|\nabla\mathbf{v}(s)\|_p^p \le c(\mathbf{f}, \mathbf{v}_0, T^*),$$

$$\text{(d)} \qquad \operatorname*{esssup}_{s\in I} \|\mathbf{v}_t(s)\|_2^2 + \int_I \mathcal{J}_p(\mathbf{v}(s))\, ds \le c(\mathbf{f}, \mathbf{v}_0, T^*).$$

The constants that are employed in the upper bounds in (b)–(d) are to stress that these estimates are valid locally in time. In particular we have that

$$\begin{aligned} \mathbf{v} &\in L^2(I, W^{2,p}(\Omega)) \cap L^p(I, W^{\frac{3p}{p+1}}(\Omega))\,, \\ \mathbf{v}_t &\in L^2(I, W^{1,\frac{4}{4-p}}(\Omega)) \cap L^\infty(I, L^2(\Omega))\,, \\ \nabla\mathbf{v} &\in W^{1,\frac{4p}{2+p}}(Q_T) \cap L^\infty(I, L^{\frac{3}{2}p}(\Omega))\,, \\ \mathbf{v}_{tt} &\in L^2(I, V_2^*)\,. \end{aligned} \tag{3.6}$$

The main difficulties in the proof of Proposition 3.5 are similar to that one in the proof of Theorem 2.9. The proof of this proposition uses again on the Galerkin approximation (cf. (2.1)). However, we want to establish the existence of strong solutions locally in time, where p has to satisfy a less restrictive lower bound than in Theorem 2.9. Moreover, we need additional informations on the solution in order

[6] For the definition of $\mathcal{I}_p(\mathbf{v})$ and $\mathcal{J}_p(\mathbf{v})$ see below.

to prove Theorem 3.4. Thus we not only test the weak formulation of the problem $(\mathrm{NS})_p$ with $\mathbf{v}$ and $-\Delta\mathbf{v}$, which gives the estimates (a) and (b), but we also use $\mathbf{v}_t$ and $\mathbf{v}_{tt}$ as test functions, which delivers (c) and (d). Note, that the crucial step for the limiting process is estimate (b). Since the technique is similar to that one used in the previous section we do not go into details here (cf. Prohl, Růžička [26], Málek, Nečas, Rokyta, Růžička [17]), but we only collect lower bounds for the terms coming from the extra stress tensor, which are needed for the above apriori estimates.

Lemma 3.7. *Let $\mathbf{v} \in C^2(\Omega)^3$ and $p \in (1,2]$. Then, there exists a constant c depending only on Ω, p, such that the following inequalities hold,*

$$\text{(a)} \qquad \|\nabla^2\mathbf{v}\|_p^2 \le c\,\mathcal{I}_p(\mathbf{v})\big(1+\|\nabla\mathbf{v}\|_p\big)^{2-p},$$

$$\text{(b)} \qquad 1+\|\nabla\mathbf{v}\|_{3p}^p+\|\nabla^2\mathbf{v}\|_{\frac{3p}{p+1}}^p \le c\left(1+\mathcal{I}_p(\mathbf{v})\right),$$

$$\text{(c)} \qquad \|\nabla\mathbf{v}_t\|_{\frac{4}{4-p}}^2 \le c\,\mathcal{J}_p(\mathbf{v})(1+\|\nabla\mathbf{v}\|_2)^{2-p},$$

$$\text{(d)} \qquad \|\nabla\mathbf{v}_t\|_{\frac{3p}{p+1}}^p \le c\left(1+\mathcal{J}_p(\mathbf{v})\right),$$

$$\text{(e)} \qquad \|\nabla\mathbf{v}\|_{1,\frac{4p}{2+p},Q_T}^p+\|\nabla\mathbf{v}\|_{2p,Q_T}^p, \le c\,\Big(1+\int_0^T\mathcal{I}_p(\mathbf{v})+\mathcal{J}_p(\mathbf{v})\,dt\Big),$$

