On the Eisenstein Class for Hilbert Modular Surfaces

June 4, 2012

Arithmetic Geometry week in Tokyo

Kenichi Bannai（Keio）

work in progress with Guido Kings (Regensburg)

Introduction

Eisenstein class: Motivic class in the motivic cohomology of an elliptic modular curve

First constructed by ***Beilinson*** using the Eisenstein symbol map

Beilinson, A.A. : *Higher regulators of modular curves*, Contemp. Math. **55**, 1-34 (1986)

Introduction

Application: Used in the proof of the Beilinson conjecture

Ex. ***C. Deninger*** proved the weak Beilinson conjecture for Hecke characters of imaginary quadratic fields

C. Deninger, *Higher regulators and Hecke L-series of imaginary quadratic fields. I*, Invent. Math. **96** (1989), no. 1, 1–69.

C. Deninger, *Higher regulators and Hecke L-series of imaginary quadratic fields. II*, Ann. of Math. (2) **132** (1990), no. 1, 131–158.

Introduction

Our Interest: Explicit calculation of the ***rigid syntomic realization***, i.e., the image of the motivic Eisenstein class with respect to the syntomic regulator

B- and G. Kings, *p-adic elliptic polylogarithm, p-adic Eisenstein series and Katz measure*, American J. Math. **132**, no. 6 (2010), 1609-1654.

B- and G. Kings, *p-adic Beilinson conjecture for ordinary Hecke motives associated to imaginary quadratic fields*, RIMS Kôkyûroku Bessatsu **B25**: Algebraic Number Theory and Related Topics 2009, eds. T. Ichikawa, M. Kida, T. Yamazaki, June (2011), 9-30.

Introduction

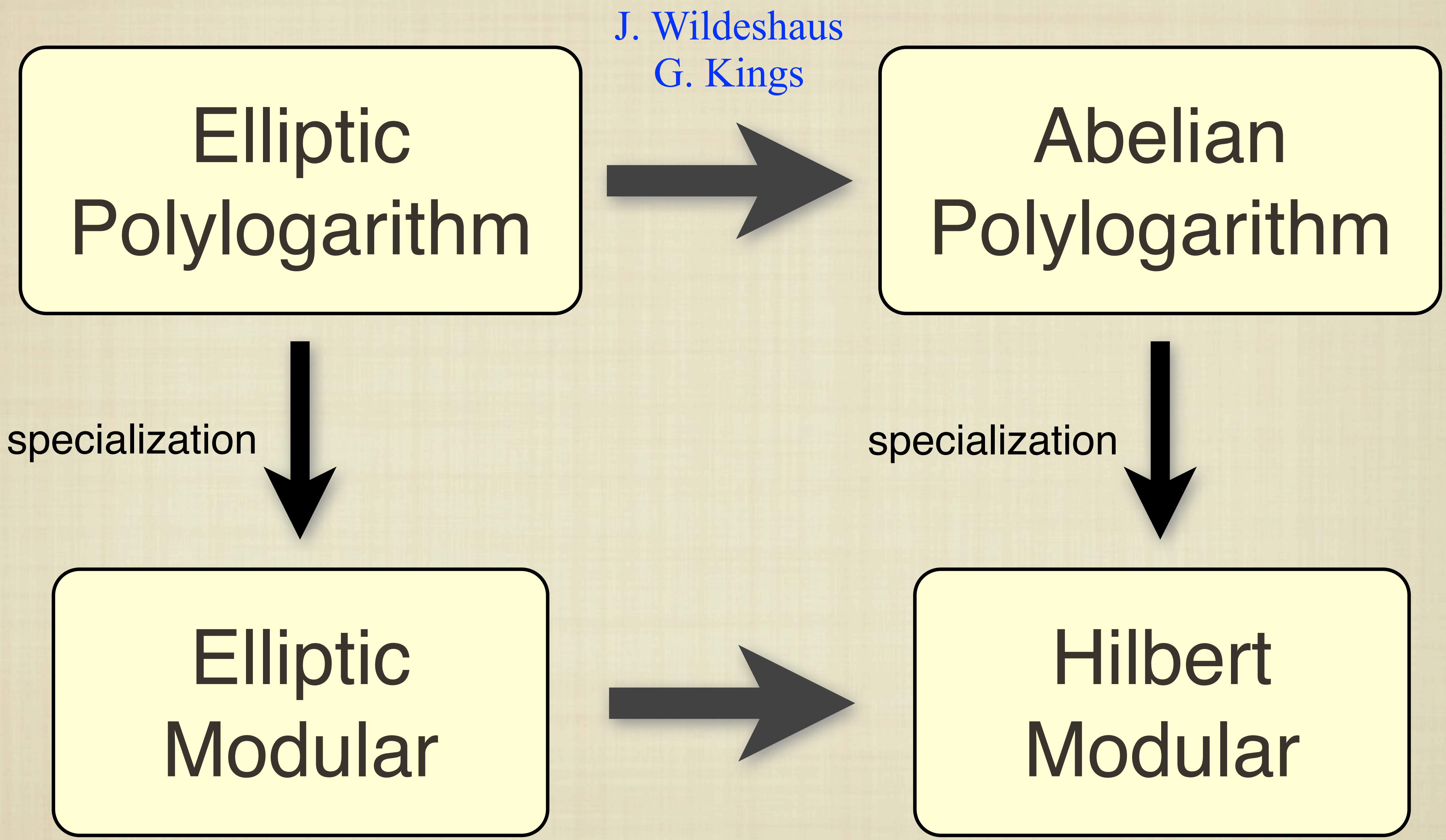
Alternative Construction: Specialization at torsion points of the ***polylogarithm*** on the universal elliptic curve