$$\text{(f)} \qquad \operatorname*{esssup}_{t\in I}\|\nabla\mathbf{v}\|_{\frac{3}{2}p}^p \le c\,\Big(1+\int_0^T\mathcal{I}_p(\mathbf{v})+\mathcal{J}_p(\mathbf{v})\,dt\Big),$$

where

$$\mathcal{I}_p(\mathbf{v}) = \int_\Omega \big(1+|\mathbf{D}(\mathbf{v})|^2\big)^{\frac{p-2}{2}}|\mathbf{D}(\nabla\mathbf{v})|^2\,dx,$$
$$\mathcal{J}_p(\mathbf{v}) = \int_\Omega \big(1+|\mathbf{D}(\mathbf{v})|^2\big)^{\frac{p-2}{2}}|\mathbf{D}(\mathbf{v}_t)|^2\,dx.$$

Concerning the existence of solutions of the discrete problem (3.3) we can not apply the same strategy as in the continuous case, since the analogy of a crucial step, i.e. the step from (2.33) to (2.36), does not work in the discrete case. However, the discrete version of the first energy estimate (cf. Proposition 3.5 (a)) is available and thus one can view the discrete time derivative as a right-hand side, which belongs to $L^2(\Omega)$ and consider the system (3.3) as a steady problem. This situation was analyzed in Frehse, Málek, Steinhauer [7] and Růžička [32] and we can use the ideas from these papers to obtain

Proposition 3.8. *Let $\mathbf{v}_0$ and $\mathbf{f}$ satisfy the same assumptions as in Proposition 3.5. Then there exists a weak solution $\mathbf{v}^{m+1}$ of (3.3) satisfying*

$$\max_{0\le m\le M}\|\mathbf{v}^{m+1}\|_2^2+k\sum_{m=0}^{M}\|\mathbf{D}(\mathbf{v}^{m+1})\|_p^p \le c(\mathbf{f},\mathbf{v}_0), \tag{3.9}$$

whenever $p>3/2$.

In order to verify Theorem 3.4 we have to deal with two problems, namely that the discrete solution $\mathbf{v}^{m+1}$ of (3.3) is only weak, and secondly that the information about $\mathbf{v}_{tt}$ is also weak. Thus we introduce an auxiliary problem to split these problems subsequently. We consider the following auxiliary problem: suppose that $\mathbf{v}$ is a strong solution to the problem $(\mathrm{NS})_p$ with the properties stated in Proposition 3.5. Then, determine $\big\{\mathbf{V}^{m+1}, \tilde{\pi}^{m+1}\big\}$ that satisfies[7]

$$\begin{aligned} d_t\mathbf{V}^{m+1} - \operatorname{div}\mathbf{S}\big(\mathbf{D}(\mathbf{V}^{m+1})\big) + [\nabla\mathbf{V}^{m+1}]\mathbf{v}(t_{m+1}) + \nabla\tilde{\pi}^{m+1} &= \mathbf{f}(t_{m+1}) \,, \\ \operatorname{div}\mathbf{V}^{m+1} &= 0 \,, \\ \mathbf{V}^0 &= \mathbf{v}_0 \,. \end{aligned} \tag{3.10}$$

The form of the linearized convective term in (3.10) enables us to use all the information on $\mathbf{v}$ established in Proposition 3.5. Thus we can prove again the existence of strong solutions for the auxiliary problem, however there appears a new lower bound for p. This is again related to the missing analogy of the steps from (2.33) to (2.36) in the discrete case.

Lemma 3.11. *Suppose that $\mathbf{v}$ is a strong solution to the problem $(NS)_p$ and $\big\{\mathbf{V}^{m+1}, \tilde{\pi}^{m+1}\big\}$ solves (3.10). Then, there exists a generic constant c independent of k but only on the given data of problem (3.10) , such that, for $p \geq \frac{3+\sqrt{29}}{5}$, holds*

$$\text{(a)} \qquad \max_{0\le m\le M} \|\mathbf{V}^{m+1}\|_2^2 + k\sum_{m=0}^{M} \|\nabla\mathbf{V}^{m+1}\|_p^p \le c(\mathbf{f}, \mathbf{v}_0) \,,$$

$$\text{(b)} \qquad \max_{0\le m\le M} \|\nabla\mathbf{V}^{m+1}\|_2^2 + k\sum_{m=0}^{M} \mathcal{I}_p(\mathbf{V}^{m+1}) \le c(\mathbf{f}, \mathbf{v}_0) \,,$$

$$\text{(c)} \qquad k\sum_{m=0}^{M} \|d_t\mathbf{V}^{m+1}\|_2^2 + \max_{0\le m\le M} \|\Phi(|\mathbf{D}(\mathbf{V}^{m+1})|^2)\|_1 \le c(\mathbf{f}, \mathbf{v}_0) \,,$$

$$\text{(d)} \qquad \max_{0\le m\le M} \|d_t\mathbf{V}^{m+1}\|_2^2 + k\sum_{m=0}^{M} \mathcal{K}_p(\mathbf{V}^{m+1}) \le c(\mathbf{f}, \mathbf{v}_0) \,,$$

where

$$\mathcal{K}_p(\mathbf{V}^{m+1}) \equiv \int_\Omega d_t\mathbf{S}(\mathbf{D}(\mathbf{V}^{m+1})) \cdot d_t\mathbf{D}(\mathbf{V}^{m+1})\, dx \,.$$

In particular we have that there exist uniform bounds $c(\mathbf{f}, \mathbf{v}_0)$ that are independent of the time-step k for

$$\begin{aligned} \mathbf{V}^{m+1} &\in l^2(I, W^{2,p}(\Omega)) \cap l^p(I, V^{\frac{3p}{p+1}}) \,, \\ d_t\mathbf{V}^{m+1} &\in l^2\big(I, V^{\frac{4}{4-p}}\big) \cap l^\infty(I, L^2(\Omega)) \,, \\ \nabla\mathbf{V}^{m+1} &\in l^\infty(I, L^{\frac{3}{2}p}(\Omega)) \,. \end{aligned} \tag{3.12}$$

[7] We use thew notation $d_t u^{m+1} = k^{-1}(u^{m+1} - u^m)$.

Now we have everything at our disposal what we need for the proof of Theorem 3.4. We split the error $\mathbf{v}(t_{m+1}) - \mathbf{v}^{m+1}$ into two parts

$$\mathbf{v}(t_{m+1}) - \mathbf{v}^{m+1} = \mathbf{v}(t_{m+1}) - \mathbf{V}^{m+1} + \mathbf{V}^{m+1} - \mathbf{v}^{m+1} =: \mathbf{E}^{m+1} + \mathbf{e}^{m+1} \,,$$

which will be investigates separately. These errors $\mathbf{E}^{m+1}$ and $\mathbf{e}^{m+1}$, respectively are governed by the following systems

$$\begin{aligned} d_t\mathbf{E}^{m+1} - \operatorname{div}\Big(\mathbf{S}\big(\mathbf{D}(\mathbf{v}(t_{m+1}))\big) - \mathbf{S}\big(\mathbf{D}(\mathbf{V}^{m+1})\big)\Big) \\ +[\nabla\mathbf{E}^{m+1}]\mathbf{v}(t_{m+1}) + \nabla\big(\pi(t_{m+1}) - \tilde{\pi}^{m+1}\big) = \mathbf{R}^{m+1} \,, \\ \operatorname{div}\mathbf{E}^{m+1} = 0 \,, \end{aligned} \tag{3.13}$$