A. Huber and G. Kings, *Degeneration of l-adic Eisenstein classes and of the elliptic polylog*, Invent. Math. **135** (1999), no. 3, 545–594.

Eisenstein Class

Introduction

Polylogarithm



Eisenstein Class

Introduction

A. Levin gave a conjectural formula for the topological realization of the abelian polylogarithm in terms of currents

A. Levin, *Polylogarithmic currents on abelian varieties*, in *Regulators in Analysis*, Geometry and Number Theory, A. Reznikov, N. Schappacher (Eds), Progr. Math. **171**, Birkhäuser (2000), 207–229.

D. Blottière proved Levin’s formula

D. Blottière, *Réalisation de Hodge du polylogarithme d'un schéma abélien, avec un appendice d'Andrey Levin*, Journal de l'Institut Mathématique de Jussieu (2009), **8** no. 1, pages 1–38

Introduction

This Talk: We give a ***conjectural formula*** for the ***Hodge*** and the ***rigid syntomic*** realizations of the Eisenstein class for Hilbert modular surfaces

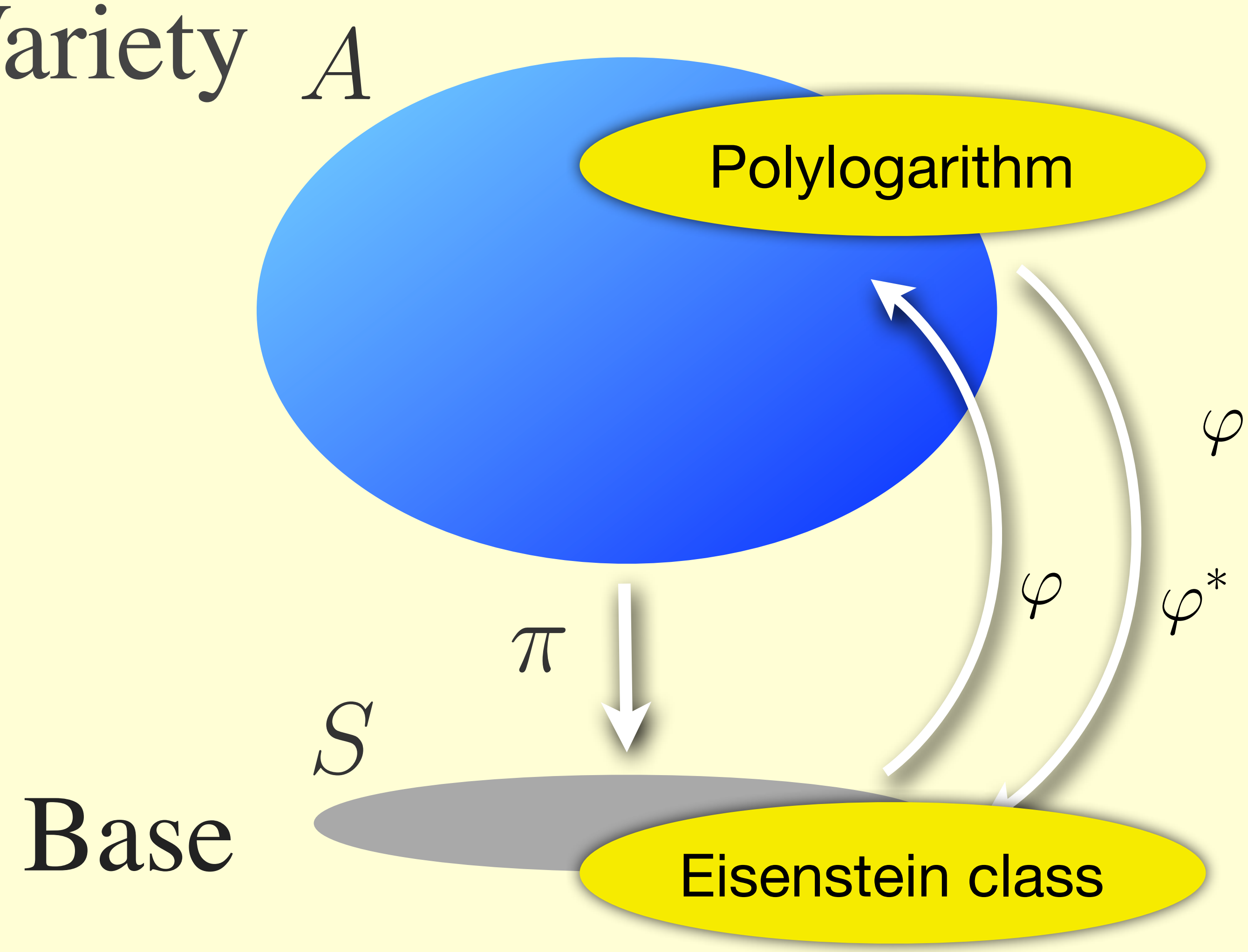
- Eisenstein Class
- Hodge Realization
- Syntomic Realization