supplemented with

$$\mathbf{R}^{m+1} \equiv \frac{1}{k}\int_{t_m}^{t_{m+1}} (s - t_m)\mathbf{v}_{tt}(s)\, ds \,, \tag{3.14}$$

and

$$\begin{aligned} d_t\mathbf{e}^{m+1} - \operatorname{div}\big(\mathbf{S}(\mathbf{D}(\mathbf{V}^{m+1})) - \mathbf{S}(\mathbf{D}(\mathbf{v}^{m+1}))\big) + \nabla\big(\tilde{\pi}^{m+1} - \pi^{m+1}\big) = \mathbf{r}^{m+1}, \\ \operatorname{div}\mathbf{e}^{m+1} = 0 \ , \end{aligned} \tag{3.15}$$

where

$$-\mathbf{r}^{m+1} = [\nabla\mathbf{V}^{m+1}]\mathbf{E}^{m+1} + [\nabla\mathbf{V}^{m+1}]\mathbf{e}^{m+1} + [\nabla\mathbf{e}^{m+1}]\mathbf{v}^{m+1} \,, \tag{3.16}$$

respectively. Let us first analyze the error $\mathbf{E}^{m+1}$. From (3.14) we compute that

$$\|\mathbf{R}^{m+1}\|_2^2 \le c \sup_{s\in[t_m,t_{m+1}]} \|\mathbf{v}_t(s)\|_2^2 \,, \tag{3.17}$$

$$\|\mathbf{R}^{m+1}\|_{-1,2}^2 \le c\,k^2 \int_{t_m}^{t_{m+1}} \|\mathbf{v}_{tt}(s)\|_{-1,2}^2 \, ds \,. \tag{3.18}$$

If we test the first equation in (3.13) with $\mathbf{E}^{m+1}$ and sum over the number of iteration steps, we obtain

$$\|\mathbf{E}^{M+1}\|_2^2 + k\sum_{m=0}^{M} \|\mathbf{D}(\mathbf{E}^{m+1})\|_p^2 \le k\sum_{m=0}^{M} (\mathbf{R}^{m+1}, \mathbf{E}^{m+1}) \,. \tag{3.19}$$

In order to get the second term on the left-hand side we used that

$$\begin{aligned} &\int_\Omega \big(\mathbf{S}(\mathbf{D}\mathbf{u}) - \mathbf{S}(\mathbf{D}\mathbf{v})\big)\cdot\mathbf{D}(\mathbf{u}-\mathbf{v})\, dx \\ &\ge c\,\|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_p^2\Big(1 + \|\mathbf{D}\mathbf{u}\|_p + \|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_p\Big)^{p-2} \,, \end{aligned} \tag{3.20}$$

together with the fact that $\nabla \mathbf{v}(t_{m+1})$ and $\nabla \mathbf{V}^{m+1}$ belong to the space $l^\infty(I, L^p(\Omega))$. Using the embedding $W^{1,p}(\Omega) \hookrightarrow W^{s,2}(\Omega)$, with $s = \frac{5p-6}{2p}$ and the interpolation of $W^{s,2}(\Omega)$ between $L^2(\Omega)$ and $W^{1,2}(\Omega)$ we get

$$\begin{aligned}(\mathbf{R}^{m+1}, \mathbf{E}^{m+1}) &\leq \|\mathbf{R}^{m+1}\|_2^{1-s} \|\mathbf{R}^{m+1}\|_{-1,2}^{s} \|\mathbf{E}^{m+1}\|_{1,p} \\ &\leq c(\mathbf{f}, \mathbf{v}_0) \|\mathbf{R}^{m+1}\|_{-1,2}^{2s} + \frac{1}{2} \|\mathbf{D}(\mathbf{E}^{m+1})\|_p^2 ,\end{aligned} \tag{3.21}$$

where we also used (3.17), $(3.6)_2$, Korn's and Young's inequality. The last term in (3.21) will be moved to the left-hand side of (3.19) and it remains to bound the first term in (3.21). From (3.18) and $(3.6)_4$ we derive

$$\begin{aligned}k \sum_{m=0}^{M} \|\mathbf{R}^{m+1}\|_{-1,2}^{2s} &\leq c\, k^{2s} \Big(\sum_{m=0}^{M} \int_{t_m}^{t_{m+1}} \|\mathbf{v}_{tt}(s)\|_{-1,2}^2 \, ds \Big)^s \\ &\leq c(\mathbf{f}, \mathbf{v}_0) k^{2s}\end{aligned}$$

which together with (3.19) yields

$$\max_{0 \leq m \leq M} \|\mathbf{E}^{m+1}\|_2^2 + k \sum_{m=0}^{M} \|\mathbf{D}(\mathbf{E}^{m+1})\|_p^2 \leq c(\mathbf{f}, \mathbf{v}_0) k^{2s} , \tag{3.22}$$

with

$$s = \frac{5p-6}{2p} , \tag{3.23}$$

which is exactly the statement of Theorem 3.4, but only for $\mathbf{E}^{m+1}$. Now we derive an estimate for $\mathbf{e}^{m+1}$. For that we test (3.15) with $\mathbf{e}^{m+1}$ and sum over all iteration steps to obtain

$$\begin{aligned}&\max_{0 \leq m \leq M} \|\mathbf{e}^{m+1}\|_2^2 + k \sum_{m=0}^{M} \frac{\|\mathbf{D}(\mathbf{e}^{m+1})\|_p^2}{c + \|\mathbf{D}(\mathbf{e}^{m+1})\|_p^{2-p}} \\ &\leq ck \sum_{m=0}^{M} \int_\Omega |\mathbf{E}^{m+1}|\, |\mathbf{e}^{m+1}|\, |\nabla \mathbf{V}^{m+1}| \, dx + ck \sum_{m=0}^{M} \int_\Omega |\mathbf{e}^{m+1}|^2 \, |\nabla \mathbf{V}^{m+1}| \, dx \\ &= ck \sum_{m=0}^{M} \left(I_1^{m+1} + I_2^{m+1} \right) .\end{aligned} \tag{3.24}$$

In the second term on the left-hand side we used again (3.20) and the uniform bound for $\nabla \mathbf{V}^{m+1} \in l^\infty(I, L^p(\Omega))$. Using Hölder's inequality and the interpolation inequality $\|\mathbf{v}\|_{\frac{6p}{3p-2}} \leq \|\mathbf{v}\|_2^{1-\lambda} \|\nabla \mathbf{v}\|_p^{\lambda}$, with $\lambda = \frac{2}{5p-6}$, and $\nabla \mathbf{V}^{m+1} \in l^\infty(I, \mathbf{L}^{\frac{3}{2}p}(\Omega))$, we derive that

$$\begin{aligned}I_1^{m+1} &\leq \|\nabla \mathbf{V}^{m+1}\|_{\frac{3}{2}p} \|\mathbf{e}^{m+1}\|_{\frac{6p}{3p-2}} \|\mathbf{E}^{m+1}\|_{\frac{6p}{3p-2}} \\ &\leq c \|\mathbf{E}^{m+1}\|_2^2 + c (c + \|\mathbf{D}(\mathbf{e}^{m+1})\|_p^p)^{\frac{2-p}{p} \frac{\lambda}{1-\lambda}} \|\mathbf{e}^{m+1}\|_2^2 \\ &\quad + c \|\mathbf{D}(\mathbf{E}^{m+1})\|_p^2 + \frac{1}{2} \frac{\|\mathbf{D}(\mathbf{e}^{m+1})\|_p^2}{(c + \|\mathbf{D}(\mathbf{e}^{m+1})\|_p^{2-p})} .\end{aligned} \tag{3.25}$$