The Eisenstein Class

The Eisenstein Class

Abelian Variety

$g := \dim A$



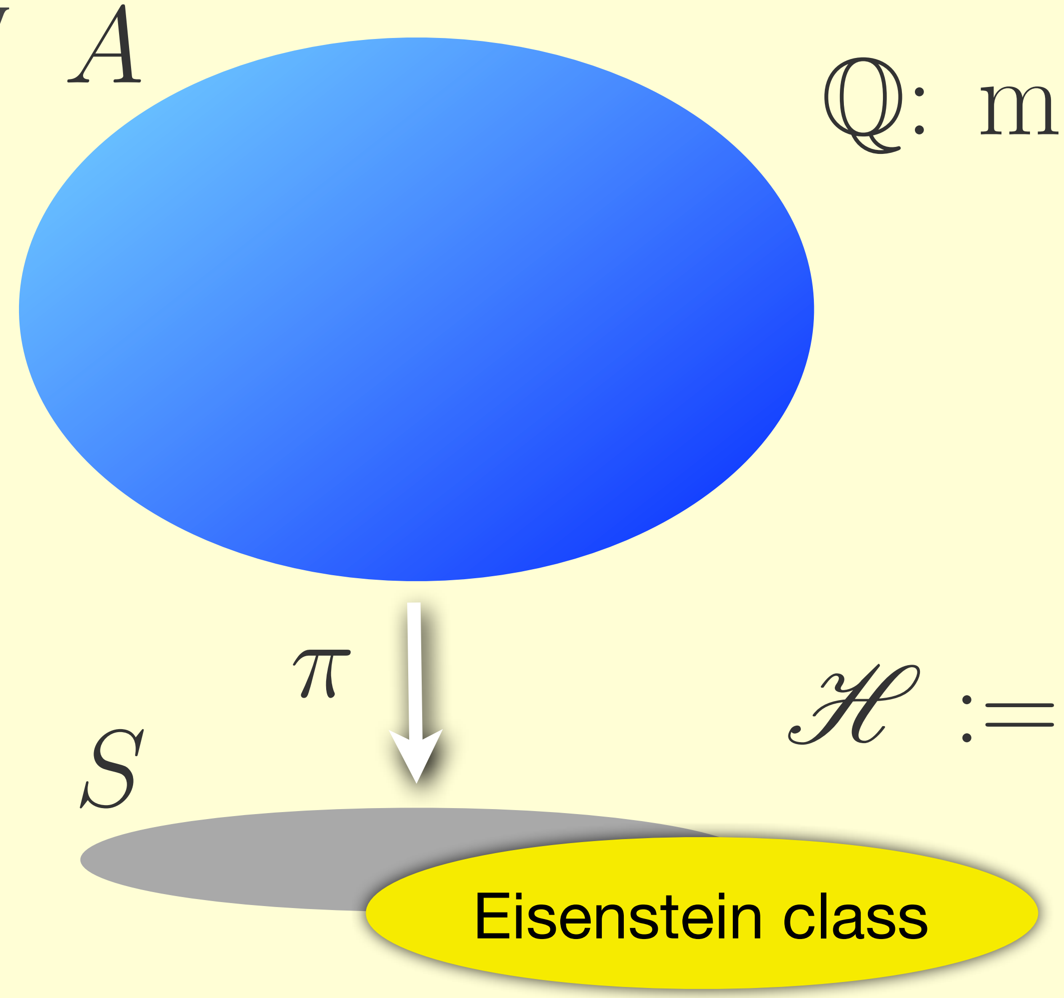
φ: torsion point

The Eisenstein Class

Abelian Variety A

$g := \dim A$

$\mathbb{Q}$: motivic sheaf



Base

$\mathscr{H} := (R^1\pi_*\mathbb{Q})^\vee$

k: integer > 0

$$\mathrm{Eis}^{k+2}_{\mathrm{mot}}(\varphi) \in H^{2g-1}_{\mathrm{mot}}(S, \mathrm{Sym}^k \mathscr{H}(g))$$

The Eisenstein Class

$$\begin{tikzcd}
\mathrm{Eis}^{k+2}_{\mathrm{mot}}(\varphi) \arrow[r, phantom, "\in"] \arrow[d, mapsto] \arrow[rrd, mapsto] & H^{2g-1}_{\mathrm{mot}}(S, \mathrm{Sym}^k \mathscr{H}(g)) \arrow[d, "r_{\mathscr{D}}"] \arrow[rr, "r_{\mathrm{syn}}"] & & H^{2g-1}_{\mathrm{syn}}(S, \mathrm{Sym}^k \mathscr{H}(g)) \\
\mathrm{Eis}^{k+2}_{\mathscr{D}}(\varphi) \arrow[r, phantom, "\in"] & H^{2g-1}_{\mathscr{D}}(S, \mathrm{Sym}^k \mathscr{H}(g)) & \textcolor{red}{?} \arrow[ur, phantom, "\in"] &
\end{tikzcd}$$

The Eisenstein Class

$$E_2^{p,q} = H^p_{\mathscr{D}}(\operatorname{Spec}\mathbb{R}, H^q(S, \operatorname{Sym}^k \mathscr{H}(g)) \Rightarrow H^{p+q}_{\mathscr{D}}(S, \operatorname{Sym}^k \mathscr{H}(g))$$

$$\begin{array}{l} \to \underset{\underset{\displaystyle \operatorname{Eis}^{k+2}_{\mathscr{D}}(\varphi)}{\rotatebox{90}{$\in$}}}{H^{2g-1}_{\mathscr{D}}}(S, \operatorname{Sym}^k \mathscr{H}(g)) \to H^0_{\mathscr{D}}(\operatorname{Spec}\mathbb{R}, H^{2g-1}(S, \operatorname{Sym}^k \mathscr{H}(g))) \to 0 \end{array}$$

The Eisenstein Class

$$\begin{array}{ccc} H^{2g-1}_{\mathscr{D}}(S, \mathrm{Sym}^k \mathscr{H}(g)) & \xrightarrow{\cong} & H^0_{\mathscr{D}}(\mathrm{Spec}\,\mathbb{R}, H^{2g-1}(S, \mathrm{Sym}^k \mathscr{H}(g))) \\ \rotatebox{90}{$\in$} & & \downarrow \\ \mathrm{Eis}^{k+2}_{\mathscr{D}}(\varphi) & & \\ & & H^{2g-1}_{\mathrm{dR}}(S, \mathrm{Sym}^k \mathscr{H}(g)) \end{array}$$

The Eisenstein Class

$$\begin{array}{ccc} H^{2g-1}_{\mathscr{D}}(S, \mathrm{Sym}^k \mathscr{H}(g)) & \hookrightarrow & H^{2g-1}_{\mathrm{dR}}(S, \mathrm{Sym}^k \mathscr{H}(g)) \\ \cup & & \cup \\ \mathrm{Eis}^{k+2}_{\mathscr{D}}(\varphi) & \longmapsto & \mathrm{Eis}^{k+2}_{\mathrm{dR}}(\varphi) \end{array}$$

The Eisenstein Class

Case $g = 1$

$$\begin{tikzcd}
\mathrm{Eis}^{k+2}_{\mathrm{dR}}(\varphi) & \in & H^1_{\mathrm{dR}}(S, \mathrm{Sym}^k \mathscr{H}(1)) & \\
 & & \Gamma(S, \mathrm{Sym}^k \mathscr{H} \otimes \Omega^1_S) \arrow[u, two heads] & \mathscr{H} := R^1\pi_*\Omega^\bullet_{A/S} \\
E^{\varphi}_{k+2}(\tau)\omega^k \dfrac{dq}{q} \arrow[uu, mapsto] & \in & \Gamma(S, \underline{\omega}^{\otimes k} \otimes \Omega^1_S) \arrow[u, hook] & \underline{\omega} := \pi_*\Omega^1_{A/S} \arrow[u, hook]
\end{tikzcd}$$

$$E^{\varphi}_{k+2}(\tau) := \sum_{(m,n)\neq(0,0)} \frac{\widehat{\varphi}(m,n)}{(m\tau + n)^{k+2}}$$

The Eisenstein Class

Case $g = 1$

$$
\begin{tikzcd}
\mathrm{Eis}^{k+2}_{\mathscr{D}}(\varphi) \arrow[d, mapsto] & \in & H^1_{\mathscr{D}}(S, \mathrm{Sym}^k \mathscr{H}(1)) \arrow[d, hook] \\
\mathrm{Eis}^{k+2}_{\mathrm{dR}}(\varphi) & \in & H^1_{\mathrm{dR}}(S, \mathrm{Sym}^k \mathscr{H}(1)) \\
\mathrm{Eis}^{k+2}_{\mathrm{syn}}(\varphi) \arrow[u, mapsto] & \in & H^1_{\mathrm{syn}}(S, \mathrm{Sym}^k \mathscr{H}(1)) \arrow[u, hook]
\end{tikzcd}
$$

The Hodge Realization

The Hodge Realization

Case $g = 2$

F: real quadratic field

$\mathcal{O}_F$: ring of integers

S: Hilbert modular surface

- dim 2 abelian variety
- real multiplication by $\mathcal{O}_F$
- level structure

$$S(\mathbb{C}) = \Gamma \setminus \mathfrak{H}^2 \qquad \Gamma \subset SL_2(\mathcal{O}_F)$$

$$\mathscr{H} = \underline{\omega} \bigoplus \underline{\omega}^\vee \qquad \underline{\omega} \cong \underline{\omega}_1 \bigoplus \underline{\omega}_2$$

$\leftarrow$ corresponding to embeddings $F \hookrightarrow \mathbb{C}$

On $\mathfrak{H}^2$

$$\mathscr{H} \cong \mathcal{O}_{\mathfrak{H}_2} y_1 \bigoplus \mathcal{O}_{\mathfrak{H}_2} y_2 \bigoplus \mathcal{O}_{\mathfrak{H}_2} \omega_1 \bigoplus \mathcal{O}_{\mathfrak{H}_2} \omega_2$$

$$\nabla(\omega_1) = y_1 \otimes d\tau_1, \quad \nabla(\omega_2) = y_2 \otimes d\tau_2, \quad \nabla(\omega_1) = \nabla(\omega_2) = 0$$