The last term on the right-hand side is absorbed on the left-hand side of (3.24). For the first and third term in (3.25) we use the estimate (3.22). The term I_2^m is treated analogously, replacing $\mathbf{E}^{m+1}$ by $\mathbf{e}^{m+1}$. Thus we arrive at

$$\begin{aligned} &\max_{0\le m\le M} \|\mathbf{e}^{m+1}\|_2^2 + k\sum_{m=0}^{M} \frac{\|\mathbf{D}(\mathbf{e}^{m+1})\|_p^2}{c+\|\mathbf{D}(\mathbf{e}^{m+1})\|_p^{2-p}} \\ &\le ck^{2s} + c\sum_{m=0}^{M} (c+\|\mathbf{D}(\mathbf{e}^{m+1})\|_p^p)^{\frac{2-p}{p}\frac{\lambda}{1-\lambda}} \|\mathbf{e}^{m+1}\|_2^2 \end{aligned} \tag{3.26}$$

and we can use the discrete Gronwall's lemma whenever $\frac{2-p}{p}\frac{\lambda}{1-\lambda} \le 1$, where $\lambda = \frac{2}{5p-6}$. One easily computes that this requirement is equivalent to $p \ge \frac{3+\sqrt{29}}{5}$. After the application of Gronwall's lemma we obtain that the left-hand side of (3.26) is bounded by ck^{2s}, with s given by (3.23). Since $2s > 1$ for the p's considered here we readily obtain that

$$\max_m \|\mathbf{D}(\mathbf{e}^{m+1})\|_p^2 \le c$$

and in turn we derive

$$\max_m \|\mathbf{e}^{m+1}\|_2^2 + k\sum_{m=0}^{M} \|\mathbf{D}(\mathbf{e}^{m+1})\|_p^2 \le ck^{2s} \,. \tag{3.27}$$

Since the same estimates hold for $\mathbf{E}^{m+1}$ we have furnished the proof of Theorem 3.4. Moreover we have also proved that

$$\max_m \|\mathbf{D}(\mathbf{v}(t_{m+1}) - \mathbf{v}^{m+1})\|_p^2 \le ck^{2s-1} \,. \tag{3.28}$$

Using this it is now possible to derive aposteriori estimates for the solution of (3.3) at the expense of further restricting the values for p.

Lemma 3.29. *Suppose that $p \ge 9/5$. Then, there exists a constant $c(\mathbf{f}, \mathbf{v}_0)$, such that the following estimates are valid for the solution of (3.3), ensured in Lemma 3.11*

(a) $$\max_{0\le m\le M} \|\nabla \mathbf{v}^{m+1}\|_2^2 + k\sum_{m=0}^{M} \mathcal{I}_p(\mathbf{v}^{m+1}) \le c(\mathbf{f}, \mathbf{v}_0) \,,$$

(b) $$k\sum_{m=0}^{M} \|d_t \mathbf{v}^{m+1}\|_2^2 + \max_{0\le m\le M} \|\Phi(|\mathbf{D}(\mathbf{v}^{m+1})|^2)\|_1 \le c(\mathbf{f}, \mathbf{v}_0) \,,$$

(c) $$\max_{0\le m\le M} \|d_t \mathbf{v}^{m+1}\|_2^2 + k\sum_{m=0}^{M} \mathcal{K}_p(\mathbf{v}^{m+1}) \le c(\mathbf{f}, \mathbf{v}_0) \,.$$

The basic observation is that (3.28) and inequality (c) of Proposition 3.5 imply

$$\max_{0\le m\le M} \|\nabla \mathbf{v}^{m+1}\|_p \le c\,. \tag{3.30}$$

Now we can test (3.3) successively with $-\Delta \mathbf{v}^{m+1}$ and $d_t\mathbf{v}^{m+1}$ and than take the discrete time derivative of (3.3) and test the result with $d_t\mathbf{v}^{m+1}$. Due to the better regularity (3.30) this procedure now works for the p's indicated in the above Lemma.

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