The Hodge Realization

Case $g = 2$

$$
\begin{array}{ccc}
H^3_{\mathscr{D}}(S, \mathrm{Sym}^k \mathscr{H}(2)) & \hookrightarrow & H^3_{\mathrm{dR}}(S, \mathrm{Sym}^k \mathscr{H}(2))) \\
\rotatebox{60}{$\in$} & & \cup \\
\mathrm{Eis}^{k+2}_{\mathscr{D}}(\varphi) & \longmapsto & \text{de Rham Class?}
\end{array}
$$

Guess

The Hodge Realization

$$H^3_{\mathscr{D}}(S, \mathrm{Sym}^k \mathscr{H}(2)) \hookrightarrow H^3_{\mathrm{dR}}(S, \mathrm{Sym}^k \mathscr{H}(2)))$$

$$\cup$$

Dolbeault class $\quad \xi := \sum_{a=0}^{k} E_{k+2}^{a+1}(\tau)\, \overline{\omega}_1^a \omega_1^{k-a} \omega_2^k \frac{d\tau_1 \wedge d\overline{\tau}_1}{A_1} \wedge d\tau_2$

Guess

$$E_{k+2}^a(\tau) = \sum_{\substack{(m,n)\neq(0,0) \\ (\mathrm{mod}\ \mathcal{O}_F^\times)}} \frac{1}{(m\overline{\tau}_1 + n)^a (m\tau_1 + n)^{k+2-a} (m'\tau_2 + n')^{k+2}}$$

$$A_1 := \frac{(\tau_1 - \overline{\tau}_1)}{2\pi i}$$

The Hodge Realization

$$
\begin{array}{ccc}
H^3_{\mathscr{D}}(S, \mathrm{Sym}^k \mathscr{H}(2)) & \hookrightarrow & H^3_{\mathrm{dR}}(S, \mathrm{Sym}^k \mathscr{H}(2))) \\
\in & & \cup \\
(\alpha, \xi) & \longmapsto & \xi := \displaystyle\sum_{a=0}^{k} E^{a+1}_{k+2}(\tau)\, \overline{\omega}_1^a \omega_1^{k-a} \omega_2^k \frac{d\tau_1 \wedge d\overline{\tau}_1}{A_1} \wedge d\tau_2
\end{array}
$$

$\nabla(\alpha) = (1 - F_\infty)\xi$ $\qquad$ F_∞: complex conjugation

$$
\alpha = \frac{(\tau_2 - \overline{\tau}_2)}{k+1} \sum_{a,b=0}^{k} E^{(a+1,b+1)}_{k+2}(\tau) \overline{\omega}_1^a \omega_1^{k-a} \overline{\omega}_2^b \omega_2^{k-b} \frac{d\tau_1 \wedge d\overline{\tau}_1}{A_1}
$$

$$
E^{(a,b)}_{k+2}(\tau) = \sum_{\substack{(m,n) \neq (0,0) \\ (\mathrm{mod}\ \mathcal{O}_F^\times)}} \frac{1}{(m\overline{\tau}_1 + n)^a (m\tau_1 + n)^{k+2-a} (m'\overline{\tau}_2 + n')^b (m'\tau_2 + n')^{k+2-b}}
$$

The Syntomic Realization

The Syntomic Realization

$$\xi := \sum_{a=0}^{k} E_{k+2}^{a+1}(\tau)\overline{\omega}_1^a \omega_1^{k-a} \omega_2^k \boxed{\frac{d\tau_1 \wedge d\overline{\tau}_1}{A_1^2}} \wedge A_1 d\tau_2$$

Chern Class

The Syntomic Realization

$$\xi := \sum_{a=0}^{k} E_{k+2}^{a+1}(\tau)\overline{\omega}_1^a \omega_1^{k-a} \omega_2^k \frac{d\tau_1 \wedge d\overline{\tau}_1}{A_1^2} \wedge A_1 d\tau_2$$

$$\frac{d\tau_1 \wedge d\overline{\tau}_1}{A_1^2} \qquad \cup \quad \boxed{\sum_{a=0}^{k} E_{k+2}^{a+1}(\tau)\, \underline{\overline{\omega}}_1^a \underline{\omega}_1^{k-a} \underline{\omega}_2^k\, A_1 d\tau_2} \quad =: \xi_0$$

$$\nabla(\xi_0) = (k+1)(1 - F_\infty^{(1)}) \left(E_{k+2}(\tau) \omega_1^k \omega_2^k d\tau_1 \wedge d\tau_2 \right)$$

$F_\infty^{(1)}$: complex conjugation in first variable

$$E_{k+2}(\tau) = \sum_{m,n \in \mathcal{O}_F} \frac{1}{(m\tau_1 + n)^{k+2}(m'\tau_2 + n')^{k+2}}$$

The Syntomic Realization

$$\begin{array}{ccc} E_{k+2}(\tau)\omega_1^k\omega_2^k d\tau_1 \wedge d\tau_2 & \in & H^2_{\mathrm{dR}}(S, \mathrm{Sym}^k \mathscr{H}) \end{array}$$

$$\begin{array}{cc} & \text{Integration after } (1 - F_\infty^{(1)}) \\ & \Big\downarrow{\scriptstyle p\text{-adic}} \\ q_1 \dfrac{d}{dq_1} & \text{Integration after } (1 - F_p^{(1)}) \end{array}$$

$F_p^{(1)}$: Frobenius in first variable

The Syntomic Realization

N. Katz

Values in
p-adic modular forms

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p} x_1^a x_2^b y^{k+1} d\mu(x_1, x_2, y) = (\text{constant}) \times E_{k+2}^{(-a,-b)}$$

$$a, b \geq 0$$

N. Katz, *p-adic L-functions for CM fields*, Invent. Math. **49** (1978), no. 3, 199–297.

$$(1 - F_p^{(1)}) \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p \times \mathbb{Z}_p} x_1^a x_2^b y^{k+1} d\mu(x_1, x_2, y)$$

Allows $a < 0$

Ordinary
Locus

May define p-adic analogues of $E_{k+2}^a(\tau) := E_{k+2}^{(-a,0)}(\tau)$

$$\xi_0 := \sum_{a=0}^{k} E_{k+2}^{a+1} u_1^a \omega_1^{k-a} \omega_2^k \frac{dq_1}{q_1}$$

The Syntomic Realization

$$\xi_0 := \sum_{a=0}^{k} E_{k+2}^{a+1} u_1^a \omega_1^{k-a} \omega_2^k \frac{dq_1}{q_1}$$

$$\begin{array}{ccc} H^3_{\mathrm{syn}}(S, \mathrm{Sym}^k \mathscr{H}(2)) & \hookrightarrow & H^3_{\mathrm{dR}}(S, \mathrm{Sym}^k \mathscr{H}(2)) \\ \rotatebox{90}{$\in$} & & \Cup \\ (\alpha, \xi) & \longmapsto & \xi := \text{Chern Class} \cup \xi_0 \end{array}$$

$$\nabla(\alpha) = (1 - F_p)\xi$$

$$(1 - F_p) \quad \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \times \mathbb{Z}_p} x_1^a x_2^b y^{k+1} d\mu(x_1, x_2, y)$$

Allows $a < 0, b < 0$

May define p-adic analogues of $E_{k+2}^{(a,b)}(\tau) := E_{k+2}^{(-a,-b)}(\tau)$

The Syntomic Realization

$$\xi_0 := \sum_{a=0}^{k} E_{k+2}^{a+1} u_1^a \omega_1^{k-a} \omega_2^k \frac{dq_1}{q_1}$$

$$\begin{array}{ccc} H^3_{\mathrm{syn}}(S, \mathrm{Sym}^k \mathscr{H}(2)) & \hookrightarrow & H^3_{\mathrm{dR}}(S, \mathrm{Sym}^k \mathscr{H}(2)) \\ \in & & \cup \\ (\alpha, \xi) & \longmapsto & \xi := \text{Chern Class} \cup \xi_0 \end{array}$$

Conclusion

$$\alpha = \text{Chern Class} \cup \frac{1}{k+1} \sum_{a,b=0}^{k} E_{k+2}^{(a+1,b+1)} u_1^a \omega_1^{k-a} u_2^b \omega_2^{k-b}$